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Singularities and MZVs in Feynman Loop Integrals

By

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Abstract

Feynman diagram is a theoretical tool for computing physical quantities in particle physics, and each diagram is given as a multiple integral. Often Feynman diagrams can be expressed by multiple zeta values (MZVs) and their generalizations. We discuss relations between singularities of diagrams and types of MZVs from a physicist's viewpoint. This provides a hint to a connection between topology of a diagram and its value in terms of MZVs. New technologies for computing complicated diagrams and unsolved problems are also discussed.

§ 1. Introduction

Feynman diagram is a theoretical tool for computing physical quantities in quantum field theory and elementary particle physics. There is a specific rule to associate a value to a given Feynman diagram. Often the value of a diagram with non-trivial topology includes multiple zeta values (MZVs) and their generalizations.

For example, at present a fundamental physical constant *the fine structure constant* α (related to the charge of electron) is determined most accurately from comparisons of experimentally measured values of the anomalous magnetic moment of electron $(g_e - 2)$ and its theoretical prediction. The current accuracy of α determined in this way is better than ten digit accuracy! ($1/\alpha = 137.035999074(44)$) The theoretical prediction for $a_e \equiv g_e - 2$ is given as a function of α by the perturbative expansion[1]

$$a_e = a_e^{(2)}\left(\frac{\alpha}{\pi}\right) + a_e^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + a_e^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + a_e^{(8)}\left(\frac{\alpha}{\pi}\right)^4 + a_e^{(10)}\left(\frac{\alpha}{\pi}\right)^5 + \cdots , \tag{1.1}$$

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[1] $\mathcal{O}(m_e/m_\mu)$ terms are suppressed.

where the expansion coefficients are given by

$$a_e^{(2)} = \frac{1}{2}, \tag{1.2}$$

$$a_e^{(4)} = \frac{197}{144} + \frac{\pi^2}{12} + \frac{3}{4}\zeta(3) - \frac{\pi^2}{2}\ln 2, \tag{1.3}$$

$$\begin{aligned} a_e^{(6)} = {} & \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left(\mathrm{Li}_4\Big(\frac{1}{2}\Big) + \frac{1}{24}\ln^4 2 - \frac{1}{24}\pi^2\ln^2 2\right) \\ & - \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}. \end{aligned} \tag{1.4}$$

By equating a_e to the experimentally measured value, we obtain the value of α. Since the expansion parameter α/π is much smaller than one, the above expansion gives a fairly accurate prediction with only first several terms. The expansion coefficients can be computed by evaluating Feynman diagrams. So far, they have been computed analytically up to the third term ($a_e^{(6)}$) [1], while they have been computed numerically up to the fifth term ($a_e^{(10)}$) [2]. The above analytic expressions include a class of transcendental numbers, which can be expressed as infinite sums:

$$\zeta(n) = \sum_{m=1}^{\infty} \frac{1}{m^n}, \qquad \ln 2 = -\sum_{m=1}^{\infty} \frac{(-1)^m}{m}, \tag{1.5}$$

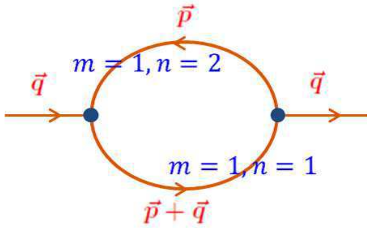
$$-2\left(\mathrm{Li}_4\Big(\frac{1}{2}\Big) + \frac{1}{24}\ln^4 2 - \frac{1}{24}\pi^2\ln^2 2\right) + \frac{\pi^4}{180} = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n}, \tag{1.6}$$

where $\zeta(x)$ and $\mathrm{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$ denote the Riemann zeta function and polylogarithm, respectively. The infinite sums belong to generalized MZVs.

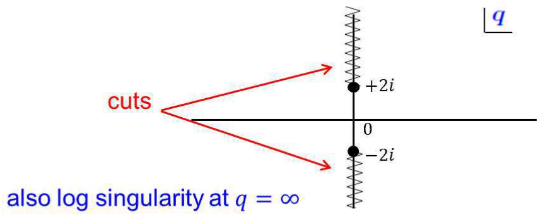
As the above example shows, it is an important theme of today's quantum field theory and elementary particle physics to compute Feynman diagrams. To compute higher orders of perturbative expansions for various physical quantities, one needs to evaluate more complicated Feynman diagrams (with more complicated topologies). Empirically, they are often expressed by more complicated (generalized) MZVs.

