RIMS Kôkyûroku Bessatsu
B65 (2017), 037–061

Regularity and lifespan of small solutions to systems of quasi-linear wave equations with multiple speeds, I: almost global existence

By

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Abstract

In this paper, we show almost global existence of small solutions to the Cauchy problem for symmetric system of wave equations with quadratic (in 3D) or cubic (in 2D) nonlinear terms and multiple propagation speeds. To measure the size of initial data, we employ a weighted Sobolev norm whose regularity index is the smallest among all the admissible Sobolev norms of integer order. We must overcome the difficulty caused by the absence of the H^1-L^p Klainerman-Sobolev type inequality, in order to obtain a required a priori bound in the low-order Sobolev norm. The introduction of good substitutes for this inequality is therefore at the core of this paper. Using the idea of showing the well-known Ladyženskaja inequality, we prove some weighted inequalities, which, together with the generalized Strauss inequality, play a role as the good substitute.

§ 1. Introduction

Let us start with some well-known results on the Cauchy problem for the quasi-linear, *scalar* wave equation in three space dimensions of the form

$$\partial_t^2 u - \Delta u = \sum_{\alpha,\beta,\gamma=0}^{3} G^{\alpha\beta\gamma}(\partial_\alpha u)\partial^2_{\beta\gamma}u + \sum_{\alpha,\beta=0}^{3} H^{\alpha\beta}(\partial_\alpha u)\partial_\beta u,\ t>0,\ x\in\mathbb{R}^3, \tag{1.1}$$

Received October 16, 2016. Revised January 18, 2017.
2010 Mathematics Subject Classification(s): 35L72
Key Words: *almost global existence, system of nonlinear wave equations, multiple propagation speeds.*
Supported by the Grant-in-Aid for Scientific Research (C) (No. 15K04955), Japan Society for the Promotion of Science (JSPS)
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where $x_0 := t$, $\partial_\alpha := \partial/\partial x_\alpha$, $\partial^2_{\alpha\beta} := \partial^2/\partial x_\alpha \partial x_\beta$, and $G^{\alpha\beta\gamma}$, $H^{\alpha\beta}$ are real constants. Though our primary concern in the present paper is on the Cauchy problem for the *system* of nonlinear wave equations with multiple speeds, we expect that to revisit some fundamental results for the scalar equation (1.1) will serve as a guide to the main problem discussed later. Just for simplicity, we suppose that $G^{\alpha 00} = 0$ for any α. Moreover, without loss of generality, we may suppose $G^{\alpha\beta\gamma} = G^{\alpha\gamma\beta}$ for any α, β, and γ because we are interested in classical solutions. It is then well known (see, e.g., page 113 of Hörmander [10]) that for any initial data in $H^4 \times H^3$ with

$$\sum_{\alpha=0}^{3} \sum_{b,c=1}^{3} |G^{\alpha bc}| \|\partial_\alpha u(0)\|_{L^\infty}$$

small enough, there exists $T > 0$ and a unique solution u to the Cauchy problem for (1.1) such that $\partial_\alpha u \in C([0,T]; H^3)$. If we consider (1.1) in $\mathbb{R}^n$ instead of $\mathbb{R}^3$ and work with fractional-order Sobolev spaces, this result of local existence remains true for the $H^{s+1} \times H^s$-data with $s > (n/2)+1$ (see, e.g, Proposition 5.2.B of Taylor [27]). Note that, for $n = 2, 3$, $H^4 \times H^3$ is the largest among all the admissible Sobolev spaces of integer order, as far as the standard local existence theorem is concerned. We also remark that, when considering (1.1) in $\mathbb{R}^n$ $(n = 2, 3)$ with more regular data in $H^{s+1} \times H^s$ for some $s \geq 4$ $(s \in \mathbb{N})$, it is possible to choose $T > 0$ depending only on $\sum_\alpha \|\partial_\alpha u(0)\|_{H^3}$ and independently of s such that the Cauchy problem for (1.1) admits a unique solution satisfying $\partial_\alpha u \in C([0,T]; H^s)$. (See, e.g., Theorem 5.8 of Racke [21]. Note that the equation (1.1) can be written in the form of the first order quasi-linear system (see, e.g., page 19 of John [13]) to which we can apply Proposition 5.2.B of [27] and Theorem 5.8 of [21].) This means that a continuation of local smooth solutions to a larger strip is reduced to the a priori H^3-bound of their first derivatives.

Concerning long-time existence, there exist positive constants C_1, ε_1 depending on the coefficients $G^{\alpha\beta\gamma}$, $H^{\alpha\beta}$ such that whenever compactly supported C^∞-initial data satisfies

$$\mathcal{W}_4^{1/2}(u(0)) \leq \varepsilon_1, \tag{1.2}$$

we can obtain the a priori L^2-bound of $\partial_\alpha \Gamma^a u(t)$, $\alpha = 0, \dots, 3$, $|a| \leq 3$ which is strong enough to show that the smooth local solution exists at least for the interval $[0, \exp(C_1 \varepsilon^{-1})]$, where $\varepsilon := \mathcal{W}_4^{1/2}(u(0))$. Here and in the following discussion, we use the notation:

$$\mathcal{W}_1(u(t)) := \frac{1}{2} \int_{\mathbb{R}^3} \bigl((\partial_t u(t,x))^2 + |\nabla u(t,x)|^2 \bigr) dx, \tag{1.3}$$

$$\mathcal{W}_\kappa(u(t)) := \sum_{|a| \leq \kappa - 1} \mathcal{W}_1(\Gamma^a u(t)), \quad \kappa = 2, 3, \dots \tag{1.4}$$

For a multi-index a, Γ^a stands for any product of the $|a|$ operators ∂_α ($\alpha = 0, \dots, 3$), $\Omega_{ij} := x_i\partial_j - x_j\partial_i$ ($1 \le i < j \le 3$), $L_k := x_k\partial_t + t\partial_k$ ($k = 1, 2, 3$), and $S := t\partial_t + x \cdot \nabla$. The proof of the a priori bound uses the Sobolev-type inequality

$$\|v(t,\cdot)\|_{L^p(\mathbb{R}^3)} \le C(1+t)^{-2(1/2-1/p)} \sum_{|a|\le 1} \|\Gamma^a v(t,\cdot)\|_{L^2(\mathbb{R}^3)} \tag{1.5}$$

(see, e.g., Ginibre and Velo [5]) as well as the Klainerman inequality [16]

$$\|v(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \le C(1+t)^{-1} \sum_{|a|\le 2} \|\Gamma^a v(t,\cdot)\|_{L^2(\mathbb{R}^3)}. \tag{1.6}$$

(Note that, while a loss of just one derivative occurs in (1.5), we lose two derivatives in applying (1.6) to the estimation of nonlinear terms.) We note that the interval mentioned above becomes exponentially large as the size ε of initial data gets smaller and smaller. Such results have been called "almost global existence theorem" in the literature since the pioneering work of John and Klainerman [14] for the equation (1.1).

Now, let us turn our attention to the main concern in the present paper: the Cauchy problem for the system of nonlinear wave equations of the form

$$(\partial_t^2 - c_l^2\Delta)u^l = G_{ij}^{l,\alpha\beta\gamma}(\partial_\alpha u^i)\partial^2_{\beta\gamma}u^j + H_{ij}^{l,\alpha\beta}(\partial_\alpha u^i)\partial_\beta u^j,\ t>0,\ x\in\mathbb{R}^3 \tag{1.7}$$

and its 2D counterpart

$$\begin{aligned}(\partial_t^2 - c_l^2\Delta)u^l = {}& G_{ijk}^{l,\alpha\beta\gamma\delta}(\partial_\alpha u^i)(\partial_\beta u^j)\partial^2_{\gamma\delta}u^k \\ &+ H_{ijk}^{l,\alpha\beta\gamma}(\partial_\alpha u^i)(\partial_\beta u^j)\partial_\gamma u^k,\ t>0,\ x\in\mathbb{R}^2.\end{aligned} \tag{1.8}$$

Here, $u = (u^1, \dots, u^N) : (0,T)\times\mathbb{R}^n \to \mathbb{R}^N$ ($n = 2, 3$, $N \in \mathbb{N}$) and, on the right-hand side of (1.7)–(1.8) and in the following discussion as well, repeated indices are summed if lowered and uppered. Greek indices range from 0 to n, and roman indices from 1 to N. Suppose the symmetry condition

$$G_{ij}^{l,\alpha\beta\gamma} = G_{ij}^{l,\alpha\gamma\beta} = G_{il}^{j,\alpha\beta\gamma} \tag{1.9}$$

for any i, j, l and α, β, γ in (1.7) or

$$G_{ijk}^{l,\alpha\beta\gamma\delta} = G_{ijk}^{l,\alpha\beta\delta\gamma} = G_{ijl}^{k,\alpha\beta\gamma\delta} \tag{1.10}$$

for any i, j, k, l and $\alpha, \beta, \gamma, \delta$ in (1.8). Then the results of local existence mentioned above for (1.1) carry over to (1.7), (1.8), because these systems can be written in the form of the first order quasi-linear symmetric system, such as (5.9) of [21], (5.2.1) of [27]. Concerning long-time existence, for compactly supported smooth initial data of the form

$$u(0) = \varepsilon f,\ \partial_t u(0) = \varepsilon g, \tag{1.11}$$

Sogge proved that there exist positive constants C_2, ε_2 depending on the speeds $c_1, \dots, c_N$, the coefficients $G_{ij}^{l,\alpha\beta\gamma}$, $H_{ij}^{l,\alpha\beta}$, and a weighted H^{10}-norm of ∇f and g such that if $0 < \varepsilon \le \varepsilon_2$, then a unique solution to (1.7), (1.11) exists at least over the time interval $[0, \exp(C_2\varepsilon^{-1})]$. (See Theorem 4.1 on page 67 of Sogge [25], the proof of which is based on that of Theorem 1.2 of Keel, Smith, and Sogge [15].) Also, Kovalyov [18] proved that there exist positive constants C_3, $\varepsilon_3 > 0$ depending on the speeds $c_1, \dots, c_N$, the coefficients $G_{ijk}^{l,\alpha\beta\gamma\delta}$, $H_{ijk}^{l,\alpha\beta\gamma}$, and a weighted H^5-norms of ∇f and g such that if $0 < \varepsilon \le \varepsilon_3$, then a unique solution to (1.8), (1.11) exists at least over $[0, \exp(C_3\varepsilon^{-2})]$. In the present paper, we aim at refining these results by employing a lower-order norm to measure the size of initial data. More precisely, we prove:

