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On the two-dimensional steady Navier-Stokes equations related to flows around a rotating obstacle

Dedicated to Professor Yoshinori Morimoto

By

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Abstract

We study a model problem related to the two-dimensional stationary Navier-Stokes equations with a rotating effect, which naturally appears in the analysis of the flows around a rotating obstacle. The unique existence of solutions and their asymptotic behavior at spatial infinity are established when the given external force is sufficiently small in a scale critical space.

§ 1. Introduction

Let $\mathcal{B}$ be a rigid body immersed in a viscous incompressible fluid that fills the whole space. Assume that the body rotates with a constant angular velocity $a \in \mathbb{R} \setminus \{0\}$ and the exterior of $\mathcal{B}(t)$ is described as $\Omega(t) \subset \mathbb{R}^2$. The time dependent domain $\Omega(t)$ is defined as

$$\begin{aligned} \Omega(t) &= \left\{ y \in \mathbb{R}^2 \mid y = O(at)x\,, x \in \Omega \right\}, \\ O(at) &= \begin{pmatrix} \cos at & -\sin at \\ \sin at & \cos at \end{pmatrix}, \end{aligned} \tag{1.1}$$

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where a given exterior domain $\Omega(0) = \Omega \subset \mathbb{R}^2$ has a smooth boundary $\partial\Omega$. The flow around the rotating body is described by the following Navier-Stokes equations:

$$\begin{cases} \partial_t v - \Delta v + v \cdot \nabla v + \nabla q \;=\; g\,, & t > 0\,,\; y \in \Omega(t)\,, \\ \operatorname{div} v \;=\; 0\,, & t > 0\,,\; y \in \Omega(t)\,. \end{cases} \tag{1.2}$$

Here $v = v(y,t) = (v_1(y,t), v_2(y,t))^\top$ and $q = q(y,t)$ are respectively unknown velocity field and pressure field, and $g = g(y,t) = (g_1(y,t), g_2(y,t))^\top$ is a given external force. We use the standard notation for derivatives: $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $\Delta = \sum_{j=1}^2 \partial_j^2$, $\operatorname{div} v = \sum_{j=1}^2 \partial_j v_j$, $v \cdot \nabla v = \sum_{j=1}^2 v_j \partial_j v$, while $x^\perp = (-x_2, x_1)^\top$ denotes the vector which is perpendicular to $x = (x_1, x_2)^\top$. To get rid of the difficulty due to the time-dependence of the domain we take the reference frame by making change of variables

$$\begin{gathered} y \;=\; O(at)x\,, \quad u(x,t) \;=\; O(at)^\top v(y,t)\,, \quad p(x,t) \;=\; q(y,t)\,, \\ f(x,t) \;=\; O(at)^\top g(y,t) \end{gathered}$$

for $t \ge 0$ and $x \in \Omega$. Then (1.2) is equivalent to the equations:

$$\begin{cases} \partial_t u - \Delta u - a(x^\perp \cdot \nabla u - u^\perp) + \nabla p \;=\; -u \cdot \nabla u + f\,, & t > 0\,,\; x \in \Omega\,, \\ \operatorname{div} u \;=\; 0\,, & t > 0\,,\; x \in \Omega\,. \end{cases} \tag{1.3}$$

In order to understand the structure of solutions at spatial infinity it is important to study this system in $\mathbb{R}^2$. The effect of the boundary is expressed as a force in this case. Motivated by this observation, as a model problem, in this paper we study the above nonlinear system in $\mathbb{R}^2$ and in the steady case. Thus, assuming that f is independent of t, we are interested in the following system:

$$\begin{cases} -\Delta u - a(x^\perp \cdot \nabla u - u^\perp) + \nabla p \;=\; -u \cdot \nabla u + f\,, & x \in \mathbb{R}^2\,, \\ \operatorname{div} u \;=\; 0\,, & x \in \mathbb{R}^2\,. \end{cases} \tag{NS$_a$}$$

Our aim is to show the existence and the asymptotic behavior of the solution to (NS$_a$). For this purpose we first consider the linearized problem

$$\begin{cases} -\Delta u - a(x^\perp \cdot \nabla u - u^\perp) + \nabla p \;=\; f\,, & x \in \mathbb{R}^2\,, \\ \operatorname{div} u \;=\; 0\,, & x \in \mathbb{R}^2\,. \end{cases} \tag{S$_a$}$$

We will show that there exists a unique solution to (S$_a$) such that the leading term of the asymptotic behavior of the flow at infinity is the rotational profile $c\frac{x^\perp}{|x|^2}$ whose coefficient c is determined by the external force f.

Before stating the main theorem, let us recall some known results on the mathematical analysis of flows around a rotating obstacle.

So far the mathematical results on this topic have been obtained mainly for the three-dimensional problem, as listed below. For the nonstationary problem the existence of global weak solutions is proved by Borchers [1], and the unique existence of time-local regular solutions is shown by Hishida [11] and Geissert, Heck, and Hieber [9], while the global strong solutions for small data are obtained by Galdi and Silvestre [8]. The spectrum of the linear operator related to this problem is studied by Farwig and Neustupa [2]; see also the linear analysis by Hishida [12]. The existence of stationary solutions to the associated system is proved in [1], Silvestre [15], Galdi [7], and Farwig and Hishida [3]. In particular, in [7] the stationary flows with the decay order $O(|x|^{-1})$ are obtained, while the work of [3] is based on the weak L^3 framework, which is another natural scale-critical space for the three-dimensional Navier-Stokes equations. In 3D case the asymptotic profiles of these stationary flows at spatial infinity are studied by Farwig and Hishida [4, 5] and Farwig, Galdi, and Kyed [6], where it is proved that the asymptotic profiles are described by the Landau solutions, stationary self-similar solutions to the Navier-Stokes equations in $\mathbb{R}^3 \setminus \{0\}$. It is worthwhile to mention that, also in the two-dimensional case, the asymptotic profile is given by the stationary self-similar solution $c\frac{x^\perp}{|x|^2}$. The stability of the above stationary solutions has been well studied in the three-dimensional case; The global L^2 stability is proved in [8], and the local L^3 stability is obtained by Hishida and Shibata [14].

All results mentioned above are considered in the three-dimensional case, while only a few results are known so far for the flow around a rotating obstacle in the two-dimensional case. An important progress has been made by Hishida [13], where the asymptotic behavior of the two-dimensional stationary Stokes flow around a rotating obstacle is investigated in details. Recently, the nonlinear problem (1.2) is analyzed in [10], and the existence of the unique solution decaying as $O(|x|^{-1})$ is proved for sufficiently small a and f when the external force f is of divergence form $f = \operatorname{div} F$ and F has a scale critical decay. Moreover, the leading profile at spatial infinity is shown as $C\frac{x^\perp}{|x|^2}$ under the additional decay condition on F such as $F = O(|x|^{-2-r})$, $r > 0$.

