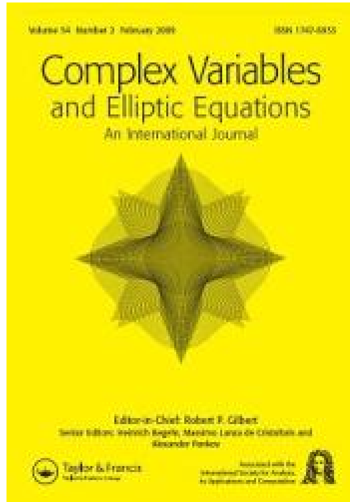

Completeness theorems for the Stokes system

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Completeness theorems for the Stokes system

A. Cialdea [*] G. Nino [†]

To Prof. Massimo Lanza de Cristoforis on occasion of his 60th birthday

Abstract The linear Stokes system is considered and the completeness (in the sense of Picone) on the boundary of a given bounded domain of polynomial solutions is proved. The completeness is obtained in both L^p and uniform norms.

Keywords Completeness theorems, Stokes system, Partial differential systems with constant coefficients.

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1 Introduction

The aim of the present paper is to prove the completeness on the boundary of a simply connected [1] bounded domain of polynomial solutions of the Stokes system:

$$\begin{cases} \Delta v - \nabla p = 0 \\ \nabla \cdot v = 0. \end{cases} \tag{1}$$

This completeness theorem concerns the Dirichlet problem and fits into the framework of the so called completeness theorems in the sense of Picone. This problem was posed long time ago by Mauro Picone and the very first results in this direction were proved by Gaetano Fichera [9] for the Laplacian.

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[1]By this we mean that both the domain and its complement are connected.

Several other results have been proved since then. They concern different boundary value problems for different partial differential equations. We refer to [6] for an history of this problem and an updated bibliography.

Here we just note two facts. First, usually these results concern with a boundary value problem for a particular partial differential equation. Quite recently in [4, 5] the Dirichlet problem for a general scalar elliptic equation of any order with constant coefficients has been considered and necessary and sufficient conditions have been established to guarantee the validity of the related completeness of polynomial solutions. Secondly, very little is known about systems. There are available results only for elasticity [3, 11] and thermoelasticity [6].

The present paper is organized as follows. Section 2 is devoted to some preliminaries concerning potential theory. In particular we give some "jump formulas" for the hydrodynamical potentials generated by measures and we prove a uniqueness result for such potentials.

Section 3 concerns the construction of the system of polynomial solutions of (1). This construction hinges on the results obtained in [1].

The completeness of polynomial solutions in $[L^p(\Sigma)]^3$ and in $[C^0(\Sigma)]^3$ are proved in Section 4 and Section 5, respectively.

Finally in Section 6 we consider multiple connected domains and we determine the closure in L^p norm of the linear space generated by the system of polynomial solutions.

2 Preliminaries

In this paper Ω is a connected bounded domain of $\mathbb{R}^3$. The boundary of Ω, which we denote by Σ is supposed to be C^1. Throughout this paper, ν is the exterior unit normal vector on Σ.

As well known, a simple layer hydrodynamical potential (v, p) with density ϕ is defined as

$$v_i(x) = -\int_\Sigma \phi_j(y)\gamma_{ij}(x-y)d\sigma_y\,, \qquad i = 1, 2, 3, \tag{2}$$

$$p(x) = -\int_\Sigma \phi_j(y)\epsilon_j(x-y)d\sigma_y\,, \tag{3}$$

where

$$\begin{aligned}\gamma_{ij}(x-y) &= -\frac{1}{4\pi}\left[\frac{\delta_{ij}}{|x-y|}-\frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}|x-y|\right], \\ \epsilon_j(x-y) &= \frac{1}{4\pi}\frac{\partial}{\partial x_j}\frac{1}{|x-y|},\end{aligned} \tag{4}$$

$(i,j=1,2,3)$ is the fundamental solution for the Stokes system (1).

The main results concerning such potentials can be found in the classical monograph by Ladyzhenskaya [13, Chapter 3]. Among them we recall the jump formulas related to the stress tensor

$$T_{ij} = -\delta_{ij}p + \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}. \tag{5}$$

Theorem 1 *Let $\Sigma \in C^1$. Let $\phi \in [L^1(\Sigma)]^3$. Let T_{ij} be the stress tensor* (5) *related to the potentials* (2), (3). *Let $x_0 \in \Sigma$ be a Lebesgue point of ϕ. Then*

$$\lim_{x\to x_0}\left(T_{ik}(x)\nu_k(x_0) - T_{ik}(x')\nu_k(x_0)\right) = \phi_i(x_0), \tag{6}$$

where x is a point on the inner normal to Σ at x_0 and x' is its symmetric with respect to x_0.