Relations between topologies of Feynman diagrams and MZVs are the subjects of common interests for mathematicians and physicists; see e.g. [3, 4, 5]. Mathematicians want to relate a topology, represented by a Feynman diagram, to MZVs as given by the value of the diagram. This may be used to characterize topologies if the relations are understood well. Physicists want to find a systematic way to relate a topology of a diagram to the value of the diagram when expressed in terms of MZVs. If this is achieved, physicists may find an efficient way to evaluate Feynman diagrams in terms of MZVs.

Each Feynman diagram is known to be characterized by various singularities which have clear physical origins. The singularities are closely tied to the topology of the



(a)



(b)

Figure 1. (a) Example of a Feynman diagram with external legs. The internal lines correspond to massive propagators with mass $m = 1$. The upper internal line has the propagator power $n = 2$ and the lower internal line has $n = 1$. (b) Analyticity of $I(q)$ in the complex q-plane.

diagram. In this article we discuss connections between these singularities in Feynman diagrams and generalized MZVs contained in the diagrams. To our knowledge there exists no systematic argument for this connection, and we present an argument as obtained from experiences in explicit computations of Feynman diagrams in particle physics. We also explain some aspects involved in contemporary computations at higher orders.

This article is organized as follows. In Sec. 2, examples of Feynman diagrams are shown. In Sec. 3, we explain singularities in Feynman diagrams and their physical origins. Then we discuss a relation between singularities of diagrams and types of MZVs in the diagrams in Sec. 4. We continue the discussion to more complicated cases in Sec. 5. A new relation among complicated MZVs is explained in Sec. 6. An example of an unsolved problem is given in Sec. 7. Finally we summarize our arguments in Sec. 8.

§ 2. Examples of Feynman Diagrams

An example of a Feynman diagram (with external legs) is shown in Fig. 1(a). To each internal line i, we assign a four-dimensional vector (momentum) $\vec{p}_i \in \mathbb{R}^4$, a mass $m_i \in \mathbb{R}$ and an integer n_i.[2] To each external line j a momentum $\vec{p}_j \in \mathbb{R}^4$ is also assigned. Each internal line represents a propagator $\left(\dfrac{1}{|\vec{p}_i|^2 + m_i^2}\right)^{n_i}$. At each vertex, momentum conservation $\sum_i \vec{p}_i = \vec{0}$ is imposed (including both internal and external momenta). We take the product of propagators and integrate over all possible internal momenta. Thus, the value of the diagram in Fig. 1(a) is given by

$$I(q) = \int d^4\vec{p}\, \frac{1}{(|\vec{p}|^2 + 1)^2(|\vec{p} + \vec{q}|^2 + 1)}, \tag{2.1}$$

where $\vec{p}, \vec{q} \in \mathbb{R}^4$. The integral region is entire $\mathbb{R}^4$. Since the integral is invariant under rotation of $\vec{q}$ (SO(4) invariant), it depends only on $q \equiv |\vec{q}|$. We can evaluate the above

[2]The four dimensions correspond to the dimensions of our space-time.

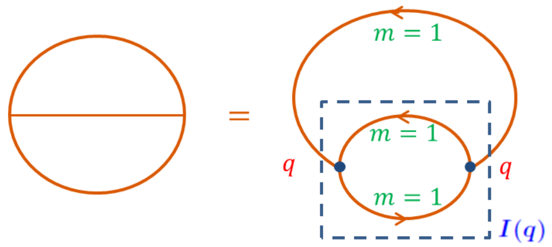


Figure 2. Example of a Feynman diagram without external legs. It is given by closing the external lines of the diagram in Fig. 1(a) after assigning $m = 1$ and $n = 2$ to the closing line.

integral easily and find

$$I(q) \propto \frac{1}{q^2\sqrt{1+4/q^2}} \log\left(\frac{\sqrt{1+4/q^2}+1}{\sqrt{1+4/q^2}-1}\right) \tag{2.2}$$

up to an irrelevant overall constant.