Theorem 1.1. *Assume the symmetry condition* (1.9), (1.10)*. Then there exist positive constants C_0, ε_0 depending on the propagation speeds and the coefficients of the equations* (1.7), (1.8) *such that if compactly supported, smooth initial data is small so that*

$$N_4(u(0)) \le \varepsilon_0 \tag{1.12}$$

may hold, then the systems (1.7), (1.8) *admit unique solutions defined on the interval $[0,T]$ such that $N_4(u(t)) \le 2N_4(u(0))$, $0 \le t \le T$. Here,*

$$T = \exp(C_0\varepsilon^{-\nu}) \quad (\varepsilon := N_4(u(0)))$$

with $\nu = 1$ for (1.7)*, $\nu = 2$ for* (1.8)*.*

Here, on the basis of the standard energy $E_1(u(t))$ associated with unperturbed wave equations

$$E_1(u(t)) = \frac{1}{2}\sum_{l=1}^{N}\int_{\mathbb{R}^n}\big(|\partial_t u^l(t,x)|^2 + c_l^2|\nabla u^l(t,x)|^2\big)dx, \tag{1.13}$$

we have defined the quantity $N_\kappa(v(t))$ for $v = (v^1, \dots, v^N)$ as

$$N_1(v(t)) = \sqrt{E_1(v(t))},\ N_2(v(t)) = \left(\sum_{|a|+|b|+d\le 1} E_1(\partial_x^a\Omega^b S^d v(t))\right)^{1/2}, \tag{1.14}$$

$$N_\kappa(v(t)) = \left(\sum_{\substack{|a|+|b|+d\le\kappa-1\\ d\le 1}} E_1(\partial_x^a\Omega^b S^d v(t))\right)^{1/2},\ \kappa = 3, 4, \dots,$$

where, for $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_m)$ ($m = 1, 3$ for $n = 2, 3$, respectively), $\partial_x^a\Omega^b S^d v := (\partial_x^a\Omega^b S^d v^1, \dots, \partial_x^a\Omega^b S^d v^N)$, $\partial_x^a := \partial_1^{a_1}\cdots\partial_n^{a_n}$, $\Omega^b := \Omega_{12}^{b_1}\cdots\Omega_{n-1\,n}^{b_m}$, $\Omega_{ij} =$

$x_i\partial_j - x_j\partial_i$ $(1 \le i < j \le n)$, and $S = t\partial_t + x \cdot \nabla$. We set $Z := \{\partial_i, \Omega_{jk}, S : i = 1, \dots, n, 1 \le j < k \le n\}$. Note that none of the operators $L_k = x_k\partial_t + t\partial_k$ $(k = 1, \dots, n)$ is an element of the set Z. Note also $\partial_t \notin Z$.

There exist some difficulties in showing the almost global existence result for (1.7), (1.8) when we employ the lower-order norm, such as N_4 defined above, to measure the size of data. Recall that, besides the standard energy inequality for the variable-coefficient wave equation, the generalized Sobolev-type inequalities (1.5)–(1.6) and the nice commutation relations between the D'Alembertian (with the propagation speed $c = 1$) and the elements of $\{\Omega_{ij}, L_k, S\}$ play an important role in showing the almost global existence for (1.1) with compactly supported, smooth data satisfying (1.2). When considering the multiple-speed system (1.7), (1.8), we must take into account the fact that the operator $\tilde{L}_k := c^{-1}x_k\partial_t + ct\partial_k$, which is a speed-dependent variant of L_k, commutes with $\Box_c := \partial_t^2 - c^2\Delta$, but it no longer does with $\Box_{\hat{c}} := \partial_t^2 - \hat{c}^2\Delta$ $(\hat{c} \neq c)$. Indeed, we have $[\tilde{L}_k, \Box_{\hat{c}}] = 2c^{-1}(\hat{c}^2 - c^2)\partial_k\partial_t$, and this commutation relation is obviously useless in our argument. We must therefore give up using such a modified operator $\tilde{L}_k$, which in turn means that we must give up using the Sobolev-type inequalities (1.5)–(1.6). On the other hand, we still enjoy the good commutation relations $[\Omega_{ij}, \Box_c] = 0$ and $[S, \Box_c] = -2\Box_c$, and some good substitutes for the Klainerman inequality (1.6) are available on the basis of the use of the operators Ω_{ij} and S and without relying upon the operators $\tilde{L}_k$. (See Lemma 6.1 of [24] and Lemma 1 of [22]. See also (4.2) of [6].) These substitutes, combined with the Klainerman-Sideris inequality (see (3.1) below), would suffice to show almost global existence theorem for (1.7), (1.8) when a suitable higher (than 4) order norm of data is small enough. See, e.g., Section 8 of Sideris and Tu [24] and Theorem 3.1 of [6]. Therefore, it is a good substitute for (1.5) that plays a key role in reducing the regularity index of norm to as low a level as in (1.2). To the best of the present author's knowledge, no substitute for (1.5) is available in the literature. We explain that our key weighted inequalities (2.7)–(2.13) are well combined with the method of [24], and they play a role as the substitute for (1.5). To prove these key inequalities, we follow the way of showing the well-known Ladyženskaja inequality [19] or use the generalization of the Strauss inequality. (See (2.20) for the generalized Strauss inequality. See also [3] for recent, another extension of the classical inequality of Strauss [26].)

The method in this paper has an application to the system of quasi-linear wave equations with *quadratic* nonlinear terms in 2D. Repeating essentially the same argument as in the proof of Theorem 1.1, we obtain the following:

Theorem 1.2. *Consider* (1.7) *in* $\mathbb{R}^2$. *Assume the symmetry condition* (1.9). *Then there exist positive constants* A_0, ε_0 *depending on the propagation speeds and the coefficients of* (1.7) *with the following property: if the compactly supported, smooth* *initial data is small so that* $N_4(u(0)) \le \varepsilon_0$ *may hold, then the problem* (1.7) *has a unique solution satisfying* $N_4(u(t)) \le 2N_4(u(0))$, $0 < t < T$. *Here* $T = A_0\varepsilon^{-2}$ $(\varepsilon := N_4(u(0)))$.

In [18], Kovalyov considered the system (1.7) not in $\mathbb{R}^3$ but in $\mathbb{R}^2$ with data of the form (1.11), and obtained the slightly weaker lower bound $T \ge C\varepsilon^{-2}(\log(1/\varepsilon))^{-2}$, while in [11], assuming $H_{ij}^{l,\alpha\beta} = 0$ for all i, j, l, α, β, Hoshiga obtained the refined lower bound $T \ge C\varepsilon^{-2}$ with a positive constant C computed explicitly from the propagation speeds $c_1, \dots, c_N$, the coefficients $G_{ij}^{l,\alpha\beta\gamma}$, and the given functions f, g. Theorem 1.2 is an improvement on the previous results of [18] and [11], for Kovalyov used a higher-order norm to measure the size of initial data and his lower bound of the lifespan is slightly weaker than ours, and Hoshiga imposed the restriction $H_{ij}^{l,\alpha\beta} = 0$ for all i, j, l, α, β, while we no longer need his strict restriction.

Here we give three remarks. Firstly, as in the books [2], [10], [13], [21], and [25], we have so far supposed that initial data is smooth and compactly supported, when considering the lifespan of small solutions. This is mainly because a continuation argument becomes considerably easier for compactly supported (in space at fixed times $t > 0$), smooth solutions. See (4.31) below. Note that the constants A_0, C_0, and ε_0 appearing Theorems 1.1 and 1.2 are completely independent of the "size" of the support of initial data. Therefore, once we have proved these theorems, we should move on to removing the compactness assumption of the support, as well as the regularity (C^∞) assumption, of initial data. The idea of doing it can be found on page 122 of [10] (see *Remark* there). In order to keep the present paper to a moderate length, we refrain from pursuing this important problem.

Secondly, in the definition of $N_\kappa(u(t))$ $(\kappa \ge 3)$ we have limited the number of occurrences of S to 1, in accordance with the idea of the earlier papers [15] and [9] that its at most 1 occurrence is actually sufficient for the proof of almost global existence. With this, there is an advantage that we can bypass the burdensome calculation of $\partial_t^j u(0,x)$ $(j = 2, 3, 4)$ when computing $N_4(u(0))$, because $\partial_t \notin Z$ and $\partial_t Su = \partial_t u + x \cdot \nabla \partial_t u$ at $t = 0$. (Compare this with the fact that we must successively calculate $\partial_t^j u(0,x)$ $(j = 2, 3, 4)$ with the help of the equation (1.1) when computing $\mathcal{W}_4^{1/2}(u(0))$ appearing in (1.2).) Another feature lies in that, when initial data $(u(0), \partial_t u(0)) = (\varphi, \psi)$ is radially symmetric about $x = 0$ (and the system of equations is not necessarily so), we easily see the condition (1.12) is satisfied whenever the norm with the "mild" weight $\langle x \rangle := \sqrt{1 + |x|^2}$

$$\sum_{1 \le l \le N} \Big(\sum_{1 \le |a| \le 4} \|\langle x \rangle \partial_x^a \varphi^l\|_{L^2} + \sum_{|a| \le 3} \|\langle x \rangle \partial_x^a \psi^l\|_{L^2} \Big) \tag{1.15}$$

is small enough. The result of almost global existence for symmetric (and not necessarily diagonal) systems of quasi-linear wave equations is new when smallness is required of only such mildly weighted Sobolev norm of radial data.

Thirdly, the proof of Theorem 1.1 obviously remains valid for the scalar equation (1.1), thus we obtain almost global existence result under the condition (1.12) with $N = 1$ which is weaker than (1.2).

We conclude this section by mentioning that, in the sequel [7], assuming the null condition in the different-speed setting proposed by Agemi and Yokoyama [1], Yokoyama [28], we will prove the global existence theorem for (1.7), (1.8) on a condition which is stronger than (1.12), but weaker than that in the previous papers [28], [12], [24], [25], [20]. In addition to the key tools used in the present paper, the proof will use the estimation lemmas due to Sideris and Tu (see Lemma 5.1 of [24]) in 3D, Lindblad, Nakamura, and Sogge (see Lemma A.4 of [20]) in 2D when handling the null-form terms.

This paper is organized as follows. In the next section, some useful inequalities of the Sobolev type or the trace type are proved. Using the Klainerman-Sideris inequality, we bound weighted space-time L^2-norms of the second or some higher-order derivatives of the local solution in Section 3. In Sections 4 and 5, we carry out the energy integral argument and complete the proof of Theorems 1.1 and 1.2.