Proof. Under more particular hypothesis this result is classical. A proof under the assumptions given here can be obtained observing that

$$\begin{aligned}T_{ik}(x) = \frac{1}{4\pi}\int_\Sigma \phi_j(y)\Big[\delta_{ik}\frac{\partial}{\partial x_j}\frac{1}{|x-y|} + \delta_{ij}\frac{\partial}{\partial x_k}\frac{1}{|x-y|} + \\ \delta_{kj}\frac{\partial}{\partial x_i}\frac{1}{|x-y|} - \frac{\partial^3}{\partial x_i\partial x_j\partial x_k}|x-y|\Big]d\sigma_y\end{aligned} \tag{7}$$

and applying [5, Theorem 3, p.6-7] (see also [4, p.84]). □

In the present paper we shall consider also potentials generated by measures, i.e.

$$v_i(x) = -\int_\Sigma \gamma_{ij}(x-y)d\mu_y^j, \qquad i=1,2,3, \tag{8}$$

$$p(x) = -\int_\Sigma \epsilon_j(x-y)d\mu_y^j, \tag{9}$$

where $\mu = (\mu^1, \mu^2, \mu^3)$ is a vector measure with $\mu^j \in M(\Sigma) = [C^0(\Sigma)]^*$.

The boundary behaviour of the stress tensor related to these potentials is more delicate. In order to describe the behaviour of such potentials, let us introduce a family of "parallel surfaces" Σ_ρ. Let us denote by $\zeta(x)$ a unit vector of class $C^1(\Sigma)$ such that $\zeta(x) \cdot \nu(x) \geqslant \beta_0 > 0$. We can choose $\rho_0 > 0$ in such a way that the surface Σ_ρ defined by $x_\rho = x + \rho\zeta(x),\ x \in \Sigma$, is the boundary of a domain containing Ω (contained in Ω) if $0 < \rho \leqslant \rho_0$ $(-\rho_0 \leqslant \rho < 0)$. One can prove that if $\Sigma \in C^1$ such a vector does exist (see [12, pp. 273–275]).

We have the following result.

Theorem 2 *Let $\Sigma \in C^1$. Let $\mu \in [M(\Sigma)]^3$. Let T_{ij} be the stress tensor* (5) *related to the potentials* (8), (9). *Then, for any $\psi \in \left[C^\lambda(\mathbb{R}^3)\right]^3$, we have the following "jump formulas":*

$$\lim_{\rho\to 0^+}\left(\int_{\Sigma_\rho} \psi_i(x_\rho)\nu_k(x_\rho)T_{ik}(x_\rho)\, d\sigma_\rho - \int_{\Sigma_{-\rho}} \psi_i(x_{-\rho})\nu_k(x_{-\rho})T_{ik}(x_{-\rho})\, d\sigma_{-\rho}\right)$$
$$= -\int_\Sigma \psi_i(x)\, d\mu^i_x\,. \tag{10}$$

Proof. This result can be proved by means of [5, Theorem 5, p.11], keeping in mind formula (7). □

We give now the following uniqueness result.

Theorem 3 *Let v be the potential* (8), *where $\mu \in [M(\Sigma]^3$. If $v(x) = 0$ for any $x \in \mathbb{R}^3 \setminus \overline{\Omega}$, then $v(x) = 0$ for $x \in \Omega$.*

Proof. Let (φ, q) be a vector function belonging to $\left[C^{2,\lambda}(\overline{\Omega})\right]^4$. We have

$$-\int_{\Omega_{-\rho}} v_i\left(\Delta\varphi_i + \frac{\partial q}{\partial x_i}\right) dx = \int_{\Sigma_{-\rho}} (T_{ij}(v)\varphi_i\nu_j - T'_{ij}(\varphi)v_i\nu_j)\, d\sigma_{-\rho}\,, \tag{11}$$

where

$$T'_{ij} = \delta_{ij}q + \frac{\partial\varphi_i}{\partial x_j} + \frac{\partial\varphi_j}{\partial x_i}$$

(see [13, p.53]).