Since we consider the problem in $\mathbb{R}^2$ in this paper, by virtue of the absence of the physical boundary, we can show the existence of solutions to (NS$_a$) without assuming the smallness of the angular velocity a. To state our result let us introduce the function space. For a fixed number $s \geq 0$ the weighted L^∞ space $L^\infty_s(\mathbb{R}^2)$ is defined as

$$L^\infty_s(\mathbb{R}^2) \;=\; \left\{ f \in L^\infty(\mathbb{R}^2) \mid (1+|x|)^s f \in L^\infty(\mathbb{R}^2) \right\}.$$

The space is equipped with the natural norm

$$\|f\|_{L^\infty_s} \;=\; \operatorname{ess.sup}_{x\in\mathbb{R}^2}(1+|x|)^s |f(x)|\,.$$

The first result of this paper is on the linear problem (S$_a$), which extends the result of [13, Proposition 5.3.2] (see also Remark 1 (2) below).

Theorem 1.1. *Let $a \in \mathbb{R} \setminus \{0\}$ and $r \in [0,1)$. Assume that $f \in L^\infty_{3+r}(\mathbb{R}^2)^2$. Then there exists a unique $(u,p) \in L^\infty_1(\mathbb{R}^2)^2 \times L^\infty(\mathbb{R}^2)$ such that:*

1. *The couple (u,p) satisfies (S_a) in the sense of distributions.*
2. *The velocity u belongs to $L^\infty_1(\mathbb{R}^2)^2$ and satisfies*

$$u(x) = \int_{|y|<\frac{|x|}{2}} y^\perp \cdot f(y)\,\mathrm{d}y\, \frac{x^\perp}{4\pi|x|^2} + \mathcal{R}[f](x)\,, \tag{1.4}$$

with

$$|\mathcal{R}[f](x)| \le \frac{C}{(1+|x|)^{1+r}}\Big(\frac{1}{1-r} + \frac{1}{|a|^{\frac{1+r}{2}}}\Big)\|f\|_{L^\infty_{3+r}}\,, \tag{1.5}$$

and in particular, it follows that

$$\|\mathcal{R}[f]\|_{L^\infty_{1+r}} \le C\Big(\frac{1}{1-r} + \frac{1}{|a|^{\frac{1+r}{2}}}\Big)\|f\|_{L^\infty_{3+r}} \tag{1.6}$$

with a numerical constant C.

3. *The pressure p is given by*

$$p(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\cdot f(y)\,\mathrm{d}y\,. \tag{1.7}$$

Remark 1. (1) The representation (1.7) leads to the regularity and the decay of the pressure such as $\nabla p \in L^q(\mathbb{R}^2)$ for $q \in (1,\infty)$ and $p \in L^\infty_1(\mathbb{R}^2)$. The solution $(u,p) \in L^\infty_1(\mathbb{R}^2)^2 \times L^\infty_1(\mathbb{R}^2)$ satisfying (S_a) in the sense of distribution is unique by virtue of the uniqueness result in Hishida [13, Lemma 5.3.5].
(2) In [13, Proposition 5.3.2] the result of Theorem 1.1 is firstly established under the conditions on f such as $f \in L^1(\mathbb{R}^2)^2 \cap L^\infty(\mathbb{R}^2)^2$, $x^\perp \cdot f \in L^1(\mathbb{R}^2)^2$, and $f(x) = O(|x|^{-3}(\log|x|)^{-1})$ as $|x| \to \infty$. Our result improves his result, and in particular, the critical case $f(x) = O(|x|^{-3})$ is treated. We note that, in the case $r = 0$, the integral in the right-hand side of (1.4) does not converge in general when $|x| \to \infty$. The reason why we can treat the critical case $f(x) = O(|x|^{-3})$ is briefly explained as follows. Both in our paper and in [13] the key is to estimate the function $\int_{\mathbb{R}^2} \Gamma_a(x,y) f(y)\,\mathrm{d}y$, where $\Gamma_a(x,y)$ is the fundamental solution to (S_a). Our improvement is achieved in the estimate of the part $\int_{\frac{|x|}{2}<|y|<2|x|} \Gamma_a(x,y) f(y)\,\mathrm{d}y$, where the loss of the logarithmic order appears in [13]. The fundamental solution $\Gamma_a(x,y)$ can be expressed in terms of the time integral, see (2.1) below, and we use the effect of the oscillation (due to the rotation) by using the integration by part in time even in the regime of $\frac{|x|}{2} < |y| < 2|x|$, while in [13] this technique is not essentially used in this regime, though it is effectively used by [13] in the other regimes $|y| \le \frac{|x|}{2}$ and $|y| \ge 2|x|$. The use of the oscillation in the term $\int_{\frac{|x|}{2}<|y|<2|x|} \Gamma_a(x,y) f(y)\,\mathrm{d}y$ enables us to remove the logarithmic loss in [13]. For details, see Section 3 and the estimate of $U_{2,1,2}$ there.

To study the nonlinear problem it is reasonable to consider the linear problem when the external force f is given by $f = \operatorname{div} F$ with $F(x) = O(|x|^{-2})$ in view of the structure of the nonlinear term $u \cdot \nabla u = \operatorname{div}(u \otimes u)$. Here the matrix $(u_i v_j)_{1\le i,j\le 2}$ is written as $u \otimes v$. The following result is essentially obtained in [10].

Theorem 1.2 ([10, Theorem 3.1(ii)]). *Let $a \in \mathbb{R} \setminus \{0\}$, $r \in [0,1)$, and $q \in (1,\infty)$. Assume that $f \in L^2(\mathbb{R}^2)^2$ is of divergence form $f = \operatorname{div} F$ with $F \in L^\infty_{2+r}(\mathbb{R}^2)^{2\times 2}$. Then there exists a unique $(u,p) \in L^\infty_1(\mathbb{R}^2)^2 \times L^q(\mathbb{R}^2)$ such that (u,p) satisfies (S_a) in the sense of distributions, and u satisfies*

$$u(x) = \int_{|y|<\frac{|x|}{2}} (F_{21}(y) - F_{12}(y))\,\mathrm{d}y \frac{x^\perp}{4\pi|x|^2} + \mathcal{R}[f](x)\,, \tag{1.8}$$

where F_{ij} is (i,j) component of the matrix $F = (F_{ij})_{1\le i,j\le 2}$ and $\mathcal{R}[f]$ satisfies

$$\begin{aligned} |\mathcal{R}[f](x)| \le C \min\Big\{\frac{1}{|a||x|^3}, \frac{1}{|x|}\Big\} \int_{|y|\le\frac{|x|}{2}} |F(y)|\,\mathrm{d}y \\ + \frac{C}{(1+|x|)^{1+r}} \frac{1}{1-r} \|F\|_{L^\infty_{2+r}}\,. \end{aligned} \tag{1.9}$$

Here C is a numerical constant independent also of a and r. In particular, it follows that

$$\|\mathcal{R}[f]\|_{L^\infty_{1+r}} \le C\Big(\frac{1}{1-r} + \frac{1+\log|a|}{|a|^{\frac{r}{2}}}\Big)\|F\|_{L^\infty_{2+r}} \tag{1.10}$$

with a numerical constant C.

