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Elliptic Ding-Iohara-Miki Algebra and Related Topics

Dedicated to Professor Akihiro Tsuchiya for his seventieth birthday

By

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Abstract

The elliptic Ding-Iohara-Miki algebra [Sa1] is an elliptic quantum group obtained from the free field realization of the elliptic Ruijsenaars operator. In this article, we review the free field realization of the elliptic Ruijsenaars operator, the elliptic Ding-Iohara-Miki algebra and related topics.

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Notations. In this paper, we use the following symbols.

$\mathbb{Z}$: the set of integers, $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$, $\mathbb{Z}_{>0} := \{1, 2, \dots\}$,
$\mathbb{Q}$: the set of rational numbers, $\mathbb{Q}(q, t)$: the field of rational functions of q, t over $\mathbb{Q}$,
$\mathbb{C}$: the set of complex numbers, $\mathbb{C}^{\times} := \mathbb{C} \setminus \{0\}$,
$\mathbb{F}[[z, z^{-1}]]$: the set of formal power series of z, z^{-1} over a field $\mathbb{F}$.

If a sequence $\lambda = (\lambda_1, \dots, \lambda_N) \in (\mathbb{Z}_{\geq 0})^N$ satisfies the condition $\lambda_i \geq \lambda_{i+1}$ $(1 \leq \forall i \leq N-1)$, λ is called a partition. We denote the set of partitions by $\mathcal{P}$. For a partition λ, we define $\ell(\lambda) := \sharp\{i \mid \lambda_i \neq 0\}$, $|\lambda| := \sum_{i=1}^{\ell(\lambda)} \lambda_i$. The symbol $\ell(\lambda)$ is called the length of λ and $|\lambda|$ is called the size of λ.

Let q, $p \in \mathbb{C}$ be complex parameters satisfying $|q| < 1$, $|p| < 1$. We define the q-infinite product as $(x; q)_\infty := \prod_{n \geq 0}(1 - xq^n)$ $(x \in \mathbb{C})$ and the theta function as

$$\Theta_p(x) := (p; p)_\infty (x; p)_\infty (px^{-1}; p)_\infty \quad (x \in \mathbb{C}^{\times}).$$

We set the double infinite product as $(x; q, p)_\infty := \prod_{m,n \geq 0}(1 - xq^m p^n)$ $(x \in \mathbb{C})$ and the elliptic gamma function as

$$\Gamma_{q,p}(x) := \frac{(qpx^{-1}; q, p)_\infty}{(x; q, p)_\infty} \quad (x \in \mathbb{C}^{\times}).$$

§ 1. Introduction

The aim of this article is to give a short review on the elliptic Ding-Iohara-Miki algebra [Sa1] and related materials. First we give backgrounds of this article.

Quantum algebraic aspects of quantum integrable systems have been studied in these decades. The Ding-Iohara-Miki algebra, which is one of main features of this article, is a quantum group which has to do with the free field realization of the Macdonald operator. Actually, first Ding and Iohara found a class of quantum groups as generalizations of the quantum affine algebra $U_q(\widehat{sl_2})$ [DI]. Then Miki [Miki] introduced a q-deformation of the $W_{1+\infty}$ algebra which has a structure of Ding and Iohara's algebra. Further Feigin-Hashizume-Hoshino-Shiraishi-Yanagida [FHHSY] obtained a quantum group which is essentially the same as Miki's quantum group, and constructed the commutative families containing the Macdonald operator by using the free field realization and the trigonometric Feigin-Odesskii algebra.

By keeping the fact that the Hamiltonian of the trigonometric Ruijsenaars model [R1] is essentially the same as the Macdonald operator in mind, we can recognize that the trigonometric Ruijsenaars model can be treated in terms of representations of the Ding-Iohara-Miki algebra. The Ding-Iohara-Miki algebra has some applications to the AGT conjecture [AFHKSY], the refined topological vertex [AFS].

On the other hand, there exists an elliptic generalization of the trigonometric Ruijsenaars model, so-called the elliptic Ruijsenaars model [R1]. Then by recalling the trigonometric case, one can have a question:

> How can we obtain an elliptic analog of the Ding-Iohara-Miki algebra which has connections to the elliptic Ruijsenaars model ?

One of main purposes is to give an answer to the above question.

Then we can find the fact that the free field realization of the Macdonald operator is based on the form of the kernel function for the operator. In addition Komori-Noumi-Shiraishi studied kernel functions for q-difference operators of Ruijsenaars type, and they obtained the kernel function for the elliptic Ruijsenaars operator [KNS]. Hence it would be a natural expect that the elliptic kernel function has important informations for the free field realization of the elliptic Ruijsenaars operator. Actually, the elliptic kernel function tells us what kind of boson is suitable. Consequently, starting from the elliptic kernel function, the free field realization of the elliptic Ruijsenaars operator was obtained, and an elliptic analog of the Ding-Iohara-Miki algebra arose. We call the elliptic quantum group **the elliptic Ding-Iohara-Miki algebra.** Furthermore it turns out that by using the free field realization and the elliptic Feigin-Odesskii algebra [FO][FHHSY], we can construct commutative families of the elliptic Ruijsenaars operator, i.e. a part of works due to Feigin-Hashizume-Hoshino-Shiraishi-Yanagida can be extended to the elliptic case.

$$\begin{array}{ccc}
\text{Elliptic Ruijsenaars operator} & \xrightarrow{\textbf{free field realization !}} & \textbf{Elliptic Ding-Iohara-Miki algebra} \\
\uparrow \text{elliptic deformation} & & \uparrow \textbf{elliptic deformation !} \\
\text{Macdonald operator} & \xrightarrow{\text{free field realization}} & \text{Ding-Iohara-Miki algebra}
\end{array}$$

Next let us explain the word the elliptic deformation. On the theta function $\Theta_p(x)$ and the elliptic gamma function $\Gamma_{q,p}(x)$, we can easily check the followings:

$$\Theta_p(x) := (p;p)_\infty (x;p)_\infty (px^{-1};p)_\infty \xrightarrow[p\to 0]{} 1-x,$$

$$\Gamma_{q,p}(x) := \frac{(qpx^{-1};q,p)_\infty}{(x;q,p)_\infty} \xrightarrow[p\to 0]{} \frac{1}{(x;q)_\infty}.$$

Thus it is natural to regard an inverse procedure of the above as the elliptic deformation. We can find such procedure as follows. Let us rewrite $1-x$ and $(x;q)_\infty^{-1}$ as

$$1-x = \exp\left(-\sum_{n>0} \frac{x^n}{n}\right), \quad \frac{1}{(x;q)_\infty} = \exp\left(\sum_{n>0} \frac{1}{1-q^n}\frac{x^n}{n}\right).$$

Furthermore we rewrite $\Theta_p(x)/(p;p)_\infty$ and $\Gamma_{q,p}(x)$ as

$$\frac{\Theta_p(x)}{(p;p)_\infty} = \exp\left(-\sum_{n\neq 0}\frac{1}{1-p^n}\frac{x^n}{n}\right), \quad \Gamma_{q,p}(x) = \exp\left(\sum_{n\neq 0}\frac{1}{(1-q^n)(1-p^n)}\frac{x^n}{n}\right).$$

Hence we can find the following procedure:

$$1-x = \exp\left(-\sum_{n>0}\frac{x^n}{n}\right) \xrightarrow[\text{deformation}]{\text{elliptic}} \exp\left(-\sum_{n\neq 0}\frac{1}{1-p^n}\frac{x^n}{n}\right) = \frac{\Theta_p(x)}{(p;p)_\infty},$$

$$\frac{1}{(x;q)_\infty} = \exp\left(\sum_{n>0}\frac{1}{1-q^n}\frac{x^n}{n}\right) \xrightarrow[\text{deformation}]{\text{elliptic}} \exp\left(\sum_{n\neq 0}\frac{1}{(1-q^n)(1-p^n)}\frac{x^n}{n}\right) = \Gamma_{q,p}(x).$$

The above operation is for functions, however not only for functions but also for boson operators there exists a similar procedure of elliptic deformation. By using the procedure, we can reproduce the theta function $\Theta_p(x)$ and the elliptic gamma function $\Gamma_{q,p}(x)$ from OPE of boson operators.

Organization of this paper.

In Section 2, we will discuss about the free field realization of the elliptic Ruijsenaars operator and the elliptic Ding-Iohara-Miki algebra $\mathcal{U}(q,t,p)$ [Sa1]. In Section 3, we will construct the commutative families of the elliptic Ruijsenaars operator by using the free field realization and the elliptic Feigin-Odesskii algebra [Sa2].