§ 2. Preliminaries

As explained in Section 1, repeated indices will be summed if lowered and uppered. Greek indices range from 0 to n ($n = 2$ or 3), and roman indices from 1 to N or 1 to n. In addition to the usual partial differential operators $\partial_\alpha = \partial/\partial x_\alpha$ ($\alpha = 0, \dots, n$), we use the generator of Euclid rotation $\Omega_{ij} = x_i\partial_j - x_j\partial_i$ and of space-time scaling $S = t\partial_0 + x \cdot \nabla$. The set of these μ ($\mu = 4$ for $n = 2$, $\mu = 7$ if $n = 3$) differential operators is denoted by $Z = \{ Z_1, \dots, Z_\mu \} = \{ \nabla, \Omega, S \}$. Note that ∂_t is not an element of Z. For a multi-index $a = (a_1, \dots, a_\mu)$, we set $Z^a := Z_1^{a_1} \cdots Z_\mu^{a_\mu}$. We also use $\bar{Z} = \{ Z_1, \dots, Z_{\mu-1} \} = \{ \nabla, \Omega \}$, with $\bar{Z}^a := \partial_1^{a_1}\partial_2^{a_2}\Omega_{12}^{a_3}$ ($a = (a_1, a_2, a_3)$), $\bar{Z}^a := \partial_1^{a_1}\partial_2^{a_2}\partial_3^{a_3}\Omega_{12}^{a_4}\Omega_{13}^{a_5}\Omega_{23}^{a_6}$ ($a = (a_1, \dots, a_6)$) for $n = 2, 3$, respectively.

We collect several results concerning commutation relations and Sobolev-type and trace-type inequalities. Let $[\cdot, \cdot]$ be the commutator: $[A, B] := AB - BA$. It is easy to verify that

$$(2.1) \qquad [Z_i, \partial_t^2 - c^2\Delta] = 0 \ \text{ for } i = 1, \dots, \mu - 1, \ \ [S, \partial_t^2 - c^2\Delta] = -2(\partial_t^2 - c^2\Delta),$$

$$(2.2) \qquad [Z_j, Z_k] = \sum_{i=1}^{\mu} C_i^{j,k} Z_i, \ \ j, \ k = 1, \dots, \mu,$$

$$(2.3) \qquad [Z_j, \partial_k] = \sum_{i=1}^{n} C_i^{j,k} \partial_i, \ \ j = 1, \dots, \mu, \ \ k = 1, \dots, n,$$

$$(2.4) \qquad [Z_j, \partial_t] = 0, \ j = 1, \dots, \mu - 1, \quad [S, \partial_t] = -\partial_t.$$

Here $C_i^{j,k}$ denotes a constant depending on i, j, and k.

The following lemma is concerned with Sobolev-type or trace-type inequalities. We use these inequalities in combination with the Klainerman-Sideris inequality (see (3.1) below). The auxiliary norms of $v = (v^1, \dots, v^N)$

$$M_2(v(t)) = \sum_{l=1}^{N} \sum_{\substack{0\le\delta\le n \\ 1\le j\le n}} \|\langle c_l t - |x|\rangle \partial^2_{\delta j} v^l(t)\|_{L^2(\mathbb{R}^n)}, \; M_4(v(t)) = \sum_{|a|\le 2} M_2(\bar{Z}^a v(t)), \tag{2.5}$$

which appear in the following discussion, play an intermediate role. We remark that S and ∂_t^2 are absent in the right-hand side above. Here and later on as well, we use the standard notation $\langle A\rangle = \sqrt{1+|A|^2}$ for a scalar or a vector A. We also use the notation $\partial_r := (x/|x|)\cdot\nabla$,

$$\begin{aligned} \|w\|_{L^\infty_r L^p_\omega(\mathbb{R}^n)} &:= \sup_{r>0} \|w(r\cdot)\|_{L^p(S^{n-1})}, \\ \|w\|_{L^2_r L^p_\omega(\mathbb{R}^n)} &:= \Bigl(\int_0^\infty \|w(r\cdot)\|^2_{L^p(S^{n-1})} r^{n-1} dr\Bigr)^{1/2}. \end{aligned} \tag{2.6}$$

Lemma 2.1. *Let v be a vector-valued function $v = (v^1, \dots, v^N) : (0,\infty)\times\mathbb{R}^n \to \mathbb{R}^N$ decaying sufficiently fast as $|x| \to \infty$. The following inequalities hold for every $l = 1, \dots, N$:*

(i) *Suppose $n = 2$. We have for $\alpha = 0, 1, 2$*

$$\|r^{1/2}\partial_\alpha v^l(t)\|_{L^\infty_r L^2_\omega(\mathbb{R}^2)} \le C N_1^{1/2}(v(t)) \bigl(\sum_{|a|=1} N_1(\partial_x^a v(t))\bigr)^{1/2}, \tag{2.7}$$

$$\|\langle c_l t - r\rangle^{1/2}\partial_\alpha v^l(t)\|_{L^4(\mathbb{R}^2)} \le C N_1^{1/2}(v(t)) \bigl(N_1(v(t)) + M_2(v(t))\bigr)^{1/2}. \tag{2.8}$$

(ii) *Suppose $n = 3$. We have for $\alpha = 0, 1, 2, 3$*

$$\|\langle c_l t - r\rangle^{1/2}\partial_\alpha v^l(t)\|_{L^3(\mathbb{R}^3)} \le C N_1^{1/2}(v(t)) \bigl(N_1(v(t)) + M_2(v(t))\bigr)^{1/2}, \tag{2.9}$$

$$\|\langle c_l t - r\rangle\partial_\alpha v^l(t)\|_{L^6(\mathbb{R}^3)} \le C \bigl(N_1(v(t)) + M_2(v(t))\bigr). \tag{2.10}$$

Moreover, for any $2 \le p < 4$ there exists a constant $C = C_p > 0$ such that we have

$$\|r\partial_\alpha v^l(t)\|_{L^\infty_r L^p_\omega(\mathbb{R}^3)} \le C \sum_{|a|\le 1} N_1(\bar{Z}^a v(t)). \tag{2.11}$$

Remark. We give three remarks. Firstly, by the Sobolev embedding $W^{1,4}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ and $W^{1,6}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, we get from (2.8) and (2.10)

$$\begin{aligned} &\langle c_l t - r\rangle^{1/2} |\partial_\alpha v^l(t,x)| \\ &\le C\Bigl(\sum_{|a|\le 1} N_1(\partial_x^a v(t))\Bigr)^{1/2} \Bigl(\sum_{|a|\le 1} N_1(\partial_x^a v(t)) + \sum_{|a|\le 1} M_2(\partial_x^a v(t))\Bigr)^{1/2} \end{aligned} \tag{2.12}$$

for $n=2$ and

$$\langle c_l t - r\rangle|\partial_\alpha v^l(t,x)| \le C\biggl(\sum_{|a|\le 1} N_1(\partial_x^a v(t)) + \sum_{|a|\le 1} M_2(\partial_x^a v(t))\biggr) \tag{2.13}$$

for $n=3$, respectively. The former (2.12) was shown by Sideris (see the last inequality on page 379 of [22]). After he submitted the manuscript, the author became aware of the recent paper of Zha [29] where the latter (2.13) had been proved (see (37) there). In addition to (2.8)–(2.10), we will also use both (2.12) and (2.13) in the following discussion. We also note that the multiplicative form of the right-hand side of (2.8) and (2.12) is very useful in our argument (see, e.g., (3.15)–(3.16) below). Secondly, we remark that we will also use for $n=2,3$

$$\|r^{(n-1)/2}\partial_\alpha v^l(t)\|_{L^\infty(\mathbb{R}^n)} \le C\sum_{|a|\le 2} N_1(\bar{Z}^a v(t)), \tag{2.14}$$

which follows immediately from the combination of (2.7), (2.11) with the Sobolev embedding $W^{1,2}(S^1)\hookrightarrow L^\infty(S^1)$, $W^{1,p}(S^2)\hookrightarrow L^\infty(S^2)$ with $p>2$, respectively. Thirdly, in fact we will use (2.9) not in the present paper but in [7]. The proof of (2.9) is similar to that of (2.8), thus we prove it here.

Proof. Applying the well-known inequality $\|\varphi\|_{L^6(\mathbb{R}^3)} \le C\|\nabla\varphi\|_{L^2(\mathbb{R}^3)}$ with $\varphi = \langle c_l t - r\rangle\partial_\alpha v^l(t,x)$, we easily obtain (2.10). The proof of (2.8) builds upon how to obtain the well-known Ladyženskaya inequality [19] $\|\varphi\|_{L^4(\mathbb{R}^2)}^4 \le 4\|\varphi\|_{L^2(\mathbb{R}^2)}^2\|\nabla\varphi\|_{L^2(\mathbb{R}^2)}^2$. Indeed, we first obtain by a direct computation

$$\begin{aligned}\langle c_l t - r\rangle|\partial_\alpha v^l(t,x)|^2 &= \int_{-\infty}^{x_1}\frac{d}{d\xi_1}\bigl(\langle c_l t-\tilde{r}\rangle|\partial_\alpha v^l(t,\xi_1,x_2)|^2\bigr)d\xi_1\\ &\le C\int_{-\infty}^{\infty}\bigl(|\partial_\alpha v^l(t,\xi_1,x_2)|^2 + \langle c_l t-\tilde{r}\rangle|\partial_\alpha v^l(t,\xi_1,x_2)||\partial_{1\alpha}^2 v^l(t,\xi_1,x_2)|\bigr)d\xi_1,\end{aligned} \tag{2.15}$$

which yields

$$\begin{aligned}&\langle c_l t - r\rangle^2|\partial_\alpha v^l(t,x)|^4\\ &\le C\int_{-\infty}^{\infty}\bigl(|\partial_\alpha v^l(t,\xi_1,x_2)|^2 + \langle c_l t-\tilde{r}\rangle|\partial_\alpha v^l(t,\xi_1,x_2)||\partial_{1\alpha}^2 v^l(t,\xi_1,x_2)|\bigr)d\xi_1\\ &\times\int_{-\infty}^{\infty}\bigl(|\partial_\alpha v^l(t,x_1,\xi_2)|^2 + \langle c_l t-\hat{r}\rangle|\partial_\alpha v^l(t,x_1,\xi_2)||\partial_{2\alpha}^2 v^l(t,x_1,\xi_2)|\bigr)d\xi_2,\end{aligned} \tag{2.16}$$

where $\tilde{r} := (\xi_1^2+x_2^2)^{1/2}$, $\hat{r} := (x_1^2+\xi_2^2)^{1/2}$. Integrating both the sides above over $\mathbb{R}^2$ and using the Fubini theorem and the Schwarz inequality, we get

$$\int_{\mathbb{R}^2}\langle c_l t - r\rangle^2|\partial_\alpha v^l(t,x)|^4dx \le C\Bigl(N_1(v(t))\bigl(N_1(v(t))+M_2(v(t))\bigr)\Bigr)^2, \tag{2.17}$$

as desired. The proof of (2.9) is similar, and we follow the proof of $\|\varphi\|_{L^3(\mathbb{R}^3)} \le \sqrt{2}\|\varphi\|^{1/2}_{L^2(\mathbb{R}^3)}\|\nabla\varphi\|^{1/2}_{L^2(\mathbb{R}^3)}$ which is a special case of the Gagliardo-Nirenberg inequality (see, e.g., page 25 of [4]). As in (2.16), we get