Remark 2. (1) In fact, the statement of [10, Theorem 3.1] is a slightly different from Theorem 1.2 above. So we give a sketch of the proof of Theorem 1.2 in Section 2, based on the key pointwise asymptotic estimate of the fundamental solution, see Lemma 2.1 below, which is due to [10, Lemma 3.3].
(2) Estimate (1.10) is derived from (1.9). Indeed, when $|x| \le 1$ the first term in the right-hand side of (1.9) is estimated as $C|x|\|F\|_{L^\infty}$, while when $|x| \ge 1$ this term is estimated by dividing into two cases (i) $|a||x|^2 \le 1$ and (ii) $|a||x|^2 \ge 1$. The factor $\log|a|$ in (1.10) is required only when r is near 0 in order to ensure that the constant C is independent of r.

The linear results of Theorem 1.1 and 1.2 are applied to the nonlinear problem (NS_a). The result for the nonlinear problem is stated as follows.

Theorem 1.3. *Let $a \in \mathbb{R} \setminus \{0\}$, $r \in [0,1)$, and $q \in (2,\infty)$. Then there exists $\delta = \delta(a,r) > 0$ such that, for any $f \in L^\infty_{3+r}(\mathbb{R}^2)^2$ satisfying $x^\perp \cdot f \in L^1(\mathbb{R}^2)$ and*

$$\|x^\perp \cdot f\|_{L^1} + \|f\|_{L^\infty_{3+r}} < \delta\,, \tag{1.11}$$

there exists a unique solution $(u,p) \in L^\infty_1(\mathbb{R}^2)^2 \times L^q(\mathbb{R}^2)$ *to* (NS_a) *such that*

$$u(x) = \alpha U(x) + v(x)\,, \tag{1.12}$$

where

$$\alpha = \frac{1}{2}\int_{\mathbb{R}^2} y^\perp \cdot f(y)\ \mathrm{d}y\,, \qquad U(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}(1-e^{-\frac{|x|^2}{4}})\,, \tag{1.13}$$

and

$$\|v\|_{L^\infty_{1+r}} \le C_r\big(\frac{1}{1-r} + \frac{1}{|a|^{\frac{1+r}{2}}}\big)\big(\|x^\perp\cdot f\|_{L^1} + \|f\|_{L^\infty_{3+r}}\big)\,, \tag{1.14}$$

and the pressure p *is given by*

$$p(x) = \nabla\cdot(-\Delta)^{-1}\nabla\cdot(u\otimes u) - \nabla\cdot(-\Delta)^{-1}f\,. \tag{1.15}$$

Here the constant C_r *depends only on* r.

Remark 3. In Theorem 1.3 the solution is constructed as the solution to the integral equation associated with (NS_a), which is formulated based on the fundamental solution to the linearized problem (S_a). The uniqueness is proved for this class of solutions.

This paper is organized as follows. In Section 2 we collect the estimates which reflect the effect of the rotation. Most of them are the abstractions from [13, 10]. Theorem 1.1 is proved in Section 3. Finally, Theorem 1.3 is proved in Section 4.

§ 2. Preliminaries

Let us consider the linear problem in the whole plane for $a \in \mathbb{R}\setminus\{0\}$:

$$-\Delta u - a(x^\perp\cdot\nabla u - u^\perp) + \nabla p = f\,, \qquad \mathrm{div}\ u = 0\,, \qquad x\in\mathbb{R}^2\,. \tag{S$_a$}$$

The couple (u,p) is said to be a weak solution to (S_a) if $(u,p)\in L^{q_1}(\mathbb{R}^2)^2\times L^{q_2}(\mathbb{R}^2)$ for some $q_1,q_2\in[1,\infty)$, and (i) $\mathrm{div}\,u = 0$ in the sense of distributions, and (ii) (u,p) satisfies

$$\int_{\mathbb{R}^2} u\cdot T_{-a}\phi\,\mathrm{d}x - \int_{\mathbb{R}^2} p\,\mathrm{div}\,\phi\,\mathrm{d}x = \int_{\mathbb{R}^2} f\cdot\phi\,\mathrm{d}x \qquad \text{for all}\ \ \phi\in\mathcal{S}(\mathbb{R}^2)^2\,,$$

where

$$T_a u = -\Delta u - a(x^\perp\cdot\nabla u - u^\perp)\,.$$

Let $\mathbb{I} = (\delta_{ij})_{1\le i,j\le 2}$ be the identity matrix. The velocity part of the fundamental solution to (S_a) plays a central role throughout this paper, which is defined as

$$\Gamma_a(x,y) = \int_0^\infty O(at)^\top K(O(at)x - y, t)\,\mathrm{d}t\,, \tag{2.1}$$

where

$$K(x,t) = G(x,t)\mathbb{I} + H(x,t)\,, \qquad H(x,t) = \int_t^\infty \nabla^2 G(x,s)\,\mathrm{d}s\,, \tag{2.2}$$

and $G(x,t)$ is the two-dimensional Gauss kernel

$$G(x,t) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}\,.$$

Similarly, the pressure part of the fundamental solution is defined as

$$Q(x-y) = \frac{1}{2\pi}\log|x-y|\,,$$

for the following identity holds.

$$\mathrm{div}\,(x^\perp\cdot\nabla u - u^\perp) = x^\perp\cdot\nabla\mathrm{div}\,u = 0\,.$$

Remark 4. We can also write $H(x,t)$ in (2.2) as follows

$$H(x,t) = -\frac{(x\otimes x)}{|x|^2}G(x,t) + \left(\frac{x\otimes x}{|x|^2} - \frac{\mathbb{I}}{2}\right)\frac{1-e^{-\frac{|x|^2}{4t}}}{\pi|x|^2}\,.$$

The next lemma is proved in [13, 10].