Since the vector v belongs to $[W^{1,1}_{\rm loc}(\mathbb{R}^3)]^3$ (see, e.g., [5, Lemma 4, p.14]) and $v(x) = 0$ for $x \in \mathbb{R}^3 \setminus \overline{\Omega}$, we have

$$\lim_{\rho\to 0^+} \int_{\Sigma_{-\rho}} T'_{ij}(\varphi) v_i \nu_j \, d\sigma_{-\rho} = 0.$$

Moreover, denoting by $\widetilde{\varphi}$ a Hölder continuous function defined in $\mathbb{R}^3$ which coincides with φ in $\overline{\Omega}$ [2], formula (10) shows that

$$\lim_{\rho\to 0^+} \int_{\Sigma_{-\rho}} T_{ij}(v)\varphi_i\nu_j \, d\sigma_{-\rho} =$$

$$\lim_{\rho\to 0^+} \left(\int_{\Sigma_{-\rho}} T_{ij}(v)\varphi_i\nu_j \, d\sigma_{-\rho} - \int_{\Sigma_\rho} T_{ij}(v)\widetilde{\varphi}_i\nu_j \, d\sigma_\rho \right) = \int_\Sigma \varphi_i \, d\mu^i$$

Letting $\rho \to 0^+$ in (11) we find

$$\int_\Omega v_i \left(\Delta\varphi_i + \frac{\partial q}{\partial x_i} \right) dx = - \int_\Sigma \varphi_i \, d\mu^i \, .$$

This leads to

$$\int_\Omega v_i \, \Delta\varphi_i \, dx = 0$$

for any $\varphi \in \left[C^{2,\lambda}(\overline{\Omega})\right]^3$ such that $\varphi = 0$ on Σ, and this implies the result. $\square$

3 Polynomials solutions

In paper [1] a system of polynomial solutions of the Stokes system (1) has been constructed. Unfortunately a small computational mistake led to a wrong expression for the function q, which was used again in [2].

[2] It suffices to take

$$\widetilde{\varphi}(x) = \inf_{y\in\overline{\Omega}} \left\{ \varphi(y) + C|x-y|^\lambda \right\},$$

where $C > 0$ and $0 < \lambda \leqslant 1$ are such that $|\varphi(x) - \varphi(y)| \leqslant C|x-y|^\lambda$, for any $x, y \in \overline{\Omega}$.

Theorem 4 *Let $v = (v_1, \ldots, v_n)$ and q be a homogeneous vector polynomial of degree k and a homogeneous scalar polynomial of degree $k-1$, respectively $(k \geqslant 2)$. The vector (v, q) satisfies Stokes system (1) if and only if*

$$
\begin{aligned}
v(x) &= H_k(x) - \frac{1}{2(k-1)} |x|^2 \nabla(\nabla \cdot H_k(x)), \\
q(x) &= -\frac{n+2k-4}{k-1} \nabla \cdot H_k(x),
\end{aligned}
\tag{12}
$$

H_k being a harmonic vector homogeneous polynomial of degree k.

Proof. We shall not repeat the proof given in [1]. We just mention that the sixth formula from the top at [1, p.317] has to be replaced by

$$2nH_{k-2}(x) + 4(x \cdot \nabla)H_{k-2}(x) = \nabla H_{k-1}.$$

Repeating the proof given in [1] the reader will find without difficulties that the right expression of the sought polynomials is (12). □

By means of the previous result, we can easily construct a complete system of polynomial solutions of Stokes system. By this we mean a system of homogeneous polynomials such that any polynomial satisfying Stokes system (1) can be written as a finite linear combination of vectors of this system.

We shall write it explicitly in the case $n = 3$. With the same idea it can be written in any dimension.

Let us denote by $\{\omega_{ks}\}(s = 1, \ldots, 2k+1; k = 0, 1, \ldots)$ a complete system of harmonic polynomials, i.e.

$$\omega_{ks} = |x|^k Y_{ks}\left(\frac{x}{|x|}\right) \qquad (s = 1, \ldots, 2k+1; k = 0, 1, \ldots) \tag{13}$$

$\{Y_{ks}\}$ being the system of spherical harmonics.