$$
\begin{aligned}
(2.18)\qquad &(\langle c_l t - r\rangle^{1/2}|\partial_\alpha v^l(t,x)|)^3\\
&\le C\Bigl(\int_{-\infty}^{\infty}\bigl(|\partial_\alpha v^l(t,X_1)|^2 + \langle c_l t-\tilde{r}\rangle|\partial_\alpha v^l(t,X_1)||\partial^2_{1\alpha}v^l(t,X_1)|\bigr)d\xi_1\Bigr)^{1/2}\\
&\times\Bigl(\int_{-\infty}^{\infty}\bigl(|\partial_\alpha v^l(t,X_2)|^2 + \langle c_l t-\hat{r}\rangle|\partial_\alpha v^l(t,X_2)||\partial^2_{2\alpha}v^l(t,X_2)|\bigr)d\xi_2\Bigr)^{1/2}\\
&\times\Bigl(\int_{-\infty}^{\infty}\bigl(|\partial_\alpha v^l(t,X_3)|^2 + \langle c_l t-\bar{r}\rangle|\partial_\alpha v^l(t,X_3)||\partial^2_{3\alpha}v^l(t,X_3)|\bigr)d\xi_3\Bigr)^{1/2},
\end{aligned}
$$

where $X_1 := (\xi_1, x_2, x_3), \dots, X_3 := (x_1, x_2, \xi_3)$, $\tilde{r} := |X_1|$, $\hat{r} := |X_2|$, and $\bar{r} := |X_3|$. Integrating both the sides above over $\mathbb{R}^3$ and using the Schwarz inequality repeatedly, we get (2.9).

The other inequalities (2.7), (2.11) follow from the well-known inequality (called the Strauss inequality, especially when we focus on radially symmetric functions; see [26] and [3])

$$
(2.19)\qquad \|r^{(n-1)/2}\varphi\|_{L^\infty_r L^2_\omega(\mathbb{R}^n)} \le \sqrt{2}\|\partial_r\varphi\|^{1/2}_{L^2(\mathbb{R}^n)}\|\varphi\|^{1/2}_{L^2(\mathbb{R}^n)}
$$

or its generalization (see (2.10) of [8]): for $2 \le q \le \infty$ and $2/p = 1/2 + 1/q$ (the reader is asked to interpret this as $p = 4$ for $q = \infty$)

$$
(2.20)\qquad \|r^{(n-1)/2}\varphi\|_{L^\infty_r L^p_\omega(\mathbb{R}^n)} \le \sqrt{p}\|\partial_r\varphi\|^{1/2}_{L^2(\mathbb{R}^n)}\|\varphi\|^{1/2}_{L^2_r L^q_\omega(\mathbb{R}^n)}.
$$

Indeed, we obtain (2.7) directly from (2.19). Moreover, we obtain (2.11) immediately from the Sobolev embedding $W^{1,2}(S^2) \hookrightarrow L^q(S^2)$ for $2 \le q < \infty$ owing to the fact that in the condition for (2.20) to hold, the condition $2 \le p < 4$ is equivalent to $2 \le q < \infty$. □

§ 3. Weighted L^2-estimates

It is necessary to bound $M_4(u(t))$ by $N_4(u(t))$ for the completion of the energy integral argument (see Lemma 3.4 below). We carry out this by starting with the next crucial inequality due to Klainerman and Sideris [17], the proof of which requires the use of the operator S; see $N_2(v(t))$ on the right-hand side of (3.1) below. In what follows we use the notation $\Box_l := \partial_t^2 - c_l^2\Delta$.

Lemma 3.1 (Klainerman–Sideris inequality). *The inequality*

$$
(3.1)\qquad M_2(v(t)) \le C\bigl(N_2(v(t)) + \sum_{l=1}^{N} t\|\Box_l v^l(t)\|_{L^2(\mathbb{R}^n)}\bigr)
$$

holds for any function $v = (v^1, \dots, v^N)$.

Proof. See Lemma 3.1 of [17] (see also Lemma 7.1 of [24]). We have only to repeat essentially the same argument as in the proof of (3.1) of [17]. Note that the proof there is obviously valid for $n = 2$ as well as $n = 3$. □

We also need the following auxiliary lemma, which compensates for the absence of ∂_t^i $(i = 2, 3, 4)$ in the norms appearing in (1.14), (2.5).

Lemma 3.2. *There exists a constant* $\varepsilon^* > 0$ *depending on the propagation speeds* $c_1, \dots, c_N$ *and the coefficients on the right-hand side of* (1.7) *or* (1.8) *with the following property: whenever a smooth solution* $u = (u^1, \dots, u^N)$ *to* (1.7) *or* (1.8) *satisfies*

$$\max\{\, |\bar{Z}^a \partial_\alpha u^i(t,x)| \,:\, |a| \le 1,\, 0 \le \alpha \le n,\, 1 \le i \le N \,\} \le \varepsilon^*, \tag{3.2}$$

the point-wise inequality

$$\sum_{i=1}^{N} |\partial_t^2 u^i(t,x)| \le C \sum_{i=1}^{N} \sum_{\alpha=0}^{n} \Big(\sum_{m=1}^{n} |\partial_{m\alpha}^2 u^i(t,x)| + |\partial_\alpha u^i(t,x)| \Big) \tag{3.3}$$

holds. Moreover, there holds for $|a| = 1, 2$

$$\sum_{i=1}^{N} |\bar{Z}^a \partial_t^2 u^i(t,x)| \le C \sum_{i=1}^{N} \sum_{|b|=1}^{|a|} \sum_{\alpha=0}^{n} \Big(\sum_{m=1}^{n} |\bar{Z}^b \partial_{m\alpha}^2 u^i(t,x)| + |\bar{Z}^b \partial_\alpha u^i(t,x)| \Big). \tag{3.4}$$

Here, $n = 2, 3$ *for the solutions to* (1.8), (1.7), *respectively.*

Proof. It suffices to prove the inequalities for the solutions to (1.7). The proof of the inequalities for the solutions to (1.8) is essentially the same. We first note the obvious equality for each $l = 1, \dots, N$

$$\begin{aligned} &\partial_t^2 u^l - G_{ij}^{l,\alpha 00}(\partial_\alpha u^i)\partial_t^2 u^j \\ &\quad = c_l^2 \Delta u^l + 2G_{ij}^{l,\alpha m\gamma}(\partial_\alpha u^i)\partial_{m\gamma}^2 u^j + H_{ij}^{l,\alpha\beta}(\partial_\alpha u^i)\partial_\beta u^j. \end{aligned} \tag{3.5}$$

Whenever $|G_{ij}^{l,\alpha 00}\partial_\alpha u^i(t,x)| \le 1/(2N)$ for any $j, l = 1, \dots, N$, we get by (3.5)

$$\begin{aligned} &|\partial_t^2 u^l| - (2N)^{-1}(|\partial_t^2 u^1| + \cdots + |\partial_t^2 u^N|) \\ &\quad \le c_l^2 |\Delta u^l| + 2|G_{ij}^{l,\alpha m\gamma}||\partial_\alpha u^i||\partial_{m\gamma}^2 u^j| + |H_{ij}^{l,\alpha\beta}||\partial_\alpha u^i||\partial_\beta u^j|. \end{aligned} \tag{3.6}$$

Summing both the sides of (3.6) over $l = 1, \dots, N$, we see that the inequality (3.3) holds whenever $\max\{|\partial_\alpha u^i(t,x)| : \alpha = 0, \dots, 3,\ i = 1, \dots, l\}$ is small enough.

Next, let us prove (3.4). Using the commutation relations (2.1) and (2.4), we get for $|a| = 1, 2$

$$\begin{aligned}(3.7)\qquad &\partial_t^2 \bar{Z}^a u^l - G_{ij}^{l,\alpha 00}(\partial_\alpha u^i)(\partial_t^2 \bar{Z}^a u^j)\\ &= c_l^2 \Delta \bar{Z}^a u^l + \sum_{\substack{b+c=a\\ c\neq a}} G_{ij}^{l,\alpha 00}(\bar{Z}^b \partial_\alpha u^i)(\partial_t^2 \bar{Z}^c u^j)\\ &+ \sum_{b+c=a} 2G_{ij}^{l,\alpha m\gamma}(\bar{Z}^b \partial_\alpha u^i)(\bar{Z}^c \partial_{m\gamma}^2 u^j) + \sum_{b+c=a} H_{ij}^{l,\alpha\beta}(\bar{Z}^b \partial_\alpha u^i)(\bar{Z}^c \partial_\beta u^j).\end{aligned}$$

Noting the obvious fact

$$\sum_{\substack{b+c=a\\ c\neq a}} G_{ij}^{l,\alpha 00}(\bar{Z}^b \partial_\alpha u^i)(\partial_t^2 \bar{Z}^c u^j) = G_{ij}^{l,\alpha 00}(\bar{Z}^a \partial_\alpha u^i)(\partial_t^2 u^j)$$

for $|a| = 1$, using (3.3) for the estimate of $|\partial_t^2 u^j(t,x)|$, and repeating the same argument as above, we see that the inequality (3.4) holds for $|a| = 1$ whenever $\max\{\,|\bar{Z}^a \partial_\alpha u^i(t,x)| : |a| \le 1,\ 0 \le \alpha \le 3,\ 1 \le i \le N\,\}$ is small enough. Finally, using (3.3) and (3.4) with $|a| = 1$ for the estimate of $|\partial_t^2 \bar{Z}^c u^j(t,x)|$ ($|c| = 0, 1$) (see the second term on the right-hand side of (3.7)) and repeating the above argument, we see that the inequality (3.4) holds for $|a| = 2$ whenever $\max\{\,|\bar{Z}^a \partial_\alpha u^i(t,x)| : |a| \le 1,\ 0 \le \alpha \le 3,\ 1 \le i \le N\,\}$ is small enough. We have finished the proof. □

Lemma 3.2 is useful in proving the following:

Lemma 3.3. *Let $u = (u^1, \ldots, u^N)$ be a smooth solution to* (1.7) *or* (1.8) *defined in $(0,T) \times \mathbb{R}^n$ satisfying*

$$(3.8)\qquad \sup_{(0,T)\times\mathbb{R}^n} \max\{\,|\bar{Z}^a \partial_\alpha u^i(t,x)| : |a| \le 1,\ 0 \le \alpha \le n,\ 1 \le i \le N\,\} \le \varepsilon^*.$$

Then the following inequalities hold: for each $l = 1, 2, \ldots, N$ and $|a| \le 2$,

$$(3.9)\qquad t\|\Box_l \bar{Z}^a u^l(t)\|_{L^2(\mathbb{R}^3)} \le CN_4^2(u(t)) + CN_4(u(t))M_4(u(t)),\ 0 < t < T$$

when u is a solution to (1.7),

$$(3.10)\qquad t\|\Box_l \bar{Z}^a u^l(t)\|_{L^2(\mathbb{R}^2)} \le CN_4^3(u(t)) + CN_4^2(u(t))M_4(u(t)),\ 0 < t < T$$

when u is a solution to (1.8).