Lemma 2.1 ([13, Proposition 5.3.1], [10, Lemma 3.3])**.** *Set*

$$L(x,y) = \frac{x^\perp\otimes y^\perp}{4\pi|x|^2}\,. \tag{2.3}$$

Then for $m=0,1$ the kernel $\Gamma_a(x,y)$ satisfies

$$\begin{aligned}&|\nabla_y^m(\Gamma_a(x,y) - L(x,y))|\\ &\le C\bigg(\delta_{0m}\min\Big\{\frac{1}{|a||x|^2}, \frac{1}{|a|^{\frac12}|x|}\Big\} + |x|^{1-m}\min\Big\{\frac{1}{|a||x|^3}, \frac{1}{|x|}\Big\} + \frac{|y|^{2-m}}{|x|^2}\bigg)\,,\\ &\qquad\qquad \text{for}\quad |x| > 2|y|\,.\end{aligned} \tag{2.4}$$

Here δ_{0m} is the Kronecker delta and C is independent of x, y, and a.

Remark 5. (1) The asymptotic estimate like (2.4) is proved in [13] when $m=0$, and then the dependence on $|a|$ is improved by [10] which is needed to solve the nonlinear problem. The detailed proof for the case $m=1$ of (2.4) is given by [10].

(2) Note that, when $|y| > 2|x|$, since $\Gamma_a(x,y) = \Gamma_{-a}(y,x)^\top$ and $(y^\perp \otimes x^\perp)^\top = x^\perp \otimes y^\perp$ we have a similar estimate:

$$
\begin{aligned}
&|\Gamma_a(x,y) - \frac{x^\perp \otimes y^\perp}{4\pi|y|^2}| \\
&\le C\Big(\min\big\{\frac{1}{|a||y|^2}, \frac{1}{|a|^{\frac12}|y|}\big\} + |y|\min\big\{\frac{1}{|a||y|^3}, \frac{1}{|y|}\big\} + \frac{|x|^2}{|y|^2}\Big), \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad |y| > 2|x|\,.
\end{aligned}
\tag{2.5}
$$

Proof of Theorem 1.2. Here we give a sketch of the proof of Theorem 1.2. The unique solution u to (S_a) decaying at spatial infinity is expressed as $u(x) = \int_{\mathbb{R}^2} \Gamma_a(x,y) f(y)\,\mathrm{d}y$, and we focus on the proof of (1.9) and (1.10). By the integration by parts we have

$$
\begin{aligned}
\int_{\mathbb{R}^2} \Gamma_a(x,y) f(y)\,\mathrm{d}y &= -\int_{\mathbb{R}^2} \nabla_y \Gamma_a(x,y) F(y)\,\mathrm{d}y \\
&= -\Big(\int_{|y|<\frac{|x|}{2}} + \int_{\frac{|x|}{2}\le|y|}\Big)\nabla_y \Gamma_a(x,y) F(y)\,\mathrm{d}y \\
&= I(x) + II(x)\,.
\end{aligned}
$$

The term I is further decomposed as

$$
\begin{aligned}
I(x) &= -\int_{|y|<\frac{|x|}{2}} \nabla_y L(x,y) F(y)\,\mathrm{d}y - \int_{|y|<\frac{|x|}{2}} \nabla_y\big(\Gamma_a(x,y) - L(x,y)\big) F(y)\,\mathrm{d}y \\
&= I_1(x) + I_2(x)\,.
\end{aligned}
$$

By the definition of $L(x,y)$ we have $-\big(\nabla_y L(x,y)\big)F = (F_{21} - F_{12})\frac{x^\perp}{4\pi|x|^2}$, which implies

$$
I_1(x) = \int_{|y|<\frac{|x|}{2}} \big(F_{21}(y) - F_{12}(y)\big)\,\mathrm{d}y \frac{x^\perp}{4\pi|x|^2}\,. \tag{2.6}
$$

As for I_2, when $|x| \ge 1$ we have from (2.4) with $m = 1$,

$$
\begin{aligned}
|I_2(x)| &\le C\min\big\{\frac{1}{|a||x|^3}, \frac{1}{|x|}\big\} \int_{|y|\le\frac{|x|}{2}} |F(y)|\,\mathrm{d}y + \frac{C}{|x|^2}\int_{|y|\le\frac{|x|}{2}} |y||F(y)|\,\mathrm{d}y \\
&\le C\min\big\{\frac{1}{|a||x|^3}, \frac{1}{|x|}\big\} \int_{|y|\le\frac{|x|}{2}} |F(y)|\,\mathrm{d}y + \frac{C}{(1+|x|)^{1+r}}\frac{1}{1-r}\|F\|_{L^\infty_{2+r}}\,,
\end{aligned}
\tag{2.7}
$$

where C is a numerical constant independent also of r. Next we have from the direct calculation

$$
|(\nabla_x K)(x,t)| \le C\big(t^{-\frac32} e^{-\frac{|x|^2}{16t}} + \int_t^\infty s^{-\frac52} e^{-\frac{|x|^2}{16s}}\,\mathrm{d}s\big)\,,
$$

which implies

$$
\int_0^\infty |(\nabla K)(O(at)x,t)|\,\mathrm{d}t \le \frac{C}{|x|}\,, \qquad x \ne 0\,.
$$

Then by the change of the variables $y = O(at)z$ we have

$$
\begin{aligned}
|II(x)| &\le \Big| \int_{|y| \ge \frac{|x|}{2}} \nabla_y \Gamma_a(x,y) F(y)\,\mathrm{d}y \Big| \\
&\le \int_0^\infty \int_{|y| \ge \frac{|x|}{2}} |(\nabla K)(O(at)x - y, t)||F(y)|\,\mathrm{d}y\,\mathrm{d}t \\
&\le C\|F\|_{L^\infty_{2+r}} \int_{|z| \ge \frac{|x|}{2}} \Big(\int_0^\infty |(\nabla K)(O(at)(x-z), t)|\,\mathrm{d}t \Big) (1+|z|)^{-2-\gamma}\,\mathrm{d}z \\
&\le C\|F\|_{L^\infty_{2+r}} \int_{|z| \ge \frac{|x|}{2}} |x-z|^{-1} (1+|z|)^{-2-\gamma}\,\mathrm{d}z \\
&\le \frac{C}{(1+|x|)^{1+\gamma}} \|F\|_{L^\infty_{2+r}} \,. \qquad (2.8)
\end{aligned}
$$

Here C is a numerical constant. From (2.6), (2.7), and (2.8), we conclude (1.8) and (1.9) for $r \in [0,1)$. The proof of Theorem 1.2 is complete.