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§ 2. Elliptic Ding-Iohara-Miki algebra

In this section, we introduce the elliptic Ding-Iohara-Miki algebra [Sa1]. For readers convenience, before starting discussions about elliptic case we give a review of the trigonometric case [FHHSY].

§ 2.1. Free field realization of the Macdonald operator

In the following, let $q,\ t \in \mathbb{C}^\times$ be complex parameters and we assume $|q| < 1$, $|t^{-1}| < 1$ and $q \neq t$. We denote the field of rational functions of $q^{1/4}$, $t^{1/4}$ over $\mathbb{Q}$ by $\mathbb{K} := \mathbb{Q}(q^{1/4}, t^{1/4})$. Set the q-shift operator by $T_{q,x}f(x) := f(qx)$. The Macdonald operator $H_N(q,t)$ $(N \in \mathbb{Z}_{>0})$ is defined by

$$H_N(q,t) := \sum_{i=1}^{N}\prod_{j\neq i}\frac{tx_i - x_j}{x_i - x_j}T_{q,x_i}.$$

Then the kernel function for the Macdonald operator defined below has the following property.

Proposition 2.1 (**Kernel function for** $H_N(q,t)$ [**Mac**]). *Define the function* $\Pi_{MN}(q,t)(x,y)$ $(M,N\in\mathbb{Z}_{>0})$ *as follows.*

$$\Pi_{MN}(q,t)(x,y) := \prod_{\substack{1\le i\le M\\ 1\le j\le N}} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}.$$

Let $H_N(q,t)_x$ *be the Macdonald operator which acts on functions of* $x_1,\ldots,x_N$. *Then the Macdonald operator* $H_N(q,t)$ *and the kernel function* $\Pi_{MN}(q,t)(x,y)$ *satisfy the following relation.*

$$H_N(q,t)_x\Pi_{NN}(q,t)(x,y) = H_N(q,t)_y\Pi_{NN}(q,t)(x,y).$$

For the free field realization of the Macdonald operator, let us consider to rewrite the kernel function $\Pi_{MN}(q,t)(x,y)$ as the following form:

$$\Pi_{MN}(q,t)(x,y) = \prod_{\substack{1\le i\le M\\ 1\le j\le N}} \exp\left(\sum_{n>0}\frac{1-t^n}{1-q^n}\frac{(x_iy_j)^n}{n}\right).$$

Next we define the $\mathbb{K}$-algebra $\mathcal{B}_\mathbb{K}$ of boson to be generated by $\{a_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ and the following relation:

$$[a_m,a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0} \quad (m,n\in\mathbb{Z}\setminus\{0\}). \tag{2.1}$$

We set the normal ordering $:\bullet:$ as

$$:a_ma_n: := \begin{cases} a_ma_n & (m<n),\\ a_na_m & (m\ge n).\end{cases}$$

Let $|0\rangle$ be the vacuum vector which satisfies $a_n|0\rangle = 0$ $(n>0)$. For a partition λ, we set $a_{-\lambda} := a_{-\lambda_1}\cdots a_{-\lambda_{\ell(\lambda)}}$ and define the boson Fock space $\mathcal{F}$ as the left $\mathcal{B}_\mathbb{K}$-module:

$$\mathcal{F} := \mathrm{span}_\mathbb{K}\left\{a_{-\lambda}|0\rangle \mid \lambda\in\mathcal{P}\right\}.$$

Let $\langle 0|$ be the dual vacuum vector which satisfies the condition $\langle 0|a_n = 0$ $(n<0)$ and define the dual boson Fock space $\mathcal{F}^*$ as the right $\mathcal{B}_\mathbb{K}$-module:

$$\mathcal{F}^* := \mathrm{span}_\mathbb{K}\left\{\langle 0|a_\lambda \mid \lambda\in\mathcal{P}\right\} \quad (a_\lambda := a_{\lambda_1}\cdots a_{\lambda_{\ell(\lambda)}}).$$

For a partition λ, we define symbols $n_\lambda(a)$, z_λ and $z_\lambda(q,t)$ by

$$n_\lambda(a) := \sharp\{i \,|\, \lambda_i = a\}, \quad z_\lambda := \prod_{a \geq 1} a^{n_\lambda(a)} n_\lambda(a)!, \quad z_\lambda(q,t) := z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

Let us define a bilinear form $\langle \bullet | \bullet \rangle : \mathcal{F}^* \times \mathcal{F} \to \mathbb{K}$ by the following condition.

$$\langle 0 | a_\lambda a_{-\mu} | 0 \rangle = \delta_{\lambda\mu} z_\lambda(q,t).$$

The answer why we need the boson (2.1) is in the following proposition.

Proposition 2.2 (**Reproduction of the kernel function** $\Pi_{MN}(q,t)(x,y)$**).** *Let us define boson operators* $\phi(z) : \mathcal{F} \to \mathcal{F}[[z,z^{-1}]]$ *and* $\phi^*(z) : \mathcal{F}^* \to \mathcal{F}^*[[z,z^{-1}]]$ *as follows.*

$$\phi(z) := \exp\left(\sum_{n>0} \frac{1-t^n}{1-q^n} a_{-n} \frac{z^n}{n} \right), \quad \phi^*(z) := \exp\left(\sum_{n>0} \frac{1-t^n}{1-q^n} a_n \frac{z^n}{n} \right).$$

We use the symbols as $\phi_N(x) := \prod_{j=1}^N \phi(x_j)$, $\phi_N^*(x) := \prod_{j=1}^N \phi^*(x_j)$ $(N \in \mathbb{Z}_{>0})$. *Then the kernel function* $\Pi_{MN}(q,t)(x,y)$ *is reproduced from the operators* $\phi_N^*(x)$ *and* $\phi_N(y)$ *as follows.*

$$\langle 0 | \phi_M^*(x) \phi_N(y) | 0 \rangle = \Pi_{MN}(q,t)(x,y).$$

For boson operators which take the following forms as

$$X(z) = \exp\left(\sum_{n<0} X_n^- a_n z^{-n} \right) \exp\left(\sum_{n>0} X_n^+ a_n z^{-n} \right) \in \mathrm{End}_{\mathbb{K}}(\mathcal{F})[[z,z^{-1}]] \quad (X_n^\pm \in \mathbb{K}),$$

we use the notations $(X(z))_\pm$ defined by

$$(X(z))_+ := \exp\left(\sum_{n>0} X_n^+ a_n z^{-n} \right), \quad (X(z))_- := \exp\left(\sum_{n<0} X_n^- a_n z^{-n} \right).$$

For the free field realization of the Macdonald operator, we prepare the following boson operators $\eta(z)$, $\xi(z)$.

Proposition 2.3 (**[FHHSY]**)**.** *Define the operator* $\eta(z) : \mathcal{F} \to \mathcal{F}[[z,z^{-1}]]$ *as follows.*

$$\eta(z) := :\exp\left(-\sum_{n\neq 0} (1-t^n) a_n \frac{z^{-n}}{n} \right): .$$

Then $\eta(z)$ *satisfies the following.*

i) $(\eta(tz))_- \phi(z) = \phi(qz), \quad \phi^*(z)(\eta(z^{-1}))_+ = \phi^*(qz).$

ii) $\eta(z)\eta(w) = \dfrac{(1-w/z)(1-qt^{-1}w/z)}{(1-qw/z)(1-t^{-1}w/z)} :\eta(z)\eta(w): \quad (|w/z| < 1).$

Proposition 2.4 (**[FHHSY]**). *Define the operator* $\xi(z) : \mathcal{F} \to \mathcal{F}[[z, z^{-1}]]$ *as follows.*

$$\xi(z) :=: \exp\left(\sum_{n \neq 0} (1 - t^n) \gamma^{|n|} a_n \frac{z^{-n}}{n} \right) : \quad (\gamma := (qt^{-1})^{-1/2}).$$

Then $\xi(z)$ *satisfies the following.*

i) $(\xi(\gamma z))_- \phi(z) = \phi(q^{-1} z), \quad \phi^*(z)(\xi(t\gamma^{-1} z^{-1}))_+ = \phi^*(q^{-1} z).$

ii) $\xi(z)\xi(w) = \dfrac{(1 - w/z)(1 - q^{-1} tw/z)}{(1 - q^{-1} w/z)(1 - tw/z)} : \xi(z)\xi(w) : \quad \left(|w/z| < |qt^{-1}|\right).$

In the following $[f(z)]_1$ denotes the constant term of $f(z)$ in z. The operators $\eta(z)$, $\xi(z)$ reproduce the Macdonald operators in the following way.