Proof. We start with the 3D case (3.9). Obviously, it suffices to deal with $|a| = 2$. Taking account of the form of the quadratic nonlinear terms of (1.7) and using the commutation relations (2.1)–(2.4) and the point-wise inequality (3.4), we get

$$\begin{aligned}(3.11)\quad &\|\Box_l \bar Z^a u(t)\|_{L^2(\mathbb{R}^3)}\\ &\le C\sum_{i,j=1}^{N}\sum_{\substack{|b|+|c|\\ \le 2}}\big(\|(\partial \bar Z^b u^i(t))\partial\partial_x \bar Z^c u^j(t)\|_{L^2}+\|(\partial \bar Z^b u^i(t))\partial \bar Z^c u^j(t)\|_{L^2}\big).\end{aligned}$$

(Here, and in the following as well, we use the notation ∂ to mean any of the standard partial differential operators ∂_a $(a=0,\dots,n)$.) It is enough to handle only the case $|b|+|c|=2$. We use the notation $B_i := \{x\in\mathbb{R}^3 : |x| < (c_i/2)t+1\}$, with B_i' being the complement of B_i. Using the triangle inequality, (2.13), (2.14), (2.11), and the Sobolev embedding $W^{1,2}(S^2)\hookrightarrow L^{\infty-}(S^2)$, we get for each $i,j=1,\dots,N$

$$\begin{aligned}(3.12)\quad &\sum_{|b|+|c|=2}\|(\partial \bar Z^b u^i(t))\partial\partial_x \bar Z^c u^j(t)\|_{L^2(\mathbb{R}^3)}\\ &\le C\sum_{|c|=2}\langle t\rangle^{-1}\big(\|\langle c_i t-r\rangle \partial u^i(t)\|_{L^\infty(B_i)}\|\partial\partial_x \bar Z^c u^j(t)\|_{L^2(\mathbb{R}^3)}\\ &\qquad\qquad + \|r\partial u^i(t)\|_{L^\infty(B_i')}\|\partial\partial_x \bar Z^c u^j(t)\|_{L^2(\mathbb{R}^3)}\big)\\ &\quad + C\sum_{|b|=|c|=1}\langle t\rangle^{-1}\big(\|\langle c_i t-r\rangle \partial \bar Z^b u^i(t)\|_{L^\infty(B_i)}\|\partial\partial_x \bar Z^c u^j(t)\|_{L^2(\mathbb{R}^3)}\\ &\qquad\qquad + \|r\partial \bar Z^b u^i(t)\|_{L^\infty_r L^{2+}_\omega(B_i')}\|\partial\partial_x \bar Z^c u^j(t)\|_{L^2_r L^{\infty-}_\omega(\mathbb{R}^3)}\big)\\ &\quad + C\sum_{|b|=2}\langle t\rangle^{-1}\big(\|\partial \bar Z^b u^i(t)\|_{L^2(\mathbb{R}^3)}\|\langle c_j t-r\rangle \partial\partial_x u^j(t)\|_{L^\infty(B_j)}\\ &\qquad\qquad + \|\partial \bar Z^b u^i(t)\|_{L^2(\mathbb{R}^3)}\|r\partial\partial_x u^j(t)\|_{L^\infty(B_j')}\big)\\ &\le C\langle t\rangle^{-1}\big(N_4(u(t))+M_4(u(t))\big)N_4(u(t)).\end{aligned}$$

(Here and in what follows, by $2+$ and $\infty-$ we mean arbitrary numbers p_2 and p_3, respectively, such that $p_2>2$, $p_3<\infty$, and $1/2=1/p_2+1/p_3$.) For the second norm on the right-hand side of (3.11), we obtain in the same way as above

$$(3.13)\quad \sum_{|b|+|c|=2}\|(\partial \bar Z^b u^i(t))\partial \bar Z^c u^j(t)\|_{L^2(\mathbb{R}^3)}\le C\langle t\rangle^{-1}\big(N_4(u(t))+M_4(u(t))\big)N_4(u(t)).$$

We have finished the proof of (3.9).

Next, let us prove (3.10). Again, we have only to handle the case $|a|=2$. As in (3.11), we get

$$\begin{aligned}(3.14)\quad \|\Box_l \bar Z^a u^l(t)\|_{L^2(\mathbb{R}^2)} \le C\sum\big(&\|(\partial \bar Z^b u^i(t))(\partial \bar Z^c u^j(t))\partial\partial_x \bar Z^d u^k(t)\|_{L^2}\\ &+\|(\partial \bar Z^b u^i(t))(\partial \bar Z^c u^j(t))\partial \bar Z^d u^k(t)\|_{L^2}\big).\end{aligned}$$

Here, the sum has been taken over $i,j,k=1,\dots,N$ and b,c,d with $|b|+|c|+|d|\le 2$. We must treat the two cases $|d|=0$ and $|d|=1,2$ separately. In the case $|d|=0$, assuming $|b|\le 1$ and $|c|\le 2$ without loss of generality, we obtain by (2.12), (2.14)

$$
\begin{aligned}
(3.15)\qquad &\|(\partial\bar Z^b u^i(t))(\partial\bar Z^c u^j(t))\partial\partial_x u^k(t)\|_{L^2(\mathbb{R}^2)}\\
&\le C\langle t\rangle^{-1}\|\langle c_i t-r\rangle^{1/2}\partial\bar Z^b u^i(t)\|_{L^\infty(B_{i,k})}\\
&\qquad\times\|\partial\bar Z^c u^j(t)\|_{L^2(\mathbb{R}^2)}\|\langle c_k t-r\rangle^{1/2}\partial\partial_x u^k(t)\|_{L^\infty(B_{i,k})}\\
&\quad+C\langle t\rangle^{-1}\|r^{1/2}\partial\bar Z^b u^i(t)\|_{L^\infty(B'_{i,k})}\|\partial\bar Z^c u^j(t)\|_{L^2(\mathbb{R}^2)}\|r^{1/2}\partial\partial_x u^k(t)\|_{L^\infty(B'_{i,k})}\\
&\le C\langle t\rangle^{-1}\big(N_4(u(t))+M_4(u(t))\big)N_4^2(u(t)),
\end{aligned}
$$

where $B_{i,k}:=\{x\in\mathbb{R}^2 : |x|<\min\{c_i t/2, c_k t/2\}+1\}$ with $B'_{i,k}$ being its complement. On the other hand, for $|d|=1,2$, we obtain by assuming $|b|=0$, $|c|\le 1$ without loss of generality

$$
\begin{aligned}
(3.16)\qquad &\|(\partial u^i(t))(\partial\bar Z^c u^j(t))\partial\partial_x\bar Z^d u^k(t)\|_{L^2(\mathbb{R}^2)}\\
&\le\|(\partial u^i(t))\partial\bar Z^c u^j(t)\|_{L^\infty(\mathbb{R}^2)}\|\partial\partial_x\bar Z^d u^k(t)\|_{L^2(\mathbb{R}^2)}\\
&\le C\langle t\rangle^{-1}\big(\|\langle c_i t-r\rangle^{1/2}\partial u^i(t)\|_{L^\infty(B_{i,j})}\|\langle c_j t-r\rangle^{1/2}\partial\bar Z^c u^j(t)\|_{L^\infty(B_{i,j})}\\
&\qquad+\|r^{1/2}\partial u^i(t)\|_{L^\infty(B'_{i,j})}\|r^{1/2}\partial\bar Z^c u^j(t)\|_{L^\infty(B'_{i,j})}\big)\|\partial\partial_x\bar Z^d u^k(t)\|_{L^2(\mathbb{R}^2)}\\
&\le C\langle t\rangle^{-1}\big(N_4(u(t))+M_4(u(t))\big)N_4^2(u(t)).
\end{aligned}
$$

By (3.15)–(3.16), we have obtained the desired estimate of the first term on the right-hand side of (3.14). The proof of the estimate for the second term on its right-hand side is quite similar. We may omit it. The proof of Lemma 3.3 has been finished. □

Lemma 3.4. *There exists a small, positive constant δ_0 with the following property: suppose that, for a local smooth solution u of* (1.7) *or* (1.8)*, the supremum of $N_4(u(t))$ over an interval $(0,T)$ is sufficiently small so that*

$$
\sup_{0<t<T} N_4(u(t))\le\delta_0 \tag{3.17}
$$

may hold. Then, the inequality

$$
M_4(u(t))\le CN_4(u(t)),\ 0<t<T \tag{3.18}
$$

holds with a constant C independent of T.

Proof. Let us denote by δ_* the supremum of $N_4(u(t))$ over the interval $(0,T)$. By the Sobolev embedding, we see that (3.8) is satisfied when δ_* is sufficiently small. Then, we see that Lemma 3.1 with $v=\bar Z^a u$ ($|a|\le 2$) and Lemma 3.3 imply for $0<t<T$

$$
\begin{aligned}
(3.19)\qquad M_4(u(t))&\le CN_4(u(t))+CN_4^\nu(u(t))\big(N_4(u(t))+M_4(u(t))\big)\\
&\le C(1+\delta_*^\nu)N_4(u(t))+C\delta_*^\nu M_4(u(t)),
\end{aligned}
$$

($\nu = 1, 2$ for (1.7), (1.8), respectively) from which we easily verify the existence of the constant δ_0, as claimed in the lemma. $\square$

Remark. In the above proof, especially when absorbing $C\delta_*^\nu M_4(u(t))$ into the left-hand side of (3.19), we have used the fact that $M_4(u(t))$ is finite for $t \in (0, T)$. Indeed, using (3.1), (3.17), and the standard Sobolev embedding, we get $M_4(u(t)) \leq CN_4(u(t)) + CtN_4^{\nu+1}(u(t)) < \infty$.

§ 4. Estimate for $N_4(u(t))$

For the given smooth and compactly supported initial data, let us assume (1.12) for a sufficiently small $\varepsilon_0 > 0$ such that $2\varepsilon_0 \leq \delta_0$ (see Lemma 3.4 for δ_0). By the local existence theorem mentioned in Section 1, a unique smooth solution exists locally in time. Note that it is compactly supported for fixed times by the finite speed of propagation. Let T^* be the supremum of the set of all $T > 0$ such that this solution to (1.7) is defined in $(0, T) \times \mathbb{R}^3$ and satisfies

$$\sup_{0<t<T} N_4(u(t)) < \infty.$$

When considering (1.8), we define T^* in the same way. When $T^* = \infty$, nothing remains to be done. We may therefore suppose $T^* < \infty$.