§ 3. Proof of linear result

In this section we prove Theorem 1.1. Set

$$
L[f](x) \;=\; \int_{\mathbb{R}^2} \Gamma_a(x,y) f(y)\,\mathrm{d}y\,, \qquad (3.1)
$$

where $\Gamma_a(x,y)$ is given by (2.1). It is known by [13, Lemma 5.3.5] that $u = L[f]$ together with p defined by (1.7) is the unique weak solution to (S_a) decaying at spatial infinity. So we focus on the proof of the estimates (1.4) and (1.5) here. To apply Lemma 2.1 we first divide (3.1) into three parts:

$$
\begin{aligned}
L[f](x) \;&=\; U_1(x) + U_2(x) + U_3(x) \\
&=\; \Big(\int_{|y| < \frac{|x|}{2}} + \int_{\frac{|x|}{2} \le |y| \le 2|x|} + \int_{2|x| < |y|} \Big) \Gamma_a(x,y) f(y)\,\mathrm{d}y\,.
\end{aligned}
$$

By Lemma 2.1 we have

$$
U_1(x) \;=\; \int_{|y| < \frac{|x|}{2}} y^\perp \cdot f(y)\,\mathrm{d}y \frac{x^\perp}{4\pi |x|^2} + W_1(x) \qquad (3.2)
$$

with

$$
\begin{aligned}
|W_1(x)| &\le C \min\big\{\frac{1}{|a||x|^2}, \frac{1}{|a|^{\frac12}|x|}\big\} \int_{|y|<\frac{|x|}{2}} |f(y)|\,\mathrm{d}y \\
&\qquad + C\min\big\{\frac{1}{|a||x|^2}, 1\big\} \int_{|y|<\frac{|x|}{2}} |f(y)|\,\mathrm{d}y + \frac{C}{|x|^2}\int_{|y|<\frac{|x|}{2}} |y|^2|f(y)|\,\mathrm{d}y \\
&\le \begin{cases} C\big(\frac{1}{|a|^{\frac12}}+1\big)\|f\|_{L^\infty}\,, & |x|\le 1\,,\\ C\big\{\frac{1}{|a|^{\frac{1+r}{2}}|x|^{1+r}} + \frac{1}{(1-r)|x|^{1+r}}\big\}\|f\|_{L^\infty_{3+r}}\,, & |x|\ge 1 \end{cases} \\
&\le \frac{C}{(1+|x|)^{1+r}}\big(\frac{1}{|a|^{\frac{1+r}{2}}} + \frac{1}{1-r}\big)\|f\|_{L^\infty_{3+r}}\,. \qquad (3.3)
\end{aligned}
$$

Similarly, by Remark 5 we have

$$
\begin{aligned}
|U_3(x)| &\le C\int_{2|x|<|y|}\Big(\min\big\{\frac{1}{|a||y|^2}, \frac{1}{|a|^{\frac12}|y|}\big\} + \min\big\{\frac{1}{|a||y|^2}, 1\big\} + \frac{|x|}{|y|}\Big)|f(y)|\,\mathrm{d}y \\
&\le \begin{cases} C\big(\frac{1}{|a|^{\frac12}}+1\big)\|f\|_{L^\infty_{3+r}}\,, & |x|\le 1\,,\\ \frac{C}{|x|^{1+r}}\big(\frac{1}{|a|^{\frac12}}+1\big)\|f\|_{L^\infty_{3+r}}\,, & |x|\ge 1\,. \end{cases} \qquad (3.4)
\end{aligned}
$$

From (3.4) we have

$$
|U_3(x)| \le \frac{C}{(1+|x|)^{1+r}}\big(\frac{1}{|a|^{\frac12}}+1\big)\|f\|_{L^\infty_{3+r}} \qquad (3.5)
$$

with a numerical constant C. Finally, we decompose $U_2(x)$ as

$$
U_2(x) \;=\; U_{2,1}(x) + U_{2,2}(x)\,, \qquad U_{2,1}(x) \;=\; U_{2,1,1}(x) + U_{2,1,2}(x) \qquad (3.6)
$$

with

$$
\begin{aligned}
U_{2,1,1}(x) \;&=\; \int_{\frac{|x|}{2}\le|y|\le 2|x|}\int_0^l O(at)^\top \frac{1}{8\pi t} e^{-\frac{|O(at)x-y|^2}{4t}} f(y)\,\mathrm{d}t\,\mathrm{d}y\,,\\
U_{2,1,2}(x) \;&=\; \int_{\frac{|x|}{2}\le|y|\le 2|x|}\int_l^\infty O(at)^\top \frac{1}{8\pi t} e^{-\frac{|O(at)x-y|^2}{4t}} f(y)\,\mathrm{d}t\,\mathrm{d}y\,,\\
U_{2,2}(x) \;&=\; \int_{\frac{|x|}{2}\le|y|\le 2|x|}\int_0^\infty O(at)^\top \big(K(O(at)x-y) - \frac{1}{8\pi t}e^{-\frac{|O(at)x-y|^2}{4t}}\mathbb{I}\big) f(y)\,\mathrm{d}t\,\mathrm{d}y\,,
\end{aligned}
$$

where $l = l(a, |x|) > 0$ will be chosen later. We start from the estimate of $U_{2,1,1}(x)$. By Fubini's theorem and changing the variable as $z = O(at)x - y$ we obtain

$$|U_{2,1,1}(x)| \le \frac{C}{(1+|x|)^{3+r}} \|f\|_{L^\infty_{3+r}} \int_{\frac{|x|}{2}\le|y|\le 2|x|} \int_0^l t^{-1} e^{-\frac{|O(at)x-y|^2}{4t}} \,\mathrm{d}t\,\mathrm{d}y$$

$$\le \frac{C}{(1+|x|)^{3+r}} \|f\|_{L^\infty_{3+r}} \int_0^l \int_{\mathbb{R}^2} t^{-1} e^{-\frac{|O(at)x-y|^2}{4t}} \,\mathrm{d}y\,\mathrm{d}t$$

$$\le \frac{C}{(1+|x|)^{3+r}} \|f\|_{L^\infty_{3+r}} \int_0^l \int_{\mathbb{R}^2} t^{-1} e^{-\frac{|z|^2}{4t}} \,\mathrm{d}z\,\mathrm{d}t$$

$$\le \frac{C}{(1+|x|)^{3+r}} l\, \|f\|_{L^\infty_{3+r}} . \tag{3.7}$$

Here C is a numerical constant. Next we estimate $U_{2,1,2}$. Since

$$O(at)^\top = -\frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}t}\dot{O}(at)^\top ,$$

the integrating by parts yields

$$U_{2,1,2}(x) = -\frac{1}{2a} \int_{\frac{|x|}{2}\le|y|\le 2|x|} \int_l^\infty \big(\frac{\mathrm{d}}{\mathrm{d}t}\dot{O}(at)^\top\big) G(O(at)x-y,t) f(y) \,\mathrm{d}t\,\mathrm{d}y$$

$$= \frac{1}{2a} \int_{\frac{|x|}{2}\le|y|\le 2|x|} \int_l^\infty \dot{O}(at)^\top \frac{\mathrm{d}}{\mathrm{d}t}\big(G(O(at)x-y,t)\big) f(y) \,\mathrm{d}t\,\mathrm{d}y + W_2(x) , \tag{3.8}$$

and the remainder term W_2 is estimated as

$$|W_2(x)| \le \frac{C}{|a|} \int_{\frac{|x|}{2}\le|y|\le|x|} |G(O(al)x-y,l) f(y)| \,\mathrm{d}y$$

$$\le \frac{C}{(1+|x|)^{1+r}} \frac{1}{l|a|} \|f\|_{L^\infty_{3+r}} . \tag{3.9}$$