Proposition 2.5 (**Free field realization of the Macdonald operator[FHHSY]**).
(1) The operator $\eta(z)$ *reproduces the Macdonald operator* $H_N(q, t)$ *as follows.*

$$[\eta(z)]_1 \phi_N(x)|0\rangle = t^{-N} \{(t - 1) H_N(q, t) + 1\} \phi_N(x)|0\rangle.$$

(2) The operator $\xi(z)$ *reproduces the Macdonald operator* $H_N(q^{-1}, t^{-1})$ *as follows.*

$$[\xi(z)]_1 \phi_N(x)|0\rangle = t^N \{(t^{-1} - 1) H_N(q^{-1}, t^{-1}) + 1\} \phi_N(x)|0\rangle.$$

We also have the dual version of Proposition 2.5.

Proposition 2.6 (**Dual version of Proposition 2.5**).
(1) The operator $\eta(z)$ *reproduces the Macdonald operator* $H_N(q, t)$ *as follows.*

$$\langle 0|\phi^*_N(x)[\eta(z)]_1 = t^{-N} \{(t - 1) H_N(q, t) + 1\} \langle 0|\phi^*_N(x).$$

(2) The operator $\xi(z)$ *reproduces the Macdonald operator* $H_N(q^{-1}, t^{-1})$ *as follows.*

$$\langle 0|\phi^*_N(x)[\xi(z)]_1 = t^N \{(t^{-1} - 1) H_N(q^{-1}, t^{-1}) + 1\} \langle 0|\phi^*_N(x).$$

By the free field realization of the Macdonald operator, we can show the functional equation of the kernel function $\Pi_{MN}(q, t)(x, y)$.

Proposition 2.7 (**Functional equation of the kernel function**). *The Macdonald operator* $H_N(q, t)$ *and the kernel function* $\Pi_{MN}(q, t)(x, y)$ *satisfy the following functional equation.*

$$\{H_M(q, t)_x - t^{M-N} H_N(q, t)_y\} \Pi_{MN}(q, t)(x, y) = \frac{1 - t^{M-N}}{1 - t} \Pi_{MN}(q, t)(x, y). \tag{2.2}$$

Here $H_M(q, t)_x$ *denotes the Macdonald operator which acts on functions of* $x_1, \cdots, x_M$.

Proof. The proof of Proposition 2.7 is very simple. By the free field realization of the Macdonald operator, we can calculate the matrix element $\langle 0|\phi_M(x)[\eta(z)]_1\phi_N(y)|0\rangle$ in different two ways as

$$\begin{aligned}&\langle 0|\phi_M(x)[\eta(z)]_1\phi_N(y)|0\rangle\\ &=t^{-M}\{(t-1)H_M(q,t)_x+1\}\Pi_{MN}(q,t)(x,y)\\ &=t^{-N}\{(t-1)H_N(q,t)_y+1\}\Pi_{MN}(q,t)(x,y).\end{aligned}$$

Consequently we have the relation (2.2). □ □

§ 2.2. Ding-Iohara-Miki algebra $\mathcal{U}(q,t)$

The Ding-Iohara-Miki algebra $\mathcal{U}(q,t)$ is defined as follows.

Definition 2.8 (**Ding-Iohara-Miki algebra** $\mathcal{U}(q,t)$ [**Miki**][**FHHSY**]**).** Let $f^{\pm}(x)$ be the polynomials in $x\in\mathbb{C}^{\times}$ and $g(x)$ be the meromorphic function in $x\in\mathbb{C}^{\times}$ defined by

$$\begin{aligned}&f^+(x):=(1-qx)(1-t^{-1}x)(1-q^{-1}tx),\quad f^-(x):=(1-q^{-1}x)(1-tx)(1-qt^{-1}x),\\ &g(x):=\frac{f^+(x)}{f^-(x)}=\frac{(1-qx)(1-t^{-1}x)(1-q^{-1}tx)}{(1-q^{-1}x)(1-tx)(1-qt^{-1}x)}.\end{aligned}$$

Let C be a central, invertible element and $x^{\pm}(z):=\sum_{n\in\mathbb{Z}}x_n^{\pm}z^{-n}$, $\psi^{\pm}(z):=\sum_{\pm n\geq 0}\psi_n^{\pm}z^{-n}$ be currents satisfying the relations:

$$\begin{gathered}[\psi^{\pm}(z),\psi^{\pm}(w)]=0,\quad \psi^+(z)\psi^-(w)=\frac{g(Cw/z)}{g(C^{-1}w/z)}\psi^-(w)\psi^+(z),\\ \psi^+(z)x^{\pm}(w)=g\Big(C^{\mp\frac{1}{2}}\frac{w}{z}\Big)^{\mp 1}x^{\pm}(w)\psi^+(z),\quad \psi^-(z)x^{\pm}(w)=g\Big(C^{\mp\frac{1}{2}}\frac{z}{w}\Big)^{\pm 1}x^{\pm}(w)\psi^-(z),\\ -(z/w)^3f^{\pm}(w/z)x^{\pm}(z)x^{\pm}(w)=f^{\pm}(z/w)x^{\pm}(w)x^{\pm}(z),\\ [x^+(z),x^-(w)]=\frac{(1-q)(1-t^{-1})}{1-qt^{-1}}\Big\{\delta\Big(C\frac{w}{z}\Big)\psi^+(C^{1/2}w)-\delta\Big(C^{-1}\frac{w}{z}\Big)\psi^-(C^{-1/2}w)\Big\}.\end{gathered}$$

We define the Ding-Iohara-Miki algebra $\mathcal{U}(q,t)$ to be an associative $\mathbb{K}$-algebra generated by $\{x_n^{\pm}\}_{n\in\mathbb{Z}}$, $\{\psi_n^{\pm}\}_{\pm n\geq 0}$, and C with the above relations.

Remark 2.9. It can be checked that the Ding-Iohara-Miki algebra has the coproduct $\Delta:\mathcal{U}(q,t)\to\mathcal{U}(q,t)\otimes\mathcal{U}(q,t)$ defined as follows [FHHSY]:

$$\begin{aligned}&\Delta(C^{\pm 1}):=C^{\pm 1}\otimes C^{\pm 1},\quad \Delta(\psi^{\pm}(z)):=\psi^{\pm}(C_{(2)}^{\pm 1/2}z)\otimes\psi^{\pm}(C_{(1)}^{\mp 1/2}z),\\ &\Delta(x^+(z)):=x^+(z)\otimes 1+\psi^-(C_{(1)}^{1/2}z)\otimes x^+(C_{(1)}z),\\ &\Delta(x^-(z)):=x^-(C_{(2)}z)\otimes\psi^+(C_{(2)}^{1/2}z)+1\otimes x^-(z).\end{aligned}$$

Here we define $C_{(1)}:=C\otimes 1$, $C_{(2)}:=1\otimes C$.

By Wick's theorem, we have the following proposition.

Proposition 2.10 (Free field realization of $\mathcal{U}(q,t)$ [FHHSY]). *Set $\gamma := (qt^{-1})^{-1/2}$ and define operators $\varphi^{\pm}(z) : \mathcal{F} \to \mathcal{F}[[z,z^{-1}]]$ as follows:*

$$\varphi^{+}(z) := :\eta(\gamma^{1/2}z)\xi(\gamma^{-1/2}z):, \quad \varphi^{-}(z) := :\eta(\gamma^{-1/2}z)\xi(\gamma^{1/2}z):.$$

Then the map

$$C \mapsto \gamma, \quad x^{+}(z) \mapsto \eta(z), \quad x^{-}(z) \mapsto \xi(z), \quad \psi^{\pm}(z) \mapsto \varphi^{\pm}(z)$$

gives a representation of the Ding-Iohara-Miki algebra $\mathcal{U}(q,t)$.

§ 2.3. Free field realization of the elliptic Ruijsenaars operator

In this subsection, we start discussions about the elliptic theory side. In the following, for parameters q, $t \in \mathbb{C}^{\times}$ assume $|q|<1$, $|t^{-1}|<1$. We regard p as a formal variable. For a fixed $N \in \mathbb{Z}_{>0}$, we define power sum polynomials by $p_n(x) := \sum_{i=1}^{N} x_i^n$, $p_n(\overline{x}) := \sum_{i=1}^{N} x_i^{-n}$ $(n \in \mathbb{Z}_{>0})$. For a partition λ, set $p_\lambda(x) := \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}(x)$, $p_\lambda(\overline{x}) := \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}(\overline{x})$.