Recall the notation $\Box_l = \partial_t^2 - c_l^2 \Delta$. We set $E_4(u(t)) = N_4^2(u(t))$ (see (1.14) for the definition of $N_4(u(t))$). For (1.8), setting $Z^a = \partial_1^{a_1} \partial_2^{a_2} \Omega_{12}^{a_3} S^{a_4}$ for $a = (a_1, \dots, a_4)$ and letting a_* stand for any multi-index $a = (a_1, \dots, a_4)$ with $a_4 \leq 1$, we have the energy equality by the standard argument

$$
\begin{aligned}
(4.1)\quad \tilde{E}_4'(u(t)) = & \sum_{\substack{1\le l\le N\\ |a_*|=3}} \int_{\mathbb{R}^2} G^{l,\alpha\beta\gamma\delta}_{ijk}(\partial_\alpha u^i)(\partial_\beta u^j)([Z^{a_*},\partial_\gamma\partial_\delta]u^k)\partial_t Z^{a_*}u^l dx\\
& + \sum_{\substack{1\le l\le N\\ |a_*|=3}} \sum_{\substack{b+c+d=a_*\\ d\neq a_*}} \int_{\mathbb{R}^2} G^{l,\alpha\beta\gamma\delta}_{ijk}(Z^b\partial_\alpha u^i)(Z^c\partial_\beta u^j)(Z^d\partial_\gamma\partial_\delta u^k)\partial_t Z^{a_*}u^l dx\\
& + \sum_{\substack{1\le l\le N\\ |a_*|\le 2}} \sum_{b+c+d=a_*} \int_{\mathbb{R}^2} G^{l,\alpha\beta\gamma\delta}_{ijk}(Z^b\partial_\alpha u^i)(Z^c\partial_\beta u^j)(Z^d\partial_\gamma\partial_\delta u^k)\partial_t Z^{a_*}u^l dx\\
& - \sum_{\substack{1\le l\le N\\ |a_*|=3}} \int_{\mathbb{R}^2} G^{l,\alpha\beta p\delta}_{ijk}\big(\partial_p((\partial_\alpha u^i)\partial_\beta u^j)\big)(\partial_\delta Z^{a_*}u^k)\partial_t Z^{a_*}u^l dx\\
& - \sum_{\substack{1\le l\le N\\ |a_*|=3}} \frac{1}{2}\int_{\mathbb{R}^2}\Big(G^{l,\alpha\beta 00}_{ijk}\big(\partial_t((\partial_\alpha u^i)\partial_\beta u^j)\big)(\partial_t Z^{a_*}u^k)\partial_t Z^{a_*}u^l\\
& \qquad\qquad - G^{l,\alpha\beta pq}_{ijk}\big(\partial_t((\partial_\alpha u^i)\partial_\beta u^j)\big)(\partial_q Z^{a_*}u^k)\partial_p Z^{a_*}u^l\Big)dx\\
& + \sum_{\substack{1\le l\le N\\ |a_*|\le 3}} \sum_{b+c+d=a_*} \int_{\mathbb{R}^2} H^{l,\alpha\beta\gamma}_{ijk}(Z^b\partial_\alpha u^i)(Z^c\partial_\beta u^j)(Z^d\partial_\gamma u^k)\partial_t Z^{a_*}u^l dx\\
& + \sum_{\substack{1\le l\le N\\ |a_*|\le 3}} \int_{\mathbb{R}^2}([\Box_l, Z^{a_*}]u^l)\partial_t Z^{a_*}u^l dx,\ 0<t<T^*.
\end{aligned}
$$

Here, taking into account the quasi-linear character of (1.8), we have introduced the modified energy

$$
\begin{aligned}
(4.2)\quad \tilde{E}_4(u(t)) := E_4(u(t)) - & \sum_{\substack{|a_*|=3\\ 1\le l\le N}} \frac{1}{2}\int_{\mathbb{R}^2}\Big(G^{l,\alpha\beta 00}_{ijk}(\partial_\alpha u^i)(\partial_\beta u^j)(\partial_t Z^{a_*}u^k)\partial_t Z^{a_*}u^l\\
& - G^{l,\alpha\beta pq}_{ijk}(\partial_\alpha u^i)(\partial_\beta u^j)(\partial_q Z^{a_*}u^k)\partial_p Z^{a_*}u^l\Big)dx.
\end{aligned}
$$

Note that, in (4.1)–(4.2), repeated indices have been summed when lowered and uppered. Precisely, the Greek indices α, β, γ, and δ run from 0 to 2, while the roman p and q from 1 to 2. The roman indices i, j, and k run from 1 to N.

Similarly, for (1.7), setting $Z^a = \partial_1^{a_1}\partial_2^{a_2}\partial_3^{a_3}\Omega_{12}^{a_4}\Omega_{13}^{a_5}\Omega_{23}^{a_6}S^{a_7}$ for $a = (a_1, \dots, a_7)$ and letting a^* stand for any multi-index $a=(a_1,\dots,a_7)$ with $a_7\le 1$, we get

$$
\begin{aligned}
(4.3)\qquad \tilde E_4'(u(t)) = &\sum_{1\le l\le N}\sum_{|a^*|=3}\int_{\mathbb{R}^3} G^{l,\alpha\beta\gamma}_{ij}(\partial_\alpha u^i)([Z^{a^*},\partial_\beta\partial_\gamma]u^j)\partial_t Z^{a^*}u^l dx\\
&+\sum_{1\le l\le N}\sum_{|a^*|=3}\sum_{\substack{b+c=a^*\\ c\ne a^*}}\int_{\mathbb{R}^3} G^{l,\alpha\beta\gamma}_{ij}(Z^b\partial_\alpha u^i)(Z^c\partial_\beta\partial_\gamma u^j)\partial_t Z^{a^*}u^l dx\\
&+\sum_{\substack{1\le l\le N\\ |a^*|\le 2}}\sum_{b+c=a^*}\int_{\mathbb{R}^3} G^{l,\alpha\beta\gamma}_{ij}(Z^b\partial_\alpha u^i)(Z^c\partial_\beta\partial_\gamma u^j)\partial_t Z^{a^*}u^l dx\\
&-\sum_{1\le l\le N}\sum_{|a^*|=3}\int_{\mathbb{R}^3} G^{l,\alpha p\gamma}_{ij}(\partial_p\partial_\alpha u^i)(\partial_\gamma Z^{a^*}u^j)\partial_t Z^{a^*}u^l dx\\
&-\sum_{1\le l\le N}\sum_{|a^*|=3}\int_{\mathbb{R}^3}\frac12\big(G^{l,\alpha 00}_{ij}(\partial_t\partial_\alpha u^i)(\partial_t Z^{a^*}u^j)\partial_t Z^{a^*}u^l\\
&\qquad\qquad -G^{l,\alpha pq}_{ij}(\partial_t\partial_\alpha u^i)(\partial_q Z^{a^*}u^j)\partial_p Z^{a^*}u^l\big)dx\\
&+\sum_{1\le l\le N}\sum_{|a^*|\le 3}\sum_{b+c=a^*}\int_{\mathbb{R}^3} H^{l,\alpha\beta}_{ij}(Z^b\partial_\alpha u^i)(Z^c\partial_\beta u^j)\partial_t Z^{a^*}u^l dx\\
&+\sum_{1\le l\le N}\sum_{|a^*|\le 3}\int_{\mathbb{R}^3}([\Box_l, Z^{a^*}]u^l)\partial_t Z^{a^*}u^l dx,\ 0<t<T^*.
\end{aligned}
$$

Here, we have defined

$$
\begin{aligned}
(4.4)\quad \tilde E_4(u(t)) := E_4(u(t)) - &\sum_{1\le l\le N}\sum_{|a^*|=3}\frac12\int_{\mathbb{R}^3}\big(G^{l,\alpha 00}_{ij}(\partial_\alpha u^i)(\partial_t Z^{a^*}u^j)\partial_t Z^{a^*}u^l\\
&\qquad -G^{l,\alpha pq}_{ij}(\partial_\alpha u^i)(\partial_q Z^{a^*}u^j)\partial_p Z^{a^*}u^l\big)dx.
\end{aligned}
$$

As in (4.1)–(4.2), repeated indices have been summed in (4.3)–(4.4) when lowered and uppered. The Greek indices α,β, and γ run from 0 to 3, while the roman p and q from 1 to 3. The roman indices i and j run from 1 to N.

We may suppose without loss of generality that, for $\tilde E_4(u(t))$ defined in (4.2), (4.4), the inequality

$$
(4.5)\qquad \frac23 E_4(u(t)) \le \tilde E_4(u(t)) \le \frac32 E_4(u(t))
$$

holds by the Sobolev embedding, whenever $N_4(u(t))$ is small enough. We also note that, owing to the commutation relations (2.3)–(2.4), the commutators $[Z^{a_*},\partial_\beta\partial_\gamma]$ and $[Z^{a^*},\partial_\beta\partial_\gamma]$, which appear in the first term on the right-hand side of (4.1), (4.3), have the form

$$
\begin{aligned}
(4.6) \qquad [Z^{a_*}, \partial_\beta \partial_\gamma] &= \sum_{|b_*| \le 2} \sum_{\alpha,\delta=0}^{2} C^{a_*, b_*}_{\beta\gamma\alpha\delta} \partial_\alpha \partial_\delta Z^{b_*},\\
[Z^{a^*}, \partial_\beta \partial_\gamma] &= \sum_{|b^*| \le 2} \sum_{\alpha,\delta=0}^{3} C^{a^*, b^*}_{\beta\gamma\alpha\delta} \partial_\alpha \partial_\delta Z^{b^*},
\end{aligned}
$$

respectively, for each a_*, a^* ($|a_*| = |a^*| = 3$), β, and γ. Here, by $C^{a_*,b_*}_{\beta\gamma\alpha\delta}$ and $C^{a^*,b^*}_{\beta\gamma\alpha\delta}$, we mean suitable constants depending on a_*, b_*, a^*, b^*, and α, β, γ, and δ. (Note that by b_* and b^*, we mean any multi-index $(b_1, \dots, b_4)$ with $b_4 \le 1$, and $(b_1, \dots, b_7)$ with $b_7 \le 1$, respectively.) Note also that, thanks to (2.1) and $a_4 \le 1$, the commutator $[\Box_l, Z^{a_*}]$ appearing in the last term on the right-hand side of (4.1) is 0 or $2\Box_l$. A similar note applies equally to $[\Box_l, Z^{a^*}]$ which appears on the right-hand side of (4.3).