To estimate the first term in the right-hand side of (3.8) we use the following calculation,

$$\frac{\mathrm{d}}{\mathrm{d}t} G(O(at)x-y,t)$$

$$= \frac{e^{-\frac{|O(at)x-y|^2}{4t}}}{4\pi} \Big\{ -t^2 + t^{-3}\frac{|O(at)x-y|^2}{4} - at^{-2}\frac{(\dot{O}(at)x)\cdot(O(at)x-y)}{2} \Big\} .$$

Hence we have

$$\Big| \int_{\frac{|x|}{2}\le|y|\le 2|x|} \int_l^\infty \dot{O}(at)^\top \frac{\mathrm{d}}{\mathrm{d}t}\big(G(O(at)x-y,t)\big) f(y) \,\mathrm{d}t\,\mathrm{d}y \Big|$$

$$\le C \int_{\frac{|x|}{2}\le|y|\le 2|x|} \int_l^\infty \big(t^{-2} + t^{-3}|O(at)x-y|^2\big) e^{-\frac{|O(at)x-y|^2}{4t}} |f(y)| \,\mathrm{d}t\,\mathrm{d}y$$

$$+ C\,|a| \int_{\frac{|x|}{2}\le|y|\le 2|x|} \int_l^\infty t^{-2}|x||O(at)x-y| e^{-\frac{|O(at)x-y|^2}{4t}} |f(y)| \,\mathrm{d}t\,\mathrm{d}y$$

$$\le \frac{C}{(1+|x|)^{3+r}} \|f\|_{L^\infty_{3+r}} \Big(\frac{1}{l}|x|^2 + |a||x| \int_{\frac{|x|}{2}\le|y|\le 2|x|} \int_0^\infty t^{-\frac{3}{2}} e^{-\frac{|O(at)x-y|^2}{8t}} \,\mathrm{d}t\,\mathrm{d}y \Big) ,$$

and then, by the change of variables as $z = O(at)^\top y$ we see

$$\begin{aligned}
&\le \frac{C}{(1+|x|)^{3+r}}\|f\|_{L^\infty_{3+r}}\big(\frac{1}{l}|x|^2 + |a||x|\int_{\frac{|x|}{2}\le|z|\le2|x|}\int_0^\infty t^{-\frac{3}{2}}e^{-\frac{|x-z|^2}{8t}}\,\mathrm{d}t\,\mathrm{d}z\big)\\
&\le \frac{C}{(1+|x|)^{3+r}}\|f\|_{L^\infty_{3+r}}\big(\frac{1}{l}|x|^2 + |a||x|\int_{|x-z|\le3|x|}\frac{\mathrm{d}z}{|x-z|}\big)
\end{aligned}$$

$$\le \frac{C}{(1+|x|)^{1+r}}\big(\frac{1}{l} + |a|\big)\|f\|_{L^\infty_{3+r}}\,. \tag{3.10}$$

Then (3.8), (3.9), and (3.10) implies that

$$|U_{2,1,2}(x)| \le \frac{C}{(1+|x|)^{1+r}}\big(\frac{1}{l|a|} + 1\big)\|f\|_{L^\infty_{3+r}}\,. \tag{3.11}$$

Here C is a numerical constant. On the other hand, the term $U_{2,2}$ converges absolutely without using the effect of rotation. Indeed, changing the variables $y = O(at)z$, we have

$$\begin{aligned}
&\Big|\int_{\frac{|x|}{2}\le|y|\le2|x|}\int_0^\infty O(at)^\top\big(K(O(at)x-y,t) - \frac{1}{8\pi t}e^{-\frac{|O(at)x-y|^2}{4t}}\mathbb{I}\big)f(y)\,\mathrm{d}t\,\mathrm{d}y\Big|\\
&\qquad\le \frac{C}{(1+|x|)^{3+r}}\|f\|_{L^\infty_{3+r}}\int_{|z|\le2|x|}\int_0^\infty |B(x-z,t)|\,\mathrm{d}t\,\mathrm{d}z\,,
\end{aligned}\tag{3.12}$$

where $B(x,t)$ is given by

$$B(x,t) \;=\; \Bigg(\frac{e^{-\frac{|x|^2}{4t}}}{8\pi t} - \frac{1-e^{-\frac{|x|^2}{4t}}}{2\pi|x|^2}\Bigg)\Bigg(\mathbb{I} - 2\frac{x\otimes x}{|x|^2}\Bigg)\,.$$

For any fixed $x - z$ we have from the change of variables as $s = \frac{|x-z|^2}{4t}$,

$$\int_0^\infty |B(x-z,t)|\,\mathrm{d}t \le \big|\mathbb{I} - 2\frac{(x-z)\otimes(x-z)}{|x-z|^2}\big|\int_0^\infty \frac{-se^{-s}+1-e^{-s}}{s^2}\,\mathrm{d}s \le C\,, \tag{3.13}$$

where C is independent of $x - z$. Here we have used the identity

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(\frac{e^{-s}-1}{s}\big) \;=\; \frac{-se^{-s}+1-e^{-s}}{s^2} > 0\,.$$

We combine (3.12) with (3.13) to conclude

$$|U_{2,2}(x)| \le \frac{C}{(1+|x|)^{3+r}}\|f\|_{L^\infty_{3+r}}\,. \tag{3.14}$$

Here C is a numerical constant. Collecting (3.6), (3.7), (3.11), and (3.14), we see

$$|U_2(x)| \le \frac{C}{(1+|x|)^{1+r}}\big\{\frac{l}{(1+|x|)^2} + \frac{1}{l|a|} + 1\big\}\|f\|_{L^\infty_{3+r}}\,,$$

and thus, by taking $l = \frac{1+|x|}{|a|^{\frac{1}{2}}}$,

$$|U_2(x)| \le \frac{C}{(1+|x|)^{1+r}} \Big\{ \frac{1}{(1+|x|)|a|^{\frac{1}{2}}} + 1 \Big\} \|f\|_{L^\infty_{3+r}} . \tag{3.15}$$

From (3.1), (3.2), (3.3), (3.5), and (3.15), we obtain (1.4) and (1.5). The proof of Theorem 1.1 is complete.