Theorem 5 *Let $v = (v_1, v_2, v_3)$ and p be homogeneous polynomials of degree k and $k-1$ respectively $(k > 1)$. The vector (v, p) satisfies the Stokes system (1) if ,and only if, it is a linear combination of the following $6k+3$*

polynomials:

$$\begin{gathered}\left(\omega_{ks}+\frac{1}{2(1-k)}|x|^2\partial_{11}\omega_{ks},\frac{1}{2(1-k)}|x|^2\partial_{21}\omega_{ks},\frac{1}{2(1-k)}|x|^2\partial_{31}\omega_{ks},\frac{2k-1}{1-k}\partial_1\omega_{ks}\right),\\ \left(\frac{1}{2(1-k)}|x|^2\partial_{12}\omega_{ks},\omega_{ks}+\frac{1}{2(1-k)}|x|^2\partial_{22}\omega_{ks},\frac{1}{2(1-k)}|x|^2\partial_{32}\omega_{ks},\frac{2k-1}{1-k}\partial_2\omega_{ks}\right),\\ \left(\frac{1}{2(1-k)}|x|^2\partial_{13}\omega_{ks},\frac{1}{2(1-k)}|x|^2\partial_{23}\omega_{ks},\omega_{ks}+\frac{1}{2(1-k)}|x|^2\partial_{33}\omega_{ks},\frac{2k-1}{1-k}\partial_3\omega_{ks}\right),\end{gathered} \tag{14}$$

$(s=1,\ldots,2k+1)$, ω_{ks} *being the harmonic polynomials* (13).

Proof. The result follows immediately from Theorem 4 and from the well known fact that any harmonic polynomial can be written as a linear combination of elements of $\{\omega_{ks}\}$. □

We denote by $\{W_k\}$ $(k=0,1,2\ldots)$ the system constituted by the vector polynomials

$$\begin{gathered}(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),\\ (x_2,0,0,0),(x_3,0,0,0),(0,x_1,0,0),(0,x_3,0,0),(0,0,x_1,0),(0,0,x_2,0),\\ (x_1,-x_2,0,0),(x_1,0,-x_3,0),\end{gathered}$$

and all the polynomials given by (14) (s=1,. . . , 2k + 1; k=0,1,. . .), ordered in one sequence.

Clearly any polynomial solution of Stokes system (1) can be written as a finite linear combination of elements of $\{W_k\}$.

4 Completeness in $L^p(\Sigma)$

.

We start with the following Lemma, in which we denote by w_k the 3-dimensional vector given by the first three components of W_k.

Lemma 1 *Let* Ω *be a bounded domain with a* C^1 *boundary and such that* $\mathbb{R}^3\setminus\overline{\Omega}$ *is connected. Let* $\mu=(\mu^1,\mu^2,\mu^3)\in[M(\Sigma)]^3$. *If*

$$\int_\Sigma w_k\,d\mu=0,\qquad k=0,1,2,\ldots, \tag{15}$$

then

$$\int_{\Sigma} \gamma_{ij}(x-y) d\mu^j_y = 0, \qquad (i=1,2,3),$$

for any $x \in \mathbb{R}^3 \setminus \overline{\Omega}$, *where* γ_{ij} *è is the fundamental velocity tensor.*

Proof. The following expansions hold:

$$\frac{1}{|x-y|} = \sum_{k=0}^{\infty} \frac{|y|^k}{|x|^{k+1}} X_k(\cos\gamma),$$

$$|x-y| = \sum_{k=0}^{\infty} \left[-\frac{1}{2k-1} \frac{|y|^k}{|x|^{k-1}} X_k(\cos\gamma) + \frac{1}{2k+3} \frac{|y|^{k+2}}{|x|^{k+1}} X_k(\cos\gamma) \right],$$

where $X_k(t)$ denotes the Legendre polynomial

$$\frac{1}{2^k k!} \frac{d^k}{dt^k} (t^2-1)^k$$

and $\cos\gamma = x \cdot y/(|x|\,|y|)$.

If we fix $x \neq 0$, these expansions uniformly converge in any compact set contained in the ball $|y| < |x|$. The first expansion is very well known. For the other one, see [8, p.196].