We may extend q to a complex variable and examine the analyticity of $I(q)$ in the complex q-plane. See Fig. 1(b). The square-root $\sqrt{1+4/q^2}$ generates branch points at $q = \pm 2i$. Branch cuts start from these branch points. One may check that there is also a logarithmic singularity at $q = \infty$.

An example of a Feynman diagram without external legs is shown in Fig. 2. It is obtained by closing the external lines of the diagram in Fig. 1(a), after assigning $m = 1$ and $n = 2$ to the closing line. By explicit computation, one finds that this diagram is given by

$$\int d\vec{q}\, \frac{1}{(|\vec{q}|^2+1)^2}\, I(q) \propto \mathrm{Cl}_2\left(\frac{\pi}{3}\right) = \mathrm{Im}\left[\sum_{n=1}^{\infty} \frac{e^{i\pi n/3}}{n^2}\right]. \tag{2.3}$$

Thus, it is given by a special value of the Clausen function, which is a special type of dilogarithm.

The generalized MZVs considered in this article are defined by a nested sum

$$Z(\infty; a_1, a_2, \ldots, a_N; \lambda_1, \lambda_2, \ldots, \lambda_N) = \sum_{n_1 > n_2 > \cdots > n_N > 0} \frac{\lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_N^{n_N}}{n_1^{a_1} n_2^{a_2} \cdots n_N^{a_N}}, \tag{2.4}$$

where $a_i \in \mathbb{N}$ and $a_1 \geq 2$. In the case $\lambda_i \in \{1\}$ it is simply called a MZV; in the case $\lambda_i \in \{-1, 1\}$ it is called a sign-alternating Euler sum. We also consider the cases that λ_i is a root of unity (e.g. $\lambda_i = e^{i\pi/3}$) and $\lambda_i \in \mathbb{C}$. In computations of Feynman diagrams, including the above example (2.3), these types of generalized MZVs appear in the analytic results. Hereafter, we refer to generalized MZV simply as MZV.

Each MZV also has a nested integral representation. By way of example,

$$\int_0^1 \frac{dx}{x} \int_0^x \frac{dy}{y-\alpha} \int_0^y \frac{dz}{z-\beta} = -Z\Big(\infty; 2, 1; \frac{1}{\alpha}, \frac{\alpha}{\beta}\Big), \tag{2.5}$$

which can be verified easily by rewriting the integrand as an infinite series expansion in y and z and integrating each term. As inferred from this example, in the case that λ_i's are roots of unity, generally the nested integral representation has singularities of the integral variables at zero or at roots of unity. We will use this property below.

§ 3. Singularities in Feynman Diagrams

A Feynman diagram is regarded as an analytic function of SO(4) invariants of the momenta of external lines. Singularities contained in a Feynman diagram can be classified according to their physical origins as follows:

- IR singularity, generated as external momenta are taken to zero. $(q \to 0)$
- UV singularity, generated as external momenta are taken to infinity. $(q \to \infty)$
- Mass singularity, generated as the masses of internal particles are taken to zero. $(m_i \to 0)$
- Threshold singularity, generated as the external energy crosses the threshold energy of an intermediate state. $(q \to \pm i \sum_{i \in I} m_i)$

Each type of singularity has clear physical origins. In a complicated Feynman diagram, often overlaps of various types of singularities are included. Algorithms for evaluating Feynman diagrams are essentially procedures to disentangle such overlaps of singularities.

Here, we explain the origin of the threshold singularity in particular. For illustration we consider the example given by Figs. 1(a)(b) and by eqs. (2.1),(2.2). Let us set $\vec{q} = (0,0,0,q)$, $\vec{p} = (p_1, p_2, p_3, p_4)$ and integrate over p_4 in eq. (2.1). The integral can be performed using Cauchy's theorem. Writing $\vec{p}\,' = (p_1, p_2, p_3)$, we find

$$I(q) \propto \int d\vec{p}\,' \left[\frac{1}{(|\vec{p}\,'|^2+1)^2 \left(q - 2i\sqrt{|\vec{p}\,'|^2+1} \right)} + \cdots \right], \tag{3.1}$$

where we have shown explicitly only one term after partial fractioning in q. If q is along the zigzag line shown in the upper half plane in Fig.1(b), the integrand diverges for some $\vec{p}\,' \in \mathbb{R}^3$. This shows that indeed there is a branch cut. The starting point of the cut (branch point) is located at $i \sum_a m_a$, where the sum is taken over the internal lines a included in an intermediate state. An intermediate state is indicated by the dashed line in Fig. 3.