Definition 2.11. Define the space $\Lambda_N(q,t,p) \subset \mathbb{K}[x_i^{\pm 1} \mid 1 \le i \le N]^{\mathfrak{S}_N}[[p]]$ $(N \in \mathbb{Z}_{>0})$ by

$$\Lambda_N(q,t,p) := \left\{ \sum_{d \ge 0} \sum_{\substack{\lambda,\, \mu \in \mathcal{P} \\ 0 \le |\mu| \le d}} c^{d}_{\lambda\mu} p_\lambda(x) p_\mu(\overline{x}) p^d \;\middle|\; \begin{array}{c} \text{For each } d \ge 0 \text{ and } \mu \in \mathcal{P}, \\ \sharp\{\lambda \in \mathcal{P} \mid c^{d}_{\lambda\mu} \neq 0\} < \infty \ (c^{d}_{\lambda\mu} \in \mathbb{K}) \end{array} \right\}.$$

Definition 2.12 (Elliptic Ruijsenaars operator $H_N(q,t,p)$ [R1]). The elliptic Ruijsenaars operator $H_N(q,t,p) \in \mathrm{End}_{\mathbb{K}[[p]]}(\Lambda_N(q,t,p))$ $(N \in \mathbb{Z}_{>0})$ is defined by

$$H_N(q,t,p) := \sum_{i=1}^{N} \prod_{j \neq i} \frac{\Theta_p(tx_i/x_j)}{\Theta_p(x_i/x_j)} T_{q,x_i}.$$

The kernel function for the elliptic Ruijsenaars operator has been introduced by Ruijsenaars [R2] and Komori-Noumi-Shiraishi [KNS].

Proposition 2.13 (Kernel function for $H_N(q,t,p)$ [R2][KNS]). *Define the function $\Pi_{MN}(q,t,p)(x,y)$ $(M,N \in \mathbb{Z}_{>0})$ as follows.*

$$\Pi_{MN}(q,t,p)(x,y) := \prod_{\substack{1 \le i \le M \\ 1 \le j \le N}} \frac{\Gamma_{q,p}(x_i y_j)}{\Gamma_{q,p}(t x_i y_j)}.$$

The symbol $H_N(q,t,p)_x$ denotes the elliptic Ruijsenaars operator which acts on functions of $x_1, \dots, x_N$. Then we have the following relation.

$$H_N(q,t,p)_x \Pi_{NN}(q,t,p)(x,y) = H_N(q,t,p)_y \Pi_{NN}(q,t,p)(x,y).$$

Let us consider to construct the free field realization of the elliptic Ruijsenaars operator. Then we pay attention to the fact again that the free field realization of the Macdonald operator is based on the kernel function for the operator. Hence also in the elliptic case, we would expect that the kernel function for the elliptic Ruijsenaars operator has important informations of the free field realization of the operator. Keeping the idea in mind, we rewrite the kernel function $\Pi_{MN}(q,t,p)(x,y)$ as the following form.

$$\begin{aligned}&\Pi_{MN}(q,t,p)(x,y)\\&=\prod_{\substack{1\le i\le M\\1\le j\le N}}\exp\left(\sum_{n>0}\frac{(qt^{-1}p)^n(1-t^n)}{(1-q^n)(1-p^n)}\frac{(x_iy_j)^{-n}}{n}\right)\exp\left(\sum_{n>0}\frac{1-t^n}{(1-q^n)(1-p^n)}\frac{(x_iy_j)^{n}}{n}\right).\end{aligned}$$

To reproduce the kernel function $\Pi_{MN}(q,t,p)(x,y)$ from OPE of boson operators, we prepare an algebra of boson $\mathcal{B}_{\mathbb{K}((p))}$ to be generated by $\{a_n\}_{n\in\mathbb{Z}\setminus\{0\}}$, $\{\overline{a}_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ and the following relations:

$$\begin{aligned}&[a_m,a_n]=m\frac{(1-q^{|m|})(1-p^{|m|})}{1-t^{|m|}}\delta_{m+n,0},\quad [\overline{a}_m,\overline{a}_n]=m\frac{(1-q^{|m|})(1-p^{|m|})}{(qt^{-1}p)^{|m|}(1-t^{|m|})}\delta_{m+n,0},\\&[a_m,\overline{a}_n]=0\quad (m,n\in\mathbb{Z}\setminus\{0\}).\end{aligned}$$

We denote the $\mathbb{K}[[p]]$-subring of $\mathcal{B}_{\mathbb{K}((p))}$ generated by $\{a_n\}_{n\in\mathbb{Z}\setminus\{0\}}$, $\{\overline{a}_np^{|n|}\}_{n\in\mathbb{Z}\setminus\{0\}}$ by $\mathcal{B}_{\mathbb{K}[[p]]}$.

We define the normal ordering $:\bullet:$ as usual:

$$:a_ma_n: := \begin{cases}a_ma_n & (m<n),\\ a_na_m & (m\ge n),\end{cases}\qquad :\overline{a}_m\overline{a}_n: := \begin{cases}\overline{a}_m\overline{a}_n & (m<n),\\ \overline{a}_n\overline{a}_m & (m\ge n).\end{cases}$$

Let $|0\rangle$ be the vacuum vector which satisfies the condition $a_n|0\rangle=\overline{a}_n|0\rangle=0$ $(n>0)$ and set the boson Fock space $\mathcal{F}$ as the left $\mathcal{B}_{\mathbb{K}[[p]]}$-module.

$$\mathcal{F}:=\left\{\sum_{d\ge 0}\sum_{\substack{\lambda,\,\mu\in\mathcal{P}\\0\le|\mu|\le d}}c^d_{\lambda\mu}a_{-\lambda}a_{-\mu}|0\rangle p^d\;\middle|\;\begin{array}{c}\text{For each } d\ge 0 \text{ and } \mu\in\mathcal{P},\\ \sharp\{\lambda\in\mathcal{P}\,|\,c^d_{\lambda\mu}\neq 0\}<\infty\ (c^d_{\lambda\mu}\in\mathbb{K})\end{array}\right\}.$$

Let $\langle 0|$ be the dual vacuum vector satisfying the condition $\langle 0|a_n=\langle 0|\overline{a}_n=0$ $(n<0)$. We define the dual boson Fock space $\mathcal{F}^*$ as the right $\mathcal{B}_{\mathbb{K}[[p]]}$-module:

$$\mathcal{F}^*:=\left\{\sum_{d\ge 0}\sum_{\substack{\lambda,\,\mu\in\mathcal{P}\\0\le|\mu|\le d}}c^d_{\lambda\mu}\langle 0|a_{\lambda}a_{\mu}p^d\;\middle|\;\begin{array}{c}\text{For each } d\ge 0 \text{ and } \mu\in\mathcal{P},\\ \sharp\{\lambda\in\mathcal{P}\,|\,c^d_{\lambda\mu}\neq 0\}<\infty\ (c^d_{\lambda\mu}\in\mathbb{K})\end{array}\right\}.$$

For a partition λ, set $z_\lambda(q,t,p) \in \mathbb{K}[[p]]$ as follows.

$$z_\lambda(q,t,p) := z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{(1-q^{\lambda_i})(1-p^{\lambda_i})}{1-t^{\lambda_i}}.$$

We define a bilinear form $\langle \bullet | \bullet \rangle : \mathcal{F}^* \times \mathcal{F} \to \mathbb{K}[[p]]$ by the following condition.

$$\langle 0| a_{\lambda_1} \overline{a}_{\lambda_2} p^{|\lambda_2|} a_{-\mu_1} \overline{a}_{-\mu_2} p^{|\mu_2|} |0\rangle = \delta_{\lambda_1 \mu_1} \delta_{\lambda_2 \mu_2} z_{\lambda_1}(q,t,p) z_{\lambda_2}(q,t,p) (qt^{-1})^{-|\lambda_2|} p^{|\lambda_2|}.$$

Then we can reproduce the kernel function $\Pi_{MN}(q,t,p)(x,y)$ as follows.

Proposition 2.14 (**Reproduction of the kernel function** $\Pi_{MN}(q,t,p)(x,y)$)**.** *Define* $\phi(p;z) : \mathcal{F} \to \mathcal{F}[[z,z^{-1}]]$ *and* $\phi^*(p;z) : \mathcal{F}^* \to \mathcal{F}^*[[z,z^{-1}]]$ *as follows.*

$$\phi(p;z) := \exp\left(\sum_{n>0} \frac{(qt^{-1}p)^n(1-t^n)}{(1-q^n)(1-p^n)} \overline{a}_{-n} \frac{z^{-n}}{n} \right) \exp\left(\sum_{n>0} \frac{1-t^n}{(1-q^n)(1-p^n)} a_{-n} \frac{z^n}{n} \right),$$

$$\phi^*(p;z) := \exp\left(\sum_{n>0} \frac{(qt^{-1}p)^n(1-t^n)}{(1-q^n)(1-p^n)} \overline{a}_{n} \frac{z^{-n}}{n} \right) \exp\left(\sum_{n>0} \frac{1-t^n}{(1-q^n)(1-p^n)} a_{n} \frac{z^n}{n} \right).$$

We use the notations $\phi_N(p;x) := \prod_{j=1}^N \phi(p;x_j)$, $\phi_N^*(p;x) := \prod_{j=1}^N \phi^*(p;x_j)$ $(N \in \mathbb{Z}_{>0})$. *Then the kernel function* $\Pi_{MN}(q,t,p)(x,y)$ *is reproduced from* $\phi_N^*(p;x)$ *and* $\phi_N(p;y)$ *as follows.*

$$\langle 0| \phi_M^*(p;x) \phi_N(p;y) |0\rangle = \Pi_{MN}(q,t,p)(x,y).$$

By attempting to construct elliptic deformations of operators $\eta(z)$, $\xi(z)$, we can find the following operators $\eta(p;z)$ and $\xi(p;z)$.