§ 4. Proof of nonlinear result

We are now in a position to give a proof of our main result. The unique existence and the asymptotic behavior of solutions to (NS_a) will be obtained by combining the results of Theorem 1.2 and Theorem 1.1 by applying the standard fixed point argument. For $r \in [0,1)$ and $\delta \in (0,1)$ we introduce the function space $X_{r,\delta}$ as follows.

$$X_{r,\delta} \; = \; \{ v \in L^\infty_{1+r}(\mathbb{R}^2)^2 \; | \; ||v||_{L^\infty_{1+r}} \le \delta, \;\; \mathrm{div}\ v = 0 \} .$$

We also set

$$\begin{aligned} U(x) \; &= \; \frac{1}{2\pi} \frac{x^\perp}{|x|^2} (1 - e^{-\frac{|x|^2}{4}}) , \\ \alpha \; &= \; \frac{\int_{\mathbb{R}^2} y^\perp \cdot f(y) \;\, \mathrm{d}y}{\int_{\mathbb{R}^2} y^\perp \cdot \Delta U(y) \;\, \mathrm{d}y} \; = \; \frac{1}{2} \int_{\mathbb{R}^2} y^\perp \cdot f(y) \;\, \mathrm{d}y , \\ w(x) \; &= \; u(x) - \alpha U(x) , \end{aligned} \tag{4.1}$$

Here we have used the fact $\int_{\mathbb{R}^2} x^\perp \cdot \Delta U \, \mathrm{d}x = 2$, which is derived from the identity

$$\Delta U \; = \; (-\partial_2 G \, , \partial_1 G)^\top \, , \qquad G(x) \; = \; \frac{1}{4\pi} e^{-\frac{|x|^2}{4}} \, .$$

The direct computation leads to the existence of a scalar function $P_U \in L^\infty(\mathbb{R}^2)$ such that

$$-a\big(x^\perp \cdot \nabla \alpha U - \alpha U^\perp\big) + \alpha^2 U \cdot \nabla U \; = \; \nabla P_U \, .$$

Then w satisfies the following equations in $\mathbb{R}^2$:

$$\begin{cases} -\Delta w - a(x^\perp \cdot \nabla w - w^\perp) + \nabla \pi \\ \qquad = \; -\alpha\big(U \cdot \nabla w + w \cdot \nabla U\big) - w \cdot \nabla w - \alpha \Delta U + f \, , \\ \mathrm{div}\, w \; = \; 0 \, . \end{cases}$$

Here $\pi = p - P_U$. Let us recall that for $f \in L^\infty_3(\mathbb{R}^2)^2$, the function $L[f](x) = \int_{\mathbb{R}^2} \Gamma_a(x,y) f(y) \, \mathrm{d}y$ defines the unique weak solution to (S_a) decaying at spatial infinity. Then we introduce the map Φ as

$$\Phi[w](x) \; = \; L\big[- \alpha(U \cdot \nabla w + w \cdot \nabla U) - w \cdot \nabla w - \alpha \Delta U + f \big](x) \, . \tag{4.2}$$

Here we consider the leading profile of (4.2). Since $U\cdot\nabla w+w\cdot\nabla U = \mathrm{div}\,(U\otimes w+w\otimes U)$ and $w\cdot\nabla w = \mathrm{div}\,(w\otimes w)$, we see that

$$\begin{aligned}&\frac{1}{4\pi}\bigg\{\int_{|y|<\frac{|x|}{2}}(U\otimes w+w\otimes U)_{2,1}-(U\otimes w+w\otimes U)_{1,2}\ \mathrm{d}y\\ &\qquad+\int_{|y|<\frac{|x|}{2}}(w\otimes w)_{2,1}-(w\otimes w)_{1,2}\ \mathrm{d}y\bigg\}\\ &= 0\,.\end{aligned}\tag{4.3}$$

From (4.2) and (4.3), Theorems 1.1 and 1.2 yield

$$\begin{aligned}\Phi[w](x) =&\,\mathcal{R}[-\alpha(U\cdot\nabla w+w\cdot\nabla U)-w\cdot\nabla w-\alpha\Delta U+f](x)\\ &+\bigg(\int_{|y|<\frac{|x|}{2}}y^{\perp}\cdot f\,\mathrm{d}y-\alpha\int_{|y|<\frac{|x|}{2}}y^{\perp}\cdot\Delta U\,\mathrm{d}y\bigg)\frac{x^{\perp}}{4\pi|x|^2}\,.\end{aligned}\tag{4.4}$$

To estimate the last term of (4.4), we have from the definition of α in (4.1) and $\int_{\mathbb{R}^2}x^{\perp}\cdot\Delta U\,\mathrm{d}x = 2$,

$$\begin{aligned}&\Big|\int_{|y|<\frac{|x|}{2}}y^{\perp}\cdot f\,\mathrm{d}y-\alpha\int_{|y|<\frac{|x|}{2}}y^{\perp}\cdot\Delta U\,\mathrm{d}y\Big|\\ &=\frac{1}{2}\Big|\int_{\mathbb{R}^2}y^{\perp}\cdot\Delta U\,\mathrm{d}y\int_{|y|<\frac{|x|}{2}}y^{\perp}\cdot f\,\mathrm{d}y-\int_{|y|<\frac{|x|}{2}}y^{\perp}\cdot\Delta U\,\mathrm{d}y\int_{\mathbb{R}^2}y^{\perp}\cdot f\,\mathrm{d}y\Big|\\ &=\frac{1}{2}\Big|\int_{\mathbb{R}^2}y^{\perp}\cdot\Delta U\,\mathrm{d}y\bigg(\int_{|y|<\frac{|x|}{2}}y^{\perp}\cdot f\,\mathrm{d}y-\int_{\mathbb{R}^2}y^{\perp}\cdot f\,\mathrm{d}y\bigg)\\ &\qquad-\bigg(\int_{|y|<\frac{|x|}{2}}y^{\perp}\cdot\Delta U\,\mathrm{d}y-\int_{\mathbb{R}^2}y^{\perp}\cdot\Delta U\,\mathrm{d}y\bigg)\int_{\mathbb{R}^2}y^{\perp}\cdot f\,\mathrm{d}y\Big|\\ &=\frac{1}{2}\Big|-\int_{\mathbb{R}^2}y^{\perp}\cdot\Delta U\,\mathrm{d}y\int_{\frac{|x|}{2}\le|y|}y^{\perp}\cdot f\,\mathrm{d}y+\int_{\frac{|x|}{2}\le|y|}y^{\perp}\cdot\Delta U\,\mathrm{d}y\int_{\mathbb{R}^2}y^{\perp}\cdot f\,\mathrm{d}y\Big|\\ &\le\frac{C_r}{(1+|x|)^r}(\|y^{\perp}\cdot f\|_{L^1}+\|f\|_{L^\infty_{3+r}})\,,\qquad\text{for}\quad|x|>1\,.\end{aligned}\tag{4.5}$$