Let us start the estimate of $E_4(u(t))$ with the case $n = 2$. We remark that, under the assumption that $N_4(u(t))$ is small enough, we have by repeating quite the same argument as in the proof of Lemma 3.2

$$
(4.7) \qquad \sum_{i=1}^{N} |\partial_t^2 Z^a u^i(t,x)| \le C \sum \bigl(|\partial^2_{m\alpha} Z^b u^i(t,x)| + |\partial_\alpha Z^b u^i(t,x)| \bigr)
$$

for any multi-index a with $|a| \le 2$, $a_4 \le 1$. Here, on the right-hand side the sum is taken over all $1 \le i \le N$, $1 \le m \le 2$, $0 \le \alpha \le 2$, and b with $b_k \le a_k$ for all $1 \le k \le 4$.

In what follows, we assume $N_4(u(t))$ is small so that (4.7) may hold. Using the energy equality (4.1) and the commutation relations (2.3), (2.4), and (4.6)–(4.7), we get

$$
\begin{aligned}
(4.8) \qquad \tilde{E}_4'(u(t)) \le{}& C \sum \|(\partial Z^b u^i)(\partial Z^c u^j) \partial \partial_x Z^d u^k\|_{L^2(\mathbb{R}^2)} N_4(u)\\
&+ C \sum \|(\partial Z^b u^i)(\partial Z^c u^j) \partial Z^d u^k\|_{L^2(\mathbb{R}^2)} N_4(u).
\end{aligned}
$$

Here, on the first term on the right-hand side above the sum is taken over all $i, j, k = 1, \dots, N$, and b, c, d with $|b| + |c| + |d| \le 3$, $|d| \le 2$, $b_4 + c_4 + d_4 \le 1$. On the second term, the sum is taken over all $i, j, k = 1, \dots, N$, and b, c, d with $|b| + |c| + |d| \le 3$, $b_4 + c_4 + d_4 \le 1$. Obviously, we have only to deal with the case $|b| + |c| + |d| = 3$. Moreover, we may focus on the case of $b_4 + c_4 + d_4 = 1$ because the argument otherwise becomes much simpler. When treating the second term on the right-hand side above, we may also assume $|b| \le |c| \le |d|$ (hence $|b|, |c| \le 1$) without loss of generality. We treat the two cases $d_4 = 0$ and $d_4 = 1$, separately. When $d_4 = 0$, we know $b_4 + c_4 = 1$

and $|d| \le 2$. If $b_4 = 0$, then we get by (2.8), (2.14)

$$\begin{aligned}(4.9)\quad &\|(\partial Z^b u^i)(\partial Z^c u^j)\partial Z^d u^k\|_{L^2}\\ &\le C\langle t\rangle^{-1}\|\langle c_i t-r\rangle^{1/2}\partial Z^b u^i\|_{L^4(B_{i,k})}\|\partial Z^c u^j\|_{L^\infty}\|\langle c_k t-r\rangle^{1/2}\partial Z^d u^k\|_{L^4(B_{i,k})}\\ &+C\langle t\rangle^{-1}\|r^{1/2}\partial Z^b u^i\|_{L^\infty(B'_{i,k})}\|r^{1/2}\partial Z^c u^j\|_{L^\infty(B'_{i,k})}\|\partial Z^d u^k\|_{L^2}\\ &\le C\langle t\rangle^{-1}N_4^3(u(t)).\end{aligned}$$

If $b_4 = 1$, then we know $c_4 = 0$ and thus we obtain the same bound as (4.9) by considering $\|\partial Z^b u^i\|_{L^\infty}\|\langle c_j t-r\rangle^{1/2}\partial Z^c u^j\|_{L^4(B_{j,k})}$ in place of $\|\langle c_i t-r\rangle^{1/2}\partial Z^b u^i\|_{L^4(B_{i,k})}\|\partial Z^c u^j\|_{L^\infty}$.

When $d_4 = 1$, we know $b_4 = c_4 = 0$ and thus obtain by (2.12), (2.14)

$$\begin{aligned}(4.10)\quad &\|(\partial Z^b u^i)(\partial Z^c u^j)\partial Z^d u^k\|_{L^2}\\ &\le C\langle t\rangle^{-1}\|\langle c_i t-r\rangle^{1/2}\partial Z^b u^i\|_{L^\infty(B_{i,j})}\|\langle c_j t-r\rangle^{1/2}\partial Z^c u^j\|_{L^\infty(B_{i,j})}\|\partial Z^d u^k\|_{L^2}\\ &+C\langle t\rangle^{-1}\|r^{1/2}\partial Z^b u^i\|_{L^\infty(B'_{i,j})}\|r^{1/2}\partial Z^c u^j\|_{L^\infty(B'_{i,j})}\|\partial Z^d u^k\|_{L^2}\\ &\le C\langle t\rangle^{-1}N_4^3(u(t)).\end{aligned}$$

Next, let us consider the bound for the first term on the right-hand side of (4.8). We may suppose $|b| \le |c|$ (thus $|b| \le 1$) without loss of generality. We discuss the two cases $d_4 = 0$ and $d_4 = 1$, separately. In the former case, we further treat the two cases $|d| \le 1$ and $|d| = 2$, separately.

Suppose $d_4 = 0$ and $|d| \le 1$. If $b_4 = 1$, then we have $c_4 = 0$, $|c| \le 2$ and thus obtain by (2.7) and the Sobolev embedding on S^1

$$\begin{aligned}(4.11)\quad &\|(\partial Z^b u^i)(\partial Z^c u^j)\partial\partial_x Z^d u^k\|_{L^2}\\ &\le C\langle t\rangle^{-1}\|\partial Z^b u^i\|_{L^\infty}\|\langle c_j t-r\rangle^{1/2}\partial Z^c u^j\|_{L^4(B_{j,k})}\|\langle c_k t-r\rangle^{1/2}\partial\partial_x Z^d u^k\|_{L^4(B_{j,k})}\\ &+C\langle t\rangle^{-1}\|r^{1/2}\partial Z^b u^i\|_{L^\infty(B'_{j,k})}\|r^{1/2}\partial Z^c u^j\|_{L^\infty_r L^2_\omega(B'_{j,k})}\|\partial\partial_x Z^d u^k\|_{L^2_r L^\infty_\omega}\\ &\le C\langle t\rangle^{-1}N_4^3(u(t)).\end{aligned}$$

If $b_4 = 0$, then $c_4 = 1$ and obtain

$$\begin{aligned}(4.12)\quad &\|(\partial Z^b u^i)(\partial Z^c u^j)\partial^2 Z^d u^k\|_{L^2}\\ &\le C\langle t\rangle^{-1}\|\langle c_i t-r\rangle^{1/2}\partial u^i\|_{L^\infty(B_{i,k})}\|\partial Z^c u^j\|_{L^2}\|\langle c_k t-r\rangle^{1/2}\partial\partial_x u^k\|_{L^\infty(B_{i,k})}\\ &+C\langle t\rangle^{-1}\|r^{1/2}\partial u^i\|_{L^\infty(B'_{i,k})}\|\partial Z^c u^j\|_{L^2}\|r^{1/2}\partial\partial_x u^k\|_{L^\infty(B'_{i,k})}\\ &\le C\langle t\rangle^{-1}N_4^3(u(t))\end{aligned}$$

for $|c|=3$ (thus $|b|=0$), and

$$
\begin{aligned}
(4.13)\quad &\|(\partial Z^b u^i)(\partial Z^c u^j)\partial^2 Z^d u^k\|_{L^2}\\
&\le C\langle t\rangle^{-1}\|\langle c_i t-r\rangle^{1/2}\partial Z^b u^i\|_{L^\infty(B_{i,k})}\|\partial Z^c u^j\|_{L^4}\|\langle c_k t-r\rangle^{1/2}\partial\partial_x Z^d u^k\|_{L^4(B_{i,k})}\\
&+C\langle t\rangle^{-1}\|r^{1/2}\partial Z^b u^i\|_{L^\infty(B'_{i,k})}\|r^{1/2}\partial Z^c u^j\|_{L^\infty_r L^2_\omega(B'_{i,k})}\|\partial\partial_x Z^d u^k\|_{L^2_r L^\infty_\omega}\\
&\le C\langle t\rangle^{-1}N_4^3(u(t))
\end{aligned}
$$

for $|c|\le 2$. (Recall that we are assuming $d_4=0$, $|d|\le 1$.)

Next, suppose $d_4=0$ and $|d|=2$. Then, we know $|b|=0$, $|c|=1$ (because of $|b|\le|c|$ and $|b|+|c|+|d|=3$), and easily obtain by (3.18)

$$
\begin{aligned}
(4.14)\quad &\|(\partial Z^b u^i)(\partial Z^c u^j)\partial^2 Z^d u^k\|_{L^2}\\
&\le C\langle t\rangle^{-1}\|\partial u^i\|_{L^\infty}\|\partial Z^c u^j\|_{L^\infty}\|\langle c_k t-r\rangle\partial\partial_x Z^d u^k\|_{L^2(B_k)}\\
&+C\langle t\rangle^{-1}\|r^{1/2}\partial u^i\|_{L^\infty(B'_k)}\|r^{1/2}\partial Z^c u^j\|_{L^\infty(B'_k)}\|\partial\partial_x Z^d u^k\|_{L^2}\\
&\le C\langle t\rangle^{-1}N_4^3(u(t)).
\end{aligned}
$$

(The definition of B_i is given in the proof of Lemma 3.3.)

Turn our attention to the case of $d_4=1$. We know $b_4=c_4=0$ and $|c|\le 2$. We discuss the two cases $|d|=1$ and $|d|=2$, separately. If $|d|=1$, then we get

$$
\begin{aligned}
(4.15)\quad &\|(\partial Z^b u^i)(\partial Z^c u^j)\partial^2 Z^d u^k\|_{L^2}\\
&\le C\langle t\rangle^{-1}\|\langle c_i t-r\rangle^{1/2}\partial Z^b u^i\|_{L^\infty(B_{i,j})}\|\langle c_j t-r\rangle^{1/2}\partial Z^c u^j\|_{L^4(B_{i,j})}\|\partial\partial_x Z^d u^k\|_{L^4}\\
&+C\langle t\rangle^{-1}\|r^{1/2}\partial Z^b u^i\|_{L^\infty(B'_{i,j})}\|r^{1/2}\partial Z^c u^j\|_{L^\infty_r L^2_\omega(B'_{i,j})}\|\partial\partial_x Z^d u^k\|_{L^2_r L^\infty_\omega}\\
&\le C\langle t\rangle^{-1}N_4^3(u(t)).
\end{aligned}
$$

If $|d|=2$, then we know $|b|=0$, $|c|=1$ and thus easily obtain

$$
\begin{aligned}
(4.16)\quad &\|(\partial Z^b u^i)(\partial Z^c u^j)\partial^2 Z^d u^k\|_{L^2}\\
&\le C\langle t\rangle^{-1}\|\langle c_i t-r\rangle^{1/2}\partial u^i\|_{L^\infty(B_{i,j})}\|\langle c_j t-r\rangle^{1/2}\partial Z^c u^j\|_{L^\infty(B_{i,j})}\|\partial\partial_x Z^d u^k\|_{L^2}\\
&+C\langle t\rangle^{-1}\|r^{1/2}\partial u^i\|_{L^\infty(B'_{i,j})}\|r^{1/2}\partial Z^c u^j\|_{L^\infty(B'_{i,j})}\|\partial\partial_x Z^d u^k\|_{L^2}\\
&\le C\langle t\rangle^{-1}N_4^3(u(t)).
\end{aligned}
$$

We have finished bounding the right-hand side of (4.8). Taking account of the equivalence between $E_4(u(t))$ and $\tilde{E}_4(u(t))$ (see (4.5)), we get from (4.9)–(4.16)

$$
\tilde{E}_4'(u(t))\le C\langle t\rangle^{-1}N_4^2(u(t))\tilde{E}_4(u(t)) \tag{4.17}
$$

as far as $N_4(u(t))$ is small enough.