Proposition 2.15 (**Elliptic current** $\eta(p;z)$ [**Sa1**])**.** *Let* $\eta(p;z) : \mathcal{F} \to \mathcal{F}[[z,z^{-1}]]$ *be the boson operator defined as follows.*

$$\eta(p;z) := \mathbf{:} \exp\left(-\sum_{n\neq 0} \frac{1-t^{-n}}{1-p^{|n|}} p^{|n|} \overline{a}_n \frac{z^n}{n} \right) \exp\left(-\sum_{n\neq 0} \frac{1-t^n}{1-p^{|n|}} a_n \frac{z^{-n}}{n} \right) \mathbf{:}\,.$$

Then $\eta(p;z)$ *satisfies the following.*

i) $(\eta(p;tz))_- \phi(p;z) = \phi(p;qz), \quad \phi^*(p;z)(\eta(p;z^{-1}))_+ = \phi^*(p;qz).$

ii) $\eta(p;z)\eta(p;w) = \dfrac{\Theta_p(w/z)\Theta_p(qt^{-1}w/z)}{\Theta_p(qw/z)\Theta_p(t^{-1}w/z)} \mathbf{:} \eta(p;z)\eta(p;w) \mathbf{:} \quad (|w/z|<1)\,.$

Proposition 2.16 (**Elliptic current** $\xi(p;z)$ [**Sa1**])**.** *Let* $\xi(p;z) : \mathcal{F} \to \mathcal{F}[[z,z^{-1}]]$ *be the boson operator defined as follows.*

$$\xi(p;z) := \mathbf{:} \exp\left(\sum_{n\neq 0} \frac{1-t^{-n}}{1-p^{|n|}} \gamma^{-|n|} p^{|n|} \overline{a}_n \frac{z^n}{n} \right) \exp\left(\sum_{n\neq 0} \frac{1-t^n}{1-p^{|n|}} \gamma^{|n|} a_n \frac{z^{-n}}{n} \right) \mathbf{:}\,.$$

Then $\xi(p;z)$ *satisfies the following.*

i) $(\xi(p;\gamma z))_- \phi(p;z) = \phi(p;q^{-1}z), \quad \phi^*(p;z)(\xi(p;t\gamma^{-1}z^{-1}))_+ = \phi^*(p;q^{-1}z).$

ii) $\xi(p;z)\xi(p;w) = \dfrac{\Theta_p(w/z)\Theta_p(q^{-1}tw/z)}{\Theta_p(q^{-1}w/z)\Theta_p(tw/z)} :\xi(p;z)\xi(p;w): \quad \left(|w/z| < |qt^{-1}|\right).$

By using operators $\eta(p;z)$, $\xi(p;z)$ we can reproduce the elliptic Ruijsenaars operator as follows.

Theorem 2.17 (**Free field realization of elliptic Ruijsenaars operators**[**Sa1**]). *(1) The elliptic Ruijsenaars operator* $H_N(q,t,p)$ *is reproduced by the operator* $\eta(p;z)$ *as follows.*

$$\begin{aligned}&[\eta(p;z) - t^{-N}(\eta(p;z))_-(\eta(p;p^{-1}z))_+]_1 \phi_N(p;x)|0\rangle \\ =& \frac{t^{-N+1}\Theta_p(t^{-1})}{(p;p)_\infty^3} H_N(q,t,p)\phi_N(p;x)|0\rangle.\end{aligned}$$

(2) The elliptic Ruijsenaars operator $H_N(q^{-1},t^{-1},p)$ *is reproduced by the operator* $\xi(p;z)$ *as follows.*

$$\begin{aligned}&[\xi(p;z) - t^{N}(\xi(p;z))_-(\xi(p;p^{-1}z))_+]_1 \phi_N(p;x)|0\rangle \\ =& \frac{t^{N-1}\Theta_p(t)}{(p;p)_\infty^3} H_N(q^{-1},t^{-1},p)\phi_N(p;x)|0\rangle.\end{aligned}$$

Theorem 2.17 is also stated as follows. First we set zero mode generators a_0, Q which satisfy the relation:

$$[a_0, Q] = 1, \quad [a_n, a_0] = [\overline{a}_n, a_0] = 0, \quad [a_n, Q] = [\overline{a}_n, Q] = 0 \quad (n \in \mathbb{Z} \setminus \{0\}).$$

We also set the condition $a_0|0\rangle = 0$. We define $|\alpha\rangle := e^{\alpha Q}|0\rangle$ $(\alpha \in \mathbb{K}[[p]])$. Then we have $a_0|\alpha\rangle = \alpha|\alpha\rangle$. We also set $\mathcal{F}_\alpha$ $(\alpha \in \mathbb{K}[[p]])$ by

$$\mathcal{F}_\alpha := \left\{ \sum_{d \geq 0} \sum_{\substack{\lambda,\mu \in \mathcal{P} \\ 0 \leq |\mu| \leq d}} c_{\lambda\mu}^d a_{-\lambda} a_{-\mu} |\alpha\rangle p^d \,\middle|\, \begin{array}{c} \text{For each } d \geq 0 \text{ and } \mu \in \mathcal{P}, \\ \sharp\{\lambda \in \mathcal{P} \mid c_{\lambda\mu}^d \neq 0\} < \infty \ (c_{\lambda\mu}^d \in \mathbb{K}) \end{array} \right\}.$$

Theorem 2.18 ([**Sa1**]). *Set the operators* $\widetilde{\eta}(p;z)$, $\widetilde{\xi}(p;z)$ *by*

$$\widetilde{\eta}(p;z) := (\eta(p;z))_-(\eta(p;p^{-1}z))_+, \quad \widetilde{\xi}(p;z) := (\xi(p;z))_-(\xi(p;p^{-1}z))_+.$$

Using these symbols we define operators $E(p;z)$, $F(p;z)$ *as follows:*

$$E(p;z) := \eta(p;z) - \widetilde{\eta}(p;z)t^{-a_0}, \quad F(p;z) := \xi(p;z) - \widetilde{\xi}(p;z)t^{a_0}. \tag{2.3}$$

Then the elliptic Ruijsenaars operators $H_N(q,t,p)$, $H_N(q^{-1},t^{-1},p)$ are reproduced by the operators $E(p;z)$, $F(p;z)$ as follows.

$$[E(p;z)]_1\phi_N(p;x)|N\rangle = \frac{t^{-N+1}\Theta_p(t^{-1})}{(p;p)_\infty^3} H_N(q,t,p)\phi_N(p;x)|N\rangle,$$
$$[F(p;z)]_1\phi_N(p;x)|N\rangle = \frac{t^{N-1}\Theta_p(t)}{(p;p)_\infty^3} H_N(q^{-1},t^{-1},p)\phi_N(p;x)|N\rangle.$$

The dual versions of Theorem 2.17, 2.18 are also available. Set the condition as $\langle 0|a_0 = 0$. For $\alpha \in \mathbb{K}[[p]]$, set $\langle\alpha| := \langle 0|e^{-\alpha Q}$. Then we have $\langle\alpha|a_0 = \alpha\langle\alpha|$. We also set $\mathcal{F}_\alpha^*$ ($\alpha \in \mathbb{K}[[p]]$) by

$$\mathcal{F}_\alpha^* := \left\{ \sum_{d\geq 0} \sum_{\substack{\lambda,\,\mu\in\mathcal{P} \\ 0\leq|\mu|\leq d}} c_{\lambda\mu}^d \langle\alpha|a_{-\lambda}a_{-\mu}p^d \;\middle|\; \begin{array}{c} \text{For each } d\geq 0 \text{ and } \mu\in\mathcal{P}, \\ \sharp\{\lambda\in\mathcal{P}\,|\,c_{\lambda\mu}^d \neq 0\} < \infty\ (c_{\lambda\mu}^d \in \mathbb{K}) \end{array} \right\}.$$