By means of the spherical harmonic addition theorem, we get

$$\frac{1}{|x-y|} = \sum_{k=0}^{\infty} \sum_{s=0}^{2k+1} \lambda_{ks}(x)\, \omega_{ks}(y),$$

$$|x-y| = \sum_{k=0}^{\infty} \sum_{s=0}^{2k+1} \left[\chi_{ks}(x)\, \omega_{ks}(y) + \tau_{ks}(x)\, |y|^2 \omega_{ks}(y) \right]$$

where ω_{ks} are the harmonic polynomials (13) and

$$\lambda_{ks}(x) = \frac{1}{|x|^{k+1}} Y_{ks}\left(\frac{x}{|x|}\right),$$

$$\chi_{ks}(x) = -\frac{1}{2k-1} \frac{1}{|x|^{k-1}} Y_{ks}\left(\frac{x}{|x|}\right), \quad \tau_{ks}(x) = \frac{1}{2k+3} \frac{1}{|x|^{k+1}} Y_{ks}\left(\frac{x}{|x|}\right).$$

From (4) we find

$$4\pi\gamma_{ij}(x-y) = -\left[\frac{\delta_{ij}}{|x-y|} - \frac{1}{2}\frac{\partial^2}{\partial y_i \partial y_j}|x-y|\right] =$$
$$-\delta_{ij}\sum_{k=0}^{\infty}\sum_{s=0}^{2k+1}\lambda_{ks}(x)\,\omega_{ks}(y)+ \tag{16}$$
$$\frac{1}{2}\sum_{k=0}^{\infty}\sum_{s=0}^{2k+1}\left(\chi_{ks}(x)\frac{\partial^2}{\partial y_i\partial y_j}\omega_{ks}(y) + \tau_{ks}(x)\frac{\partial^2}{\partial y_i\partial y_j}(|y|^2\omega_{ks}(y))\right).$$

We remark that

$$\nabla\frac{\partial}{\partial y_i}\omega_{ks}(y)$$

is a divergence-free harmonic vector. Therefore it satisfies Stokes system (1) ($p=0$). Consider now the sum of the first and the last series in the right hand side of (16). The generic term we obtain is equal to

$$-\delta_{ij}\lambda_{ks}(x)\,\omega_{ks}(y) + \frac{1}{2}\tau_{ks}(x)\frac{\partial^2}{\partial y_i\partial y_j}(|y|^2\omega_{ks}(y)) =$$
$$\tau_{ks}(x)\left(-\delta_{ij}(2k+3)\,\omega_{ks}(y) + \frac{1}{2}\frac{\partial^2}{\partial y_i\partial y_j}(|y|^2\omega_{ks}(y))\right).$$

Let us fix k and s and denote by $\Phi_{ij}(y)$ the function

$$-\delta_{ij}(2k+3)\,\omega_{ks}(y) + \frac{1}{2}\frac{\partial^2}{\partial y_i\partial y_j}(|y|^2\omega_{ks}(y))$$

and by Φ_i the vector $(\Phi_{i1},\Phi_{i2},\Phi_{i3})$. Since $\Delta(|y|^2\omega_{ks}(y)) = 2(2k+3)\omega_{ks}(y)$, we have

$$\Delta\left(-\delta_{ij}(2k+3)\,\omega_{ks}(y) + \frac{1}{2}\frac{\partial^2}{\partial y_i\partial y_j}(|y|^2\omega_{ks}(y))\right) = (2k+3)\frac{\partial^2\omega_{ks}}{\partial y_i\partial y_j}.$$

This shows that $\Delta\Phi_i = \nabla p$, where

$$p = (2k+3)\frac{\partial\omega_{ks}}{\partial y_i}. \tag{17}$$

Moreover Φ_i is divergence-free, since

$$\frac{\partial\Phi_{ij}}{\partial y_j} = -(2k+3)\frac{\partial\omega_{ks}}{\partial y_i} + \frac{1}{2}\frac{\partial}{\partial y_i}\Delta(|y|^2\omega_{ks}(y)) = 0.$$

We have then shown that the vector Φ_i satisfies Stokes system (1) (where p is given by (17)). In view of (16) we can write

$$\gamma_i(x-y) = \sum_{h=0}^{\infty} c_{ih}(x) v_h(y)$$

uniformly for $y \in \overline{\Omega}$, provided that $|x| > R$, where $R = (\max_{x\in\overline{\Omega}} |x|)/r$. Here c_{ih} are scalar functions and v_h are vector homogenous polynomials satisfying Stokes system.