In general, for a given Feynman diagram, it can be shown that threshold singularities arise as branch points, each of which corresponds to the sum of the masses of an

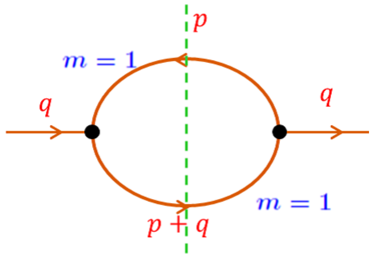


Figure 3. Intermediate state of $I(q)$ indicated by the dashed line, which gives rise to the branch cuts from $q = \pm i \sum_a m_a = \pm 2i$.

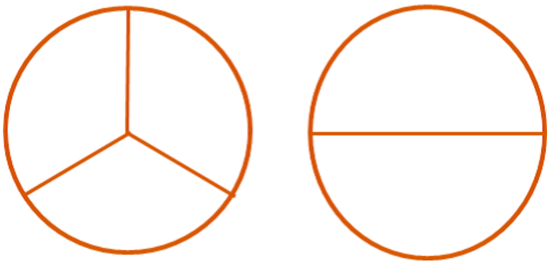


Figure 4. Examples of diagrams which include MZVs with sixth roots of unity $\lambda_i = e^{i n_i \pi/3}$. Each line corresponds to either a massive propagator with mass $m = 1$ or a massless ($m = 0$) propagator.

intermediate state of the diagram.[3] Since collection of intermediate states determines the topology of a diagram, the threshold singularties carry information on the topology of the diagram.

§ 4. Relation between Singularities of Diagram and λ_i's of MZVs

In this section we discuss what kind of MZVs given by eq. (2.4) (in particular which λ_i's) are associated with a diagram. We discuss some empirical aspects gained through experiences in explicit computations. For instance, the diagrams shown in Fig. 4 are known to be expressed by MZVs with λ_i's given by sixth roots of unity. We have already seen this for the second diagram in eq. (2.3). It is expected that λ_i's are closely connected to singularities in a Feynman diagram, which are also closely tied to the topology of the diagram, as we have seen in the previous section.

Let us clarify the connection in the case of the second diagram of Fig. 4. On the left-hand-side of eq. (2.3), we can integrate over the angular variables and reduce the integral to a one-dimensional integral over q. The square-root $\sqrt{1+4/q^2}$ in the integrand can be eliminated by an Euler transformation $x = \frac{1}{2}(1 + \sqrt{1+4/q^2})$. We may further express the logarithm as an integral of a rational function. In this way, we

[3]The locations of threshold singularities can be identified using the cut diagram method. This is a generalization of the optical theorem, which follows from unitarity of the S-matrix in quantum field theory.

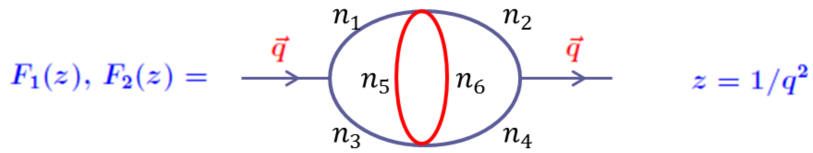


Figure 5. Cat's-eye diagram. Lines $n_1, \ldots, n_4$ have mass $m = 1$ and lines n_5,n_6 have $m = 0$.

can express this diagram as a nested double integral of rational functions, which is in fact a MZV. The map of the singularities by the Euler transformation is given by

$$\{\pm i, \pm 2i, \infty\} \xrightarrow[x = \frac{1+\sqrt{1+4/q^2}}{2}]{\hspace{4cm}} \{e^{\pm \pi i/3}, \text{—}, (0,1)\}. \tag{4.1}$$

The singularities at $\pm i$ stem from the propagator $1/(q^2+1)^2$ attached to $I(q)$ and those at $\{\pm 2i, \infty\}$ are the ones included in $I(q)$. The map is double-valued, and the singularities at $\pm 2i$ are mapped to regular points. As mentioned, the ratios of the singularities of rational function in the integrand become λ_i's of MZVs. Thus, we see that sixth-roots of unity are generated by this map, and that they indeed originate from physical singularities in the diagram.