Theorem 2.19 (**Dual versions of Theorem 2.17, 2.18**). *(1) The operators $\eta(p;z)$, $\xi(p;z)$ reproduce the elliptic Ruijsenaars operators $H_N(q,t,p)$, $H_N(q^{-1},t^{-1},p)$ as follows.*

$$\begin{aligned}&\langle 0|\phi_N^*(p;x)[\eta(p;z) - t^{-N}(\eta(p;z))_-(\eta(p;p^{-1}z))_+]_1 \\ &= \frac{t^{-N+1}\Theta_p(t^{-1})}{(p;p)_\infty^3} H_N(q,t,p)\langle 0|\phi_N^*(p;x),\end{aligned}$$

$$\begin{aligned}&\langle 0|\phi_N^*(p;x)[\xi(p;z) - t^{N}(\xi(p;z))_-(\xi(p;p^{-1}z))_+]_1 \\ &= \frac{t^{N-1}\Theta_p(t)}{(p;p)_\infty^3} H_N(q^{-1},t^{-1},p)\langle 0|\phi_N^*(p;x).\end{aligned}$$

(2) The operators $E(p;z)$, $F(p;z)$ reproduce the elliptic Ruijsenaars operators $H_N(q,t,p)$, $H_N(q^{-1},t^{-1},p)$ as follows.

$$\langle N|\phi_N^*(p;x)[E(p;z)]_1 = \frac{t^{-N+1}\Theta_p(t^{-1})}{(p;p)_\infty^3} H_N(q,t,p)\langle N|\phi_N^*(p;x),$$
$$\langle N|\phi_N^*(p;x)[F(p;z)]_1 = \frac{t^{N-1}\Theta_p(t)}{(p;p)_\infty^3} H_N(q^{-1},t^{-1},p)\langle N|\phi_N^*(p;x).$$

By using the free field realization of the elliptic Ruijsenaars operator, we obtain the functional equation of the elliptic kernel function $\Pi_{MN}(q,t,p)(x,y)$ as follows.

Theorem 2.20 (**Functional equation of the elliptic kernel function [Sa2]**).

We define $C_{MN}(p;x,y)$ *as*

$$C_{MN}(p;x,y) := \frac{\langle 0|\phi^*_M(p;x)[(\eta(p;z))_-(\eta(p;p^{-1}z))_+]_1\phi_N(p;y)|0\rangle}{\Pi_{MN}(q,t,p)(x,y)}$$

$$= \oint_C \frac{dz}{2\pi i z}\prod_{i=1}^{M}\frac{\Theta_p(t^{-1}x_i z)}{\Theta_p(x_i z)}\prod_{j=1}^{N}\frac{\Theta_p(z/y_j)}{\Theta_p(t^{-1}z/y_j)}, \tag{2.4}$$

where the integral contour C *is chosen by*

$$C: |z| < \min\{|x_1|^{-1},\dots,|x_M|^{-1},|y_1|,\dots,|y_N|\}.$$

For the elliptic Ruijsenaars operator and the kernel function $\Pi_{MN}(q,t,p)(x,y)$*, we have the following functional equation.*

$$\{H_M(q,t,p)_x - t^{M-N}H_N(q,t,p)_y\}\Pi_{MN}(q,t,p)(x,y)$$

$$= \frac{(1-t^{M-N})(p;p)^3_\infty}{\Theta_p(t)}C_{MN}(p;x,y)\Pi_{MN}(q,t,p)(x,y). \tag{2.5}$$

Here the symbol $H_M(q,t,p)_x$ *denotes the elliptic Ruijsenaars operator which acts on functions of* $x_1,\cdots,x_M$*.*

Proof. The proof is straightforward. What we have to do is to calculate the matrix element $\langle 0|\phi^*_M(p;x)[\eta(p;z)]_1\phi_N(p;y)|0\rangle$ by Theorem 2.17, 2.19 in different two ways as follows:

$$\begin{aligned}&\langle 0|\phi^*_M(p;x)[\eta(p;z)]_1\phi_N(p;y)|0\rangle\\ &= \frac{t^{-M+1}\Theta_p(t^{-1})}{(p;p)^3_\infty}H_M(q,t,p)_x\Pi_{MN}(q,t,p)(x,y) + t^{-M}C_{MN}(p;x,y)\Pi_{MN}(q,t,p)(x,y)\\ &= \frac{t^{-N+1}\Theta_p(t^{-1})}{(p;p)^3_\infty}H_N(q,t,p)_y\Pi_{MN}(q,t,p)(x,y) + t^{-N}C_{MN}(p;x,y)\Pi_{MN}(q,t,p)(x,y).\end{aligned}$$

Therefore we obtain Theorem 2.20. □ □

Remark 2.21. We can check the following:

$$\begin{aligned}C_{MN}(p;x,y) &= \oint_C \frac{dz}{2\pi i z}\prod_{i=1}^{M}\frac{\Theta_p(t^{-1}x_i z)}{\Theta_p(x_i z)}\prod_{j=1}^{N}\frac{\Theta_p(z/y_j)}{\Theta_p(t^{-1}z/y_j)}\\ &\xrightarrow[p\to 0]{} \oint_C \frac{dz}{2\pi i z}\prod_{i=1}^{M}\frac{1-t^{-1}x_i z}{1-x_i z}\prod_{j=1}^{N}\frac{1-z/y_j}{1-t^{-1}z/y_j} = 1.\end{aligned}$$

Hence by taking the limit $p \to 0$ the formula (2.5) reduces to the equation (2.2).

§ 2.4. Commutator of operators $E(p;z)$, $F(p;z)$

The commutator $[\eta(p;z),\xi(p;w)]$ is given by

$$[\eta(p;z),\xi(p;w)]=\frac{\Theta_p(q)\Theta_p(t^{-1})}{(p;p)_\infty^3\Theta_p(qt^{-1})}\left\{\delta\Big(\gamma\frac{w}{z}\Big)\varphi^+(p;\gamma^{1/2}w)-\delta\Big(\gamma^{-1}\frac{w}{z}\Big)\varphi^-(p;\gamma^{-1/2}w)\right\}.$$

Since $[\varphi^+(p;z)]_1\neq[\varphi^-(p;z)]_1$, we have $[[\eta(p;z)]_1,[\xi(p;w)]_1]\neq 0$. On the other hand, the operators $E(p;z)$, $F(p;z)$ defined in (2.3) satisfy the following.

Proposition 2.22 (**[Sa1]**). *(1) For operators $E(p;z)$, $F(p;z)$ we have*

$$E(p;z)E(p;w)=g_p\Big(\frac{z}{w}\Big)E(p;w)E(p;z),$$
$$F(p;z)F(p;w)=g_p\Big(\frac{z}{w}\Big)^{-1}F(p;w)F(p;z).$$

(2) The commutator of operators $E(p;z)$, $F(p;z)$ takes the form as

$$[E(p;z),F(p;w)]=\frac{\Theta_p(q)\Theta_p(t^{-1})}{(p;p)_\infty^3\Theta_p(qt^{-1})}\delta\Big(\gamma\frac{w}{z}\Big)\{\varphi^+(p;\gamma^{1/2}w)-\varphi^+(p;\gamma^{1/2}p^{-1}w)\}. \tag{2.6}$$

From the relation (2.6) we have $[[E(p;z)]_1,[F(p;w)]_1]=0$. This corresponds to the commutativity of the elliptic Ruijsenaars operators $[H_N(q,t,p),H_N(q^{-1},t^{-1},p)]=0$. The above proposition is needed in Section 3.

§ 2.5. Elliptic Ding-Iohara-Miki algebra $\mathcal{U}(q,t,p)$

The elliptic Ding-Iohara-Miki algebra is an elliptic analog of the Ding-Iohara-Miki algebra introduced by the author.