Here the constant C_r depends only on r. Note that the boundedness of $\|y^{\perp}\cdot f\|_{L^1}$ is always valid when $r>0$. When $|x|\le 1$ it is easy to see

$$\Big|\int_{|y|<\frac{|x|}{2}}y^{\perp}\cdot f\,\mathrm{d}y-\alpha\int_{|y|<\frac{|x|}{2}}y^{\perp}\cdot\Delta U\,\mathrm{d}y\Big|\le C(\|y^{\perp}\cdot f\|_{L^1}+\|f\|_{L^\infty})\,.\tag{4.6}$$

Estimates (4.5) and (4.6) combined with (4.4) imply that

$$\begin{aligned}\Phi[w](x) =&\,\mathcal{R}[-\alpha(U\cdot\nabla w+w\cdot\nabla U)-w\cdot\nabla w-\alpha\Delta U+f](x)\\ &+O\Big((\|y^{\perp}\cdot f\|_{L^1}+\|f\|_{L^\infty_{3+r}})(1+|x|)^{-1-r}\Big)\,.\end{aligned}\tag{4.7}$$

By (4.7) and Theorems 1.1 and 1.2, we can verify that

$$
\begin{aligned}
\|\Phi[w]\|_{L^\infty_{1+r}} &\le C\big(\frac{1}{1-r}+\frac{1+\log|a|}{|a|^{\frac{r}{2}}}\big)\|\alpha(U\otimes w+w\otimes U)+\frac12 w\otimes w\|_{L^\infty_{2+r}}\\
&\quad + C\big(\frac{1}{1-r}+\frac{1}{|a|^{\frac{1+r}{2}}}\big)\|-\alpha\Delta U+f\|_{L^\infty_{3+r}}\\
&\quad + C_r\big(\|y^\perp\cdot f\|_{L^1}+\|f\|_{L^\infty_{3+r}}\big)\\
&\le C\big(\frac{1}{1-r}+\frac{1+\log|a|}{|a|^{\frac{r}{2}}}\big)\big(|\alpha|\|w\|_{L^\infty_{1+r}}+\|w\|^2_{L^\infty_{1+r}}\big)\\
&\quad + C_r\big(\frac{1}{1-r}+\frac{1}{|a|^{\frac{1+r}{2}}}\big)\big(\|f\|_{L^\infty_{3+r}}+\|y^\perp\cdot f\|_{L^1}\big)\,,
\end{aligned}
\tag{4.8}
$$

where C_r depends only on r, while C is a numerical constant. We may take C and C_r larger than 1, and note that $|\alpha|\le 2^{-1}\|y^\perp\cdot f\|_{L^1}$. If

$$0<\delta\le \frac{1}{3C(\frac{1}{1-r}+\frac{1+\log|a|}{|a|^{\frac{r}{2}}})}$$

and

$$\lambda(f)\;=\;\|f\|_{L^\infty_{3+r}}+\|y^\perp\cdot f\|_{L^1}\le \frac{\delta}{3C_r(\frac{1}{1-r}+\frac{1}{|a|^{\frac{1+r}{2}}})}\,,$$

then we see that $\Phi[w]$ becomes a mapping from $X_{r,\delta}$ into $X_{r,\delta}$. Moreover, from (1.10) there is a numerical constant $C'>0$ such that

$$
\begin{aligned}
&\|\Phi[w_1]-\Phi[w_2]\|_{L^\infty_{1+r}}\\
&\quad=\|\mathcal{R}\big[-\alpha\big\{U\cdot\nabla(w_1-w_2)+(w_1-w_2)\cdot\nabla U\big\}-w_1\cdot\nabla w_1+w_2\cdot\nabla w_2\big]\|_{L^\infty_{1+r}}\\
&\quad\le C'\big(\frac{1}{1-r}+\frac{1+\log|a|}{|a|^{\frac{r}{2}}}\big)\|-\alpha\big\{(U\otimes(w_1-w_2)+(w_1-w_2)\otimes U)\big\}\\
&\qquad\qquad -\frac12(w_1\otimes w_1-w_2\otimes w_2)\|_{L^\infty_{2+r}}\\
&\quad\le C'\big(\frac{1}{1-r}+\frac{1+\log|a|}{|a|^{\frac{r}{2}}}\big)(|\alpha|+\|w_1\|_{L^\infty_{1+r}}+\|w_2\|_{L^\infty_{1+r}})\|w_1-w_2\|_{L^\infty_{1+r}}\\
&\quad\le C'\big(\frac{1}{1-r}+\frac{1+\log|a|}{|a|^{\frac{r}{2}}}\big)\big(\frac{\lambda(f)}{2}+2\delta\big)\|w_1-w_2\|_{L^\infty_{1+r}}\\
&\quad\le 3\delta C'\big(\frac{1}{1-r}+\frac{1+\log|a|}{|a|^{\frac{r}{2}}}\big)\|w_1-w_2\|_{L^\infty_{1+r}}\\
&\quad=\tau\|w_1-w_2\|_{L^\infty_{1+r}}\,,
\end{aligned}
$$

for all $w_1,w_2\in X_{r,\delta}$, where we have set $\tau=3\delta C'(\frac{1}{1-r}+\frac{1+\log|a|}{|a|^{\frac{r}{2}}})$. Hence, if δ (and thus, also $\lambda(f)$) is sufficiently small so that $\tau\in(0,1)$ is justified, then we can conclude that Φ is a contraction on $X_{r,\delta}$. By the fixed point theorem, there exists a fixed point v, which is unique in $X_{r,\delta}$, such that

$$u(x) = \alpha U(x) + v(x), \qquad v \in X_{r,\delta}$$

is a unique solution to (NS_a) with the pressure p defined by (1.15). Finally, the estimate (1.14) follows from (4.8) for the fixed point v of Φ by virtue of the smallness of $|\alpha|$ and δ. The proof of Theorem 1.3 is complete.

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