We discuss more complicated cases below.

§ 5. Computation of Diagrams: Method of Differential Equation

In computing Feynman diagrams, there is no known systematic method to disentangle general overlapping singularities in such a way to cast them into MZVs (if this is possible at all).[4,5] In simple cases this can be achieved, for instance, by iteratively applying the method of differential equation [8]. We explain the method in this section.

Let us consider the Cat's-eye diagram shown in Fig. 5, with two kinds of masses $m = 1$ (red lines) and $m = 0$ (purple lines) for the internal lines. We can choose various combinations of the powers of six propagators, $(n_1, \ldots, n_6)$. It turns out, however, that only two of the combinations are independent, in the sense that any diagram with arbitrary $(n_1, \ldots, n_6)$ can be expressed as a linear combination of two bases, say, $F_1(z)$ and $F_2(z)$ corresponding to $(n_1, \ldots, n_6) = (1,1,1,1,1,1)$ and $(1,1,1,1,1,2)$, respectively, modulo diagrams with simpler topologies (see below).

[4] In numerical evaluations of Feynman diagrams disentanglement of overlapping singularities is always possible in finite steps, using sector decomposition and the theorem on resolution of singularities [6].

[5] Although Feynman diagrams without external legs and with integer masses cannot always be expressed by MZVs, they can always be expressed as periods [7].

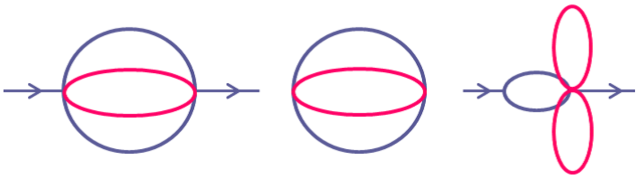


Figure 6. Diagrams which appear on the right-hand side of eq. (5.1). These diagrams are obtained by pinching some of the internal lines of the Cat's-diagram (Fig. 5).

If we differentiate $F_1(z)$ or $F_2(z)$ with respect to $z = 1/q^2$, it can be expressed by the same diagram with either n_5 or n_6 raised by one.[6] Hence, it can be expressed by a linear combination of $F_1(z)$ and $F_2(z)$, modulo diagrams with simpler topologies. The latter diagrams are those, where some of the internal lines of the Cat's-eye diagram are pinched. These are shown in Fig. 6. In this way we obtain a coupled differential equation

$$\frac{d}{dz}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} - \begin{pmatrix} 0 & 4 \\ \dfrac{6}{z(1-16z)} & -\dfrac{2(1-28z)}{z(1-16z)} \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{5.1}$$

where on the right-hand side f_1, f_2 are given by linear combinations of the diagrams in Fig. 6.

The solution to the differential equation is given by

$$\begin{pmatrix} F_1(z) \\ F_2(z) \end{pmatrix} = M(z) \int dz\, M(z)^{-1} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} \tag{5.2}$$

with

$$M(z) = \begin{pmatrix} F_{1a}^{(\mathrm{hom})}(z) & F_{1b}^{(\mathrm{hom})}(z) \\ F_{2a}^{(\mathrm{hom})}(z) & F_{2b}^{(\mathrm{hom})}(z) \end{pmatrix}, \tag{5.3}$$

where $\left(F_{1i}^{(\mathrm{hom})}, F_{2i}^{(\mathrm{hom})}\right)^T$ represents a solution to the homogeneous differential equation [eq. (5.1) with $f_1 = f_2 = 0$]. Thus, $F_i(z)$ can be expressed as an integral of simpler diagrams. The above solution depends on constants of integration. They can be determined from boundary conditions, such as expected asymptotic behaviors of $F_i(z)$ as $z \to 0$ or $z \to \infty$.