Definition 2.23 (**Elliptic Ding-Iohara-Miki algebra $\mathcal{U}(q,t,p)$ [Sa1]**). Set the $\mathbb{K}[[p]]$-valued holomorphic functions $f^{\pm}(p;x)$ in $x\in\mathbb{C}^\times$ and the $\mathbb{K}[[p]]$-valued meromorphic function $g_p(x)$ in $x\in\mathbb{C}^\times$ as

$$f^+(p;x):=\Theta_p(qx)\Theta_p(t^{-1}x)\Theta_p(q^{-1}tx),\quad f^-(p;x):=\Theta_p(q^{-1}x)\Theta_p(tx)\Theta_p(qt^{-1}x),$$
$$g_p(x):=\frac{f^+(p;x)}{f^-(p;x)}=\frac{\Theta_p(qx)\Theta_p(t^{-1}x)\Theta_p(q^{-1}tx)}{\Theta_p(q^{-1}x)\Theta_p(tx)\Theta_p(qt^{-1}x)}.$$

Let $x^{\pm}(p;z)$, $\psi^{\pm}(p;z)$ be currents, i.e. generating functions of generators having the form

$$x^{\pm}(p;z)=\sum_{d\geq 0}x_d^{\pm}(z)p^d=\sum_{d\geq 0}\sum_{n\in\mathbb{Z}}x_d^{\pm}[n]z^{-n}p^d,$$
$$\psi^{\pm}(p;z)=\sum_{d\geq 0}\psi_d^{\pm}(z)p^d=\sum_{d\geq 0}\sum_{n\in\mathbb{Z}}\psi_d^{\pm}[n]z^{-n}p^d.$$

and C be a central, invertible element. We impose the following defining relations:

$$[\psi^{\pm}(p;z),\psi^{\pm}(p;w)]=0,\quad \psi^{+}(p;z)\psi^{-}(p;w)=\frac{g_p(Cw/z)}{g_p(C^{-1}w/z)}\psi^{-}(p;w)\psi^{+}(p;z),$$
$$\psi^{+}(p;z)x^{\pm}(p;w)=g_p\Big(C^{\mp\frac{1}{2}}\frac{w}{z}\Big)^{\mp 1}x^{\pm}(p;w)\psi^{+}(p;z),$$
$$\psi^{-}(p;z)x^{\pm}(p;w)=g_p\Big(C^{\mp\frac{1}{2}}\frac{z}{w}\Big)^{\pm 1}x^{\pm}(p;w)\psi^{-}(p;z),$$
$$-(z/w)^3f^{\pm}(p;w/z)x^{\pm}(p;z)x^{\pm}(p;w)=f^{\pm}(p;z/w)x^{\pm}(p;w)x^{\pm}(p;z),$$
$$[x^{+}(p;z),x^{-}(p;w)]=\frac{\Theta_p(q)\Theta_p(t^{-1})}{(p;p)_\infty^3\Theta_p(qt^{-1})}\Big\{\delta\Big(C\frac{w}{z}\Big)\psi^{+}(p;C^{\frac{1}{2}}w)-\delta\Big(C^{-1}\frac{w}{z}\Big)\psi^{-}(p;C^{-\frac{1}{2}}w)\Big\}.$$

We define the elliptic Ding-Iohara-Miki algebra $\mathcal{U}(q,t,p)$ to be an associative $\mathbb{K}[[p]]$-algebra generated by $\{x_d^{\pm}[n]\}_{n\in\mathbb{Z}}^{d\geq 0}$, $\{\psi_d^{\pm}[n]\}_{n\in\mathbb{Z}}^{d\geq 0}$ and C with the above relations.

Theorem 2.24 (Free field realization of $\mathcal{U}(q,t,p)$ [Sa1]). *Set $\gamma:=(qt^{-1})^{-1/2}$ and operators $\varphi^{\pm}(p;z):\mathcal{F}\to\mathcal{F}[[z,z^{-1}]]$ as*

$$\varphi^{+}(p;z):=:\eta(p;\gamma^{1/2}z)\xi(p;\gamma^{-1/2}z):,\quad \varphi^{-}(p;z):=:\eta(p;\gamma^{-1/2}z)\xi(p;\gamma^{1/2}z):.$$

Then the map

$$C\mapsto\gamma,\quad x^{+}(p;z)\mapsto\eta(p;z),\quad x^{-}(p;z)\mapsto\xi(p;z),\quad \psi^{\pm}(p;z)\mapsto\varphi^{\pm}(p;z)$$

gives a representation of the elliptic Ding-Iohara-Miki algebra $\mathcal{U}(q,t,p)$.

Remark 2.25. (1) By the definition, the trigonometric limit $p\to 0$ of the elliptic Ding-Iohara-Miki algebra $\mathcal{U}(q,t,p)$ is the Ding-Iohara-Miki algebra $\mathcal{U}(q,t)$.

(2) The elliptic Ding-Iohara-Miki algebra has a coproduct

$$\Delta:\mathcal{U}(q,t,p)\to\mathcal{U}(q,t,p)\otimes\mathcal{U}(q,t,p)$$

defined by

$$\Delta(C^{\pm 1}):=C^{\pm 1}\otimes C^{\pm 1},\quad \Delta(\psi^{\pm}(p;z)):=\psi^{\pm}(p;C_{(2)}^{\pm 1/2}z)\otimes\psi^{\pm}(p;C_{(1)}^{\mp 1/2}z),$$
$$\Delta(x^{+}(p;z)):=x^{+}(p;z)\otimes 1+\psi^{-}(p;C_{(1)}^{1/2}z)\otimes x^{+}(p;C_{(1)}z),$$
$$\Delta(x^{-}(p;z)):=x^{-}(p;C_{(2)}z)\otimes\psi^{+}(p;C_{(2)}^{1/2}z)+1\otimes x^{-}(p;z).$$

§ 3. Relation to the elliptic Feigin-Odesskii algebra

In this section, we introduce the elliptic Feigin-Odesskii algebra [FO][FHHSY]. We utilize the algebra for the construction of commutative families of the elliptic Ruijsenaars operator [Sa2].

§ 3.1. Elliptic Feigin-Odesskii algebra $\mathcal{A}(p)$

Definition 3.1 (Star product $*$). Set a meromorphic function $\omega_p(x,y)$ in x, $y \in \mathbb{C}^\times$ as

$$\omega_p(x,y) := \frac{\Theta_p(q^{-1}y/x)\Theta_p(ty/x)\Theta_p(qt^{-1}y/x)}{\Theta_p(y/x)^3}.$$

Define the star product $*$ as

$$(f * g)(x_1,\dots,x_{m+n}) := \mathrm{Sym}\Bigg[f(x_1,\dots,x_m)g(x_{m+1},\dots,x_{m+n}) \prod_{\substack{1\le\alpha\le m\\ m+1\le\beta\le m+n}} \omega_p(x_\alpha,x_\beta)\Bigg].$$

Proposition 3.2 ([FO][FHHSY]). *Define an n-variable meromorphic function $\varepsilon_n(q,p;x)$ $(n \in \mathbb{Z}_{>0})$ as follows.*

$$\varepsilon_n(q,p;x) := \prod_{1\le a<b\le n} \frac{\Theta_p(qx_a/x_b)\Theta_p(q^{-1}x_a/x_b)}{\Theta_p(x_a/x_b)^2}.$$

Then we have $\varepsilon_m(q,p;\bullet) * \varepsilon_n(q,p;\bullet)(x) = \varepsilon_n(q,p;\bullet) * \varepsilon_m(q,p;\bullet)(x)$ $(m,n \in \mathbb{Z}_{>0})$.

Definition 3.3 (Elliptic Feigin-Odesskii algebra $\mathcal{A}(p)$ [FO][FHHSY]). For a partition λ, we set $\varepsilon_\lambda(q,p;x)$ as

$$\varepsilon_\lambda(q,p;x) := (\varepsilon_{\lambda_1}(q,p;\bullet) * \cdots * \varepsilon_{\lambda_{\ell(\lambda)}}(q,p;\bullet))(x).$$

Set $\mathcal{A}_0(p) := \mathbb{K}[[p]]$, $\mathcal{A}_n(p) := \mathrm{span}\{\varepsilon_\lambda(q,p;x) \,|\, |\lambda| = n\}$ $(n \ge 1)$. We define the elliptic Feigin-Odesskii algebra as $\mathcal{A}(p) := \bigoplus_{n\ge0} \mathcal{A}_n(p)$ whose algebra structure is given by the star product $*$.

Proposition 3.4 ([FHHSY]). *The elliptic Feigin-Odesskii algebra $(\mathcal{A}(p), *)$ is unital, associative, and commutative.*

§ 3.2. Commutative families $\mathcal{M}(p)$, $\mathcal{M}'(p)$

Recall that the boson operators $E(p;z)$, $F(p;z)$ defined by (2.3) in Section 2 satisfy

$$E(p;z)E(p;w) = g_p\Big(\frac{z}{w}\Big)E(p;w)E(p;z), \tag{3.1}$$

$$F(p;z)F(p;w) = g_p\Big(\frac{z}{w}\Big)^{-1}F(p;w)F(p;z), \tag{3.2}$$

where $g_p(x)$ is defined by

$$g_p(x) = \frac{\Theta_p(qx)\Theta_p(t^{-1}x)\Theta_p(q^{-1}tx)}{\Theta_p(q^{-1}x)\Theta_p(tx)\Theta_p(qt^{-1}x)}.$$

Set a meromorphic function $\omega_p'(x,y)$ in x, $y \in \mathbb{C}^\times$ as

$$\omega_p'(x,y) := \frac{\Theta_p(qy/x)\Theta_p(t^{-1}y/x)\Theta_p(q^{-1}ty/x)}{\Theta_p(y/x)^3}.$$

Due to the relations (3.1), (3.2), operator-valued functions as

$$\prod_{1\le i<j\le N} \omega_p(x_i,x_j)^{-1} E(p;x_1)\cdots E(p;x_N),$$
$$\prod_{1\le i<j\le N} \omega_p'(x_i,x_j)^{-1} F(p;x_1)\cdots F(p;x_N)$$

are symmetric in $x_1, \dots, x_N$.