Orthogonality conditions (15) imply

$$\int_\Sigma \gamma_{ij}(x-y) d\mu_y^j = \sum_{h=0}^{\infty} c_{ih}(x) \int_\Sigma v_h(y)\, d\mu_y = 0$$

for any x such that $|x| > R$. The potentials on the left hand side being analytic in the connected domain $\mathbb{R}^3 \setminus \overline{\Omega}$, the thesis follows. $\square$

We are now in a position to prove the completeness of $\{w_k\}$ in the L^p-norm.

Theorem 6 *Let $\Omega \in \mathbb{R}^3$ a bounded connected domain with a C^1 boundary and such that $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected. The system $\{w_k\}$ is complete in the subspace of $[L^p(\Sigma)]^3$*

$$\left\{ f \in [L^p(\Sigma)]^3 \mid \int_\Sigma f \cdot \nu \, d\sigma = 0 \right\} \tag{18}$$

($1 \leqslant p < \infty$).

Proof. In order to prove this result, it is sufficient to show that if $\phi \in [L^q(\Sigma)]^3$ ($q = p/(p-1)$) is such that

$$\int_\Sigma \phi \cdot w_k \, d\sigma = 0, \qquad k = 0, 1, 2, \dots,$$

then $\phi = c\nu$. Thanks to Lemma 1, we have

$$\int_\Sigma \phi_j(y) \gamma_{ij}(x-y) d\sigma_y = 0, \qquad x \in \mathbb{R}^3 \setminus \overline{\Omega},\ i = 1, 2, 3.$$

The simple layer potential (v_1, v_2, v_3, p), where

$$v_i(x) = v_i(\phi, x) = -\int_\Sigma \phi_j(y)\gamma_{ij}(x-y)d\sigma_y\,,$$
$$p(x) = p(\phi, x) = -\int_\Sigma \phi_i(y)\epsilon_i(x-y)d\sigma_y\,,$$

satisfies Stokes system (1) for $x \notin \Sigma$. In $\mathbb{R}^3 \setminus \overline{\Omega}$ we have $v = 0$, and then $\nabla p = 0$. The set $\mathbb{R}^3 \setminus \overline{\Omega}$ being connected, we find $p = c$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. On the other hand, we have that the potential representing p vanishes at infinity and then $p = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$.

Theorem 3 shows that $v = 0$ (and then $\nabla p = 0$) also in Ω. Then the potential (3) is equal to a real constant $-c$ in Ω and vanishes in $\mathbb{R}^3 \setminus \overline{\Omega}$.

Therefore

$$T_{ij} = \begin{cases} \delta_{ij}c & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases}$$

In view of jump formula (6), we get $\phi = c\nu$ a.e. on Σ and the Theorem is proved. $\square$

We note that this completeness Theorems implies the Runge property for Stokes system (1). Indeed we have:

Theorem 7 *Let A be a domain such that $\mathbb{R}^3 \setminus A$ is connected. Let $U = (u, p) \in [C^\infty(A)]^4$ be a solution of system (1) in A. For any compact set $K \subset A$ there exists a sequence v_n of polynomials solutions of (1) such that $v_n \to u$ uniformly in K.*

Proof. Let Ω be a bounded domain with smooth boundary such that $K \subset \Omega$, $\overline{\Omega} \subset A$ and $\mathbb{R}^3 \setminus \Omega$ is connected. Because of the completeness property proved in Theorem 6 there exists a sequence v_n of polynomial solutions of (1) such that $v_n \to u$ in $[L^2(\partial\Omega)]^3$. In view of classical results, this implies that $v_n \to u$ uniformly in K. $\square$

5 Completeness in the uniform norm

The problem of the completeness in uniform norm is much more delicate, as it involves potentials generated by measures.

Theorem 8 *Let $\Omega \subset \mathbb{R}^3$ be a bounded connected domain with a C^1 boundary such that $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected. The system $\{w_k\}$ is complete in the subspace of $[C^0(\Sigma)]^3$*

$$\left\{ f \in \left[C^0(\Sigma)\right]^3 \mid \int_\Sigma f \cdot \nu \, d\sigma = 0 \right\} .$$

Proof. In order to prove this theorem, it is sufficient to show that if $\mu \in [M(\Sigma)]^3$ is such that

$$\int_\Sigma w_k \, d\mu = 0 \qquad k = 0, 1, 2, \dots, \tag{19}$$

then

$$d\mu = c\nu d\sigma \tag{20}$$

where c is a real constant and σ is the $(n-1)$-dimensional Lebesgue measure on Σ. Equation (20) means that $\mu \ll \sigma$ (i.e. μ is absolutely continuous with respect to σ) and that the Radon-Nikodym derivative of μ with respect to σ is $c\nu$, i.e.

$$\mu^i(A) = c \int_A \nu^i \, d\sigma, \quad i = 1, 2, 3,$$

for any Borel set $A \subset \Sigma$.