The diagrams in Fig. 6 can be expressed by simpler diagrams in a similar manner. Thus, this procedure can be iterated recursively to express $F_i(z)$ in terms of even simpler diagrams. Finally diagrams become so simple that they can be evaluated easily in terms

[6] If we retain $m(=1)$, F_i is a function of m^2/q^2. Hence, differentiating F_i with respect to m^2 raises the power of a massive propagator by one.

of simple functions. Using this method, the Cat's-eye diagram can be expressed by a combination of nested integrals of the form

$$\int_0^z dz_1\, g_{i,1}(z_1) \int_0^{z_1} dz_2\, g_{i,2}(z_2) \cdots \int_0^{z_{N-1}} dz_N\, g_{i,N}(z_N). \tag{5.4}$$

Although $g_{i,j}(z)$'s include square-roots, they can be eliminated by successive Euler transformations and the integrand can be expressed by rational functions. Then the integral can be written easily by combinations of MZVs.

Using the above method, we can convert Feynman diagrams to combinations of nested integrals. In simple cases, $g_{i,j}(z)$'s can be written in simple functions and by transformations of integral variables we can convert diagrams to combinations of MZVs. On the other hand, the locations of singularities in terms of external momentum-invariants can be identified easily, for each diagram that appear at intermediate stages of computations. Hence, we can trace the origins of λ_i's and they can be associated to physical singularities of diagrams, in the cases that this method can be used to reduce Feynman diagrams to MZVs.

We anticipate that the above argument would be useful for elucidating the connection between the topology of a diagram and its value in terms of MZVs.

§ 6. Proliferation of λ_i's and Relations among Generalized MZVs

It is well known that various MZVs can be expressed by a small set of basis MZVs. More precisely, one may consider the vector space over $\mathbb{Q}$ spanned by MZVs at each given weight, and the dimension of the vector space is much smaller than the number of MZVs. This fact enables us to obtain compact analytic expressions for higher-order terms of perturbative expansions of physical quantities, although they originally stem from vast numbers of Feynman diagrams. (e.g. The anomalous magnetic moment of electron in Sec. 1.)

In computing diagrams with complicated topologies, we encounter the cases where MZVs with more general λ_i's are required. In such cases, reduction of MZVs to smaller sets become quite non-trivial. Although shuffle relations [9] are known to be powerful in the case $\lambda_i \in \{1\}$, these relations become no longer sufficient, since there are too many variables and constraints obtained by the shuffle relations are insufficient. In [10] we have found a new class of relations among MZVs, which can be used to reduce MZVs in such cases.

Let us give an example. Suppose that $\lambda \in \mathbb{C}$ satisfies

$$|\lambda| \le 1 \quad \text{and} \quad \lambda \neq 1. \tag{6.1}$$

We rearrange the order of the summation of a weight-two MZV:

$$
\begin{aligned}
(6.2)\qquad Z(\infty;1,1;\lambda,-\lambda) &= \sum_{n>m>0}\frac{\lambda^n(-\lambda)^m}{nm} = \sum_{s=3}^{\infty}\sum_{m=1}^{\lfloor\frac{s-1}{2}\rfloor}\lambda^s\,\frac{(-1)^m}{m(s-m)} \\
&= \sum_{r=1}^{\infty}\lambda^{2r+1}\sum_{m=1}^{r}\frac{(-1)^m}{m(2r+1-m)} + \sum_{r=1}^{\infty}\lambda^{2r+2}\sum_{m=1}^{r}\frac{(-1)^m}{m(2r+2-m)}.
\end{aligned}
$$

We set $s = n + m$. The upper bound of m follows from the condition $m \le n - 1 = s - m - 1$. In the last line we separate s to odd and even numbers and set $s = 2r + 1$ and $s = 2r + 2$, respectively. The sum over m in the last line can be converted to a combination of nested sums using Algorithm I of [10]. Hence, the right-hand side is also given as a combination of MZVs. In this way, we find a relation among MZVs:

$$
\begin{aligned}
(6.3)\quad z(\lambda,-\lambda) = &-\frac{1}{2}z(-2\lambda^2) + \frac{3}{4}z(-\lambda,-1) - z(-\lambda,-i) - z(-\lambda,i) - \frac{1}{4}z(-\lambda,1) \\
&- \frac{1}{4}z(\lambda,-1) + z(\lambda,-i) + z(\lambda,i) - \frac{1}{4}z(\lambda,1) + \frac{1}{4}z(\lambda^2,1).
\end{aligned}
$$

A short-hand notation for MZV is used, where the original forms can be reproduced via $z(x_1,\ldots,x_n) \to Z(\infty;\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)$ with $\alpha_i = |x_i(\lambda = 1)|$ and $\beta_i = x_i/\alpha_i$. For instance, in the case $\lambda = e^{i\pi/4}$ (a primitive eighth root of unity), we obtain a relation between MZVs with eighth roots of unity, which is independent of the shuffle relations or other known relations (to the best of our knowledge). In fact, this reduces the number of new bases that appear at weight two for eighth roots of unity to two (in contrast to the previously known three new bases).