Definition 3.5 (**Maps $\mathcal{O}_p$, $\mathcal{O}_p'$**). We define linear maps

$$\mathcal{O}_p : \mathcal{A}(p) \to \bigoplus_{\alpha\in\mathbb{K}[[p]]} \mathrm{End}_{\mathbb{K}[[p]]}(\mathcal{F}_\alpha) \quad \text{and} \quad \mathcal{O}_p' : \mathcal{A}(p) \to \bigoplus_{\alpha\in\mathbb{K}[[p]]} \mathrm{End}_{\mathbb{K}[[p]]}(\mathcal{F}_\alpha)$$

as follows.

$$\mathcal{O}_p(f) := \left[f(z_1,\dots,z_n) \prod_{1\le i<j\le n} \omega_p(z_i,z_j)^{-1} E(p;z_1)\cdots E(p;z_n)\right]_1 \quad (f\in\mathcal{A}_n(p)),$$
$$\mathcal{O}_p'(f) := \left[f(z_1,\dots,z_n) \prod_{1\le i<j\le n} \omega_p'(z_i,z_j)^{-1} F(p;z_1)\cdots F(p;z_n)\right]_1 \quad (f\in\mathcal{A}_n(p)).$$

Here $[f(z_1,\dots,z_n)]_1$ denotes the constant term of $f(z_1,\dots,z_n)$ in $z_1,\dots,z_n$. We extend the maps linearly.

Proposition 3.6. *For any f, $g\in\mathcal{A}(p)$, we have*

$$\mathcal{O}_p(f * g) = \mathcal{O}_p(f)\mathcal{O}_p(g), \quad \mathcal{O}_p'(f * g) = \mathcal{O}_p'(f)\mathcal{O}_p'(g).$$

From Proposition 3.6, we obtain the following theorem.

Theorem 3.7 (**Commutative families $\mathcal{M}(p)$, $\mathcal{M}'(p)$ [Sa2]**). *(1) Set $\mathcal{M}(p) := \mathcal{O}_p(\mathcal{A}(p))$, $\mathcal{M}'(p) := \mathcal{O}_p'(\mathcal{A}(p))$. Then*

$$\mathcal{M}(p) \subset \bigoplus_{\alpha\in\mathbb{K}[[p]]} \mathrm{End}_{\mathbb{K}[[p]]}(\mathcal{F}_\alpha), \quad \mathcal{M}'(p) \subset \bigoplus_{\alpha\in\mathbb{K}[[p]]} \mathrm{End}_{\mathbb{K}[[p]]}(\mathcal{F}_\alpha).$$

The spaces are commutative subalgebras of boson operators.

(2) The commutative families $\mathcal{M}(p)$, $\mathcal{M}'(p)$ satisfy $[\mathcal{M}(p),\mathcal{M}'(p)]=0$.

By the free field realization of the elliptic Ruijsenaars operator, we have the following.

Corollary 3.8 (**[Sa2]**). *Elements in $\mathcal{M}(p)$, $\mathcal{M}'(p)$ act on $\phi_N(p;x)|N\rangle$ as q-difference operators in $\mathrm{End}_{\mathbb{K}[[p]]}(\Lambda_N(q,t,p))$ commuting with each other, where $\Lambda_N(q,t,p)$ is the space of functions of $x_1,\dots,x_N$ defined in Definition 2.11. By the free field realization of the elliptic Ruijsenaars operators, the commuting q-difference operators also commute with the elliptic Ruijsenaars operators $H_N(q,t,p)$, $H_N(q^{-1},t^{-1},p)$.*

The elliptic Feigin-Odesskii algebra originally appeared in Feigin-Odesskii [FO]. In this section, we have regarded the elliptic Feigin-Odesskii algebra as a machinery for the construction of the commutative families of the elliptic Ruijsenaars operator. For combinatorial properties of the trigonometric and the elliptic Feigin-Odesskii algebra, see [FHHSY].

§ 4. Perspectives

In this article, we have seen some topics related to the elliptic Ding-Iohara-Miki algebra. Here we give some comments about open problems which should be studied more in a future.

§ 4.1. An elliptic analog of the Macdonald symmetric functions

The trigonometric Ruijsenaars model is solvable due to the theory of the Macdonald symmetric functions. In the elliptic case, Ruijsenaars showed Liouville integrability of the elliptic Ruijsenaars model [R1]. Then one would have some questions:

> Does there exists eigenfunctions for the elliptic Ruijsenaars operator ? Can we obtain an elliptic analog of the Macdonald symmetric functions ?

However this problem is very difficult because of the emergence of the elliptic functions, thus the problem remains open since Ruijsenaars found the elliptic Ruijsenaars model.

On the other hand, there are some interesting works due to Langmann on the elliptic Calogero-Sutherland model [L1][L2][L3]. This model is a degeneration of the elliptic Ruijsenaars model in the limit $q \to 1$. In the papers [L1][L2][L3], Langmann gave the free field realization of the elliptic Calogero-Sutherland Hamiltonian and the kernel function for the operator, and researched an elliptic analog of the Jack symmetric functions [L3]. He also showed the functional equation of the kernel function for the elliptic Calogero-Sutherland Hamiltonian. Then it is remarkable there appears the p-derivative $D_p := p\dfrac{\partial}{\partial p}$ (p is the elliptic parameter) in the functional equation. Further by keeping the relation between the elliptic Calogero-Sutherland model and the elliptic Ruijsenaars model in mind, the functional equation of the kernel function for the elliptic

Ruijsenaars operator (2.5) in Section 2 as

$$\begin{aligned}&\{H_M(q,t,p)_x - t^{M-N}H_N(q,t,p)_y\}\Pi_{MN}(q,t,p)(x,y)\\&=\frac{(1-t^{M-N})(p;p)_\infty^3}{\Theta_p(t)}C_{MN}(p;x,y)\Pi_{MN}(q,t,p)(x,y)\end{aligned}$$

should be recognized as a q-deformation of Langmann's functional equation. By comparing the above equation and Langmann's result, it seems to be natural that the right hand side of the above as

$$\frac{(1-t^{M-N})(p;p)_\infty^3}{\Theta_p(t)}C_{MN}(p;x,y)\Pi_{MN}(q,t,p)(x,y)$$

is written in terms of the p-derivative D_p. Thus it would be nice if there exists a function $f_{MN}(x)$ such that

$$\begin{aligned}&f_{MN}(D_p)\Pi_{MN}(q,t,p)(x,y)\\&=\frac{(1-t^{M-N})(p;p)_\infty^3}{\Theta_p(t)}C_{MN}(p;x,y)\Pi_{MN}(q,t,p)(x,y).\end{aligned}$$

If as such function $f_{MN}(x)$ is obtained, perhaps circumstances concerning an elliptic analog of the Macdonald symmetric functions would be more clear.

§ 4.2. Modular double of the Ding-Iohara-Miki algebra

As a material which is in progress, the author would like to mention the study of the modular double of the Ding-Iohara-Miki algebra.

Modular doubles arised from studies of relations between quantum groups and modularities in mathematical physics [F][FKV]. In the theories of modular doubles, the double sine function which is a modular analog of the q-infinite product plays important roles. Let ω_1, $\omega_2 \in \mathbb{C}$ be parameters satisfying $\mathrm{Re}(\omega_1)>0$, $\mathrm{Re}(\omega_2)>0$. The double sine function $S(\omega_1,\omega_2;u)$ is defined by

$$S(\omega_1,\omega_2;u) := \exp\left(\int_{\mathbb{R}+i0}\frac{e^{ku}}{(1-e^{\omega_1 k})(1-e^{\omega_2 k})}\frac{dk}{k}\right). \tag{4.1}$$

The integral is well-defined in $0<\mathrm{Re}(u)<\mathrm{Re}(\omega_1+\omega_2)$. If $\boldsymbol{q} := e^{2\pi i\frac{\omega_1}{\omega_2}}$ satisfies $|\boldsymbol{q}|<1$, $S(\omega_1,\omega_2;u)$ has the q-infinite product representation as follows.

$$S(\omega_1,