Lemma 1 shows that conditions (19) imply

$$v_i(x) = - \int_\Sigma \gamma_{ij}(x-y) \, d\mu_y^j = 0 \, , \qquad x \in \mathbb{R}^3 \setminus \overline{\Omega}, \; i = 1, 2, 3.$$

Theorem 3 allows to conclude that $v(x) = 0$ also in Ω.

As in the proof of Theorem 6, there exists a real constant c such that

$$p(x) = - \int_\Sigma \epsilon_j(x-y) \, d\mu_y^j = \begin{cases} -c & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^3 \setminus \overline{\Omega}, \end{cases}$$

and then

$$T_{ij} = \begin{cases} \delta_{ij} c & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases}$$

Let $\psi \in \left[C^\lambda(\mathbb{R}^3)\right]^3$. In view of Theorem 2, we get

$$\int_\Sigma \psi_i \, d\mu^i =$$

$$-\lim_{\rho\to 0^+}\left(\int_{\Sigma_\rho} \psi_i(x_\rho)\nu_k(x_\rho)T_{ik}(x_\rho)\, d\sigma_\rho - \int_{\Sigma_{-\rho}} \psi_i(x_{-\rho})\nu_k(x_{-\rho})T_{ik}(x_{-\rho})\, d\sigma_{-\rho}\right) =$$

$$\lim_{\rho\to 0^+} c\int_{\Sigma_{-\rho}} \psi_i(x_{-\rho})\,\nu_i(x_{-\rho})\, d\sigma_{-\rho} = c\int_\Sigma \psi_i\,\nu_i\, d\sigma\,.$$

Because of the arbitrariness of ψ, we have $\mu \ll \sigma$ and $d\mu = c\nu d\sigma$. $\square$

6 Multiple connected domains

In this section we consider domains which are connected but not simply connected. Saying that Ω is a multiple connected domain, or - more precisely - an $(m+1)$-connected domain, we mean that Ω is an open connected set of the form

$$\Omega = \Omega_0 \setminus \bigcup_{j=1}^{m} \overline{\Omega}_j, \tag{21}$$

where each Ω_j $(j = 0, \ldots, m)$ is a bounded domain of $\mathbb{R}^3$ with connected boundaries $\Sigma_j \in C^1$ $(j = 0, \ldots, m)$ and such that

$$\overline{\Omega}_j \subset \Omega_0 \;\text{ and }\; \overline{\Omega}_j \cap \overline{\Omega}_k = \emptyset, \quad j,k = 1,\ldots,m,\; j \neq k.$$

For such domains the polynomial solutions are not complete in the subspace (18) anymore. We are going to determine the closure of the space generated by such systems. What comes out is that such a closure can be described by means of the boundary values of the solutions of some transmission problems.

In the following Theorem, writing $T_{ij}(v,p)$ we mean the stress tensor (5) related to the vector (v,p) and the suffix $^+$ $(^-)$ will denote the limit from the interior (exterior) of Ω.

Theorem 9 *Let $1 \leqslant p < \infty$. Let Ω be the $(m+1)$-connected domain* (21)*. Suppose $f = (f_1, f_2, f_3) \in [L^q(\Sigma)]^3$ $(q = p/(p-1)$ if $p > 1$; $q = \infty$ if $p = 1)$*

is such that

$$\int_\Sigma f \cdot w_k \, d\sigma = 0, \quad k = 0, 1, 2, \dots . \tag{22}$$

Then there exists (v, p) *solution of Stokes system* [3]

$$\begin{cases} \Delta v - \nabla p = 0, & \text{in } \Omega \\ \nabla \cdot v = 0, & \text{in } \Omega \end{cases} \tag{23}$$

satisfying the boundary condition

$$v = 0, \quad \text{on } \Sigma_0, \tag{24}$$

and such that, denoting by $(v^{(h)}, p^{(h)})$ *the solution of the BVP*

$$\begin{cases} \Delta v^{(h)} - \nabla p^{(h)} = 0, & \text{in } \Omega_h \\ \nabla \cdot v^{(h)} = 0, & \text{in } \Omega_h \\ v^{(h)} = v, & \text{on } \Sigma_h, \end{cases} \tag{25}$$

$(h = 1, \dots, m)$*, we have*

$$\begin{cases} f_i = T^+_{ij}(v, p)\nu_j, & \text{on } \Sigma_0 \\ f_i = T^+_{ij}(v, p)\nu_j - T^-_{ij}(v^{(h)}, p^{(h)})\nu_j\,, & \text{on } \Sigma_h, \ h = 1, \dots, m. \end{cases} \tag{26}$$

Proof. As in the proof of Theorem 6, conditions (22) imply that the potentials

$$v_i(x) = -\int_\Sigma f_j(y)\gamma_{ij}(x - y)d\sigma_y\,, \qquad i = 1, 2, 3, \tag{27}$$

$$p(x) = -\int_\Sigma f_j(y)\epsilon_j(x - y)d\sigma_y\,, \tag{28}$$

vanish in $\mathbb{R}^3 \setminus \overline{\Omega_0}$.

Let us denote by (v, p) the vector given by (27)-(28) for $x \in \Omega$. This vector satisfies Stokes system (1) and the boundary condition (24). In view of (6) we get also the first equation in (26).

[3] This and the other BVPs are considered in suitable spaces of potentials with L^p densities. See [7, Theorem 23] for further details.

For any $1 \leqslant h \leqslant m$, the vector $(v^{(h)}, p^{(h)})$ given by (27)-(28) for $x \in \Omega_h$, is solution of the BVP (25) and f satisfies the condition (26) on Σ_h because of (6).

Conversely, suppose (23)-(26) are satisfied. From the Green formula (see [13, formula (10), p.53] and writing the vector W_k as $(w_{k1}, w_{k2}, w_{k,3}, w_{k4}) = (w_k, w_{k4})$, we get

$$\int_\Sigma f \cdot w_k \, d\sigma = \int_{\Sigma_0} f \cdot w_k \, d\sigma + \sum_{h=1}^m \int_{\Sigma_h} f \cdot w_k \, d\sigma =$$

$$\int_{\Sigma_0} T_{ij}^+(v,p)\nu_j w_{ki} \, d\sigma + \sum_{h=1}^m \int_{\Sigma_h} \left(T_{ij}^+(v,p) - T_{ij}^-(v^{(h)}, p^{(h)})\right) \nu_j w_{ki} \, d\sigma =$$

$$\int_\Sigma T_{ij}^+(v,p)\nu_j w_{ki} \, d\sigma - \sum_{h=1}^m \int_{\Sigma_h} T_{ij}^-(v^{(h)}, p^{(h)})\nu_j w_{ki} \, d\sigma =$$

$$\int_\Sigma T_{ij}'^+(w_k, w_{k4})\nu_j v_i \, d\sigma - \sum_{h=1}^m \int_{\Sigma_h} T_{ij}'^-(w_k, w_{k4})\nu_j v_i^{(h)} d\sigma =$$

$$\sum_{h=1}^m \int_{\Sigma_h} \left(T_{ij}'^+(w_k, w_{k4}) - T_{ij}'^-(w_k, w_{k4})\right) \nu_j v_i \, d\sigma \, .$$

The integrals in the last hand side vanish, because

$$T_{ij}'^-(w_k, w_{k4}) = T_{ij}'^+(w_k, w_{k4}).$$

In fact the first derivatives of vectors W_k have no jumps across Σ, since they are C^∞ all over the space $\mathbb{R}^3$. This shows that (22) are satisfied and the theorem is proved. $\square$

As a Corollary we can describe the linear space generated by $\{w_k\}$ as the orthogonal complement of a certain linear space.

Theorem 10 *The closure in $[L^p(\Sigma)]^3$ of the linear space generated by the system $\{w_k\}$ is constituted by the vectors $G \in [L^p(\Sigma)]^3$ such that*

$$\int_\Sigma F \cdot G \, d\sigma = 0,$$

for any $F \in [L^q(\Sigma)]^3$ given by (26), with (v,p) and $(v^{(h)}, p^{(h)})$ $(h = 1, \ldots, m)$ satisfying (23)-(25).

Proof. It follows immediately from Theorem 9. □

Finally we mention that a complete system in a $(m+1)$-connected domain could be obtained as done in [9, 10] for Laplace equation and elasticity.

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