§ 7. Challenge: Integral not reducible to MZV (as yet)

The following integral includes a constant whose value has not been expressed in terms of MZVs up to now:

$$
\begin{aligned}
(7.1)\qquad &\int_0^{\Lambda} dx\,\frac{1}{\sqrt{x}\sqrt{1-4x}}\int_{1/4}^{x} dy\,\frac{1+2y}{y\sqrt{1-y}\sqrt{1-4y}}\int_1^{y} dz\,\frac{\operatorname{Arctan} z}{z\sqrt{1-z}} \\
&\qquad = C_2\,\log^2\Lambda + C_1\,\log\Lambda + C_0 + \mathcal{O}\Big(\frac{1}{\Lambda}\log^n\Lambda\Big).
\end{aligned}
$$

We consider the limit $\Lambda \to \infty$ and extract the coefficients C_0, C_1, C_2 as above. (We omit the terms which vanish as $\Lambda \to \infty$.) C_2 and C_1 can be expressed by MZVs, which is already a result of a non-trivial analysis. The constant C_0 has not been expressed by MZVs and its nature is still unknown. This is a part of the third-order perturbative corrections to the static QCD potential, whose full analytic expression is as yet unknown. Colleagues are invited to reveal the nature of C_0.

§ 8. Summary

In particle physics, evaluation of Feynman diagrams in terms of MZVs and their extensions is a fundamental subject. There are many known examples, in which coefficients of perturbative expansions of physical quantities are given by (generalized) MZVs defined by eq. (2.4). In particular, we encounter the cases in which λ_i's are roots of unity and more generally complex numbers.

At present, there is no known algorithm applicable to evaluate general Feynman diagrams analytically. The method of differential equation is one of the powerful methods for analytical evaluation. It converts Feynman diagrams to combinations of nested integrals. In simple cases, integrands can be transformed to rational functions and therefore the integrals can be expressed in terms of MZVs. In such cases, λ_i's can be related to singularities of Feynman diagrams which have clear physical interpretations. In this way we may elucidate connections between the topology of a Feynman diagram and the types of MZVs included in the diagram. In evaluating complicated diagrams, overlaps of various singularities and variable transformations to cast integrals to MZVs can cause proliferation of λ_i's. In such cases, reduction of MZVs becomes quite non-trivial and new relations between MZVs become requisite. We have found a new class of relations which are useful for this purpose.

Many aspects of complicated Feynman diagrams are still unknown. We quoted an example in Sec. 7, which appears in a three-loop Feynman diagram for the static QCD potential. We encourage our colleagues to reveal its nature.

References

[1] S. Laporta and E. Remiddi, Phys. Lett. B **379** (1996) 283 [hep-ph/9602417].

[2] T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio, Phys. Rev. Lett. **109** (2012) 111807 [arXiv:1205.5368 [hep-ph]].

[3] D. J. Broadhurst, hep-th/9604128.

[4] D. J. Broadhurst, Eur. Phys. J. C **8** (1999) 311 [hep-th/9803091].

[5] F. Brown and O. Schnetz, arXiv:1006.4064 [math.AG].

[6] C. Bogner and S. Weinzierl, Comput. Phys. Commun. **178** (2008) 596 [arXiv:0709.4092 [hep-ph]].

[7] C. Bogner and S. Weinzierl, J. Math. Phys. **50** (2009) 042302 [arXiv:0711.4863 [hep-th]].

[8] E. Remiddi, Nuovo Cim. A **110** (1997) 1435 [hep-th/9711188].

[9] K. Ihara, M. Kaneko, and D. Zagier, *Derivation and double shuffle relations for multiple zeta values*, Compos. Math., **142** (2006) 307.

[10] C. Anzai and Y. Sumino, J. Math. Phys. **54** (2013) 033514 [arXiv:1211.5204 [hep-th]].