We turn our attention to the case $n = 3$. In the same way as we got (4.8), we obtain by (4.3)

$$\tilde{E}_4'(u(t)) \leq C \sum \|(\partial Z^b u^i)\partial\partial_x Z^c u^j\|_{L^2} N_4(u) + C \sum \|(\partial Z^b u^i)\partial Z^c u^j\|_{L^2} N_4(u), \tag{4.18}$$

where in the right-hand side, the sum is taken over all $i, j = 1, \dots, N$ and b, c with $|b| + |c| \leq 3$ ($|c| \leq 2$ for the first term), $b_7 + c_7 \leq 1$. As in the case of $n = 2$, it suffices to treat the terms with $|b| + |c| = 3$ and $b_7 + c_7 = 1$.

Let us first treat the second term on the right-hand side above. We may suppose $|b| \leq |c|$ (thus $|b| \leq 1$) without loss of generality. When $c_7 = 0$, we know $b_7 = 1$, $|c| \leq 2$ and thus obtain by (2.10), (2.11) and the Sobolev embedding on S^2

$$\begin{aligned} \|(\partial Z^b u^i)\partial Z^c u^j\|_{L^2} &\leq C\langle t\rangle^{-1} \|\partial Z^b u^i\|_{L^3} \|\langle c_j t - r\rangle \partial Z^c u^j\|_{L^6(B_j)} \\ &\quad + C\langle t\rangle^{-1} \|r\partial Z^b u^i\|_{L_r^\infty L_\omega^{2+}(B_j')} \|\partial Z^c u^j\|_{L_r^2 L_\omega^{\infty-}} \\ &\leq C\langle t\rangle^{-1} N_4^2(u(t)). \end{aligned} \tag{4.19}$$

When $c_7 = 1$, we know $b_7 = 0$ and thus obtain by (2.13), (2.14)

$$\begin{aligned} &\|(\partial Z^b u^i)\partial Z^c u^j\|_{L^2} \\ &\leq C\langle t\rangle^{-1} \big(\|\langle c_i t - r\rangle \partial Z^b u^i\|_{L^\infty(B_i)} + \|r\partial Z^b u^i\|_{L^\infty(B_i')} \big) \|\partial Z^c u^j\|_{L^2} \\ &\leq C\langle t\rangle^{-1} N_4^2(u(t)). \end{aligned} \tag{4.20}$$

Next, let us turn to the estimate of the first term on the right-hand side of (4.18). Again, we discuss the two cases $c_7 = 0$ and $c_7 = 1$, separately.

Suppose $c_7 = 0$. We handle the two cases $|c| \leq 1$ and $|c| = 2$, separately. When $c_7 = 0$ and $|c| \leq 1$, we know $b_7 = 1$ and thus obtain

$$\begin{aligned} &\|(\partial Z^b u^i)\partial\partial_x Z^c u^j\|_{L^2} \\ &\leq C\langle t\rangle^{-1} \|\partial Z^b u^i\|_{L^2} \big(\|\langle c_j t - r\rangle \partial\partial_x u^j\|_{L^\infty(B_j)} + \|r\partial\partial_x u^j\|_{L^\infty(B_j')} \big) \\ &\leq C\langle t\rangle^{-1} N_4^2(u(t)) \end{aligned} \tag{4.21}$$

for $|b| = 3$ (thus $|c| = 0$), and

$$\begin{aligned} \|(\partial Z^b u^i)\partial\partial_x Z^c u^j\|_{L^2} &\leq C\langle t\rangle^{-1} \|\partial Z^b u^i\|_{L^3} \|\langle c_j t - r\rangle \partial\partial_x Z^c u^j\|_{L^6(B_j)} \\ &\quad + C\langle t\rangle^{-1} \|r\partial Z^b u^i\|_{L_r^\infty L_\omega^{2+}(B_j')} \|\partial\partial_x Z^c u^j\|_{L_r^2 L_\omega^{\infty-}} \\ &\leq C\langle t\rangle^{-1} N_4^2(u(t)) \end{aligned} \tag{4.22}$$

for $|b| \leq 2$. When $c_7 = 0$ and $|c| = 2$, we know $b_7 = 1$, $|b| \leq 1$ and thus easily get by

(3.18)

$$\|(\partial Z^b u^i)\partial\partial_x Z^c u^j\|_{L^2} \le C\langle t\rangle^{-1}\|\partial Z^b u^i\|_{L^\infty}\|\langle c_j t - r\rangle\partial\partial_x Z^c u^j\|_{L^2(B_j)} + C\langle t\rangle^{-1}\|r\partial Z^b u^i\|_{L^\infty(B_j')}\|\partial\partial_x Z^c u^j\|_{L^2} \le C\langle t\rangle^{-1}N_4^2(u(t)). \tag{4.23}$$

Finally, suppose $c_7 = 1$. We know $b_7 = 0$ and $|b| \le 2$. Let us discuss the two cases $|c| = 1$ and $|c| = 2$, separately. If $|c| = 1$, then

$$\|(\partial Z^b u^i)\partial\partial_x Z^c u^j\|_{L^2} \le C\langle t\rangle^{-1}\|\langle c_i t - r\rangle\partial Z^b u^i\|_{L^6(B_i)}\|\partial\partial_x S u^j\|_{L^3} + C\langle t\rangle^{-1}\|r\partial Z^b u^i\|_{L^\infty_r L^{2+}_\omega(B_i')}\|\partial\partial_x S u^j\|_{L^2_r L^{\infty-}_\omega} \le C\langle t\rangle^{-1}N_4^2(u(t)). \tag{4.24}$$

If $|c| = 2$, then we know $|b| \le 1$ (and $b_7 = 0$) and thus easily get

$$\|(\partial Z^b u^i)\partial\partial_x Z^c u^j\|_{L^2} \le C\langle t\rangle^{-1}\big(\|\langle c_i t - r\rangle\partial Z^b u^i\|_{L^\infty(B_i)} + \|r\partial Z^b u^i\|_{L^\infty(B_i')}\big)\|\partial\partial_x Z^c u^j\|_{L^2} \le C\langle t\rangle^{-1}N_4^2(u(t)). \tag{4.25}$$

We have finished the required estimates of the two terms on the right-hand side of (4.18). Combining (4.19)–(4.25) and recalling (4.5), we get

$$\tilde E_4'(u(t)) \le C\langle t\rangle^{-1}N_4(u(t))\tilde E_4(u(t)) \tag{4.26}$$

as far as $N_4(u(t))$ is small enough.

We are in a position to complete the proof of Theorem 1.1. We prove Theorem 1.1 for the solutions to (1.7), because the proof for those to (1.8) is similar. Let T_* be the supremum of the set of all $T > 0$ such that this solution to (1.7) is defined in $(0, T) \times \mathbb{R}^3$ and small so that

$$\sup_{0<t<T} N_4(u(t)) \le 2\varepsilon, \tag{4.27}$$

where $\varepsilon := N_4(u(0))$. By definition, we know $T_* \le T^*$.

If we assume

$$\varepsilon \log(1 + T_*) < B \tag{4.28}$$

for the constant B defined via $\exp\{C_1 B\} = 7/6$ (see (4.29) below for the constant $C_1 > 0$), then we will get a contradiction. Indeed, we get by (4.26)

$$\tilde E_4'(u(t)) \le 2\varepsilon C_1(1 + t)^{-1}\tilde E_4(u(t)),\ 0 < t < T_* \tag{4.29}$$

for a suitable constant $C_1 > 0$. This together with (4.5), (4.27)–(4.28) immediately yields

$$N_4(u(t)) \leq \frac{7}{4}\varepsilon < 2\varepsilon,\ 0 < t < T_*. \tag{4.30}$$

We note that, thanks to the fact that $u(t,x)$ is smooth and compactly supported for fixed times, we easily see that

$$N_4(u(t)) \in C([0,T^*)). \tag{4.31}$$

(It is this simple proof of (4.31) that needs the smoothness of the solution and the compactness of the support for fixed times.) Recall $T_* \leq T^*$ by definition. If $T_* < T^*$, then in view of (4.31) the bound (4.30) obviously contradicts the definition of T_*. We thus see $T_* = T^*$. Recall that the system (1.7) is invariant under the translation of the time variable, and that the length of the interval of existence of C^∞-solutions to (1.7) with data (φ,ψ) given at $t = t_0$ depends only on the H^3-norm of $(\nabla\varphi,\psi)$ but it does not on t_0. Thanks to the bound $N_4(u(t)) \leq 7\varepsilon/4$ $(0 < t < T^*)$, we can therefore extend this solution $u(t,x)$ to a larger strip, say, $(0, T^* + T') \times \mathbb{R}^3$ (for some $T' > 0$) with

$$\sup_{0<t<T^*+T'} N_4(u(t)) < \infty$$

by solving (1.7) subject to the compactly supported C^∞-data $(u(T^*-\delta,x), \partial_t u(T^*-\delta,x))$ given at $t = T^* - \delta$. (Here, by $\delta > 0$, we mean a sufficiently small positive number.) This, however, contradicts the definition of T^*. We thus see that (4.28) is false and there holds

$$\varepsilon \log(1 + T_*) \geq B, \tag{4.32}$$

by which we have finished the proof of Theorem 1.1 for the solutions to (1.7).

§ 5. Proof of Theorem 1.2

The proof of Theorem 1.2 requires only obvious modifications of that of Theorem 1.1. We may therefore leave the details to the reader.

Acknowledgement. The author is grateful to Professors Hideo Kubo and Hideo Takaoka for inviting him to the conference "Harmonic Analysis and Nonlinear Partial Differential Equations" held in July 2016 at the Research Institute for Mathematical Sciences (RIMS), Kyoto University. He also thanks Professor Chengbo Wang for information on low regularity well-posedness for quasi-linear wave equations, which was very helpful to him. His thanks go as well to the referee for reading the manuscript carefully and making a valuable comment on the continuation argument for local solutions in Section 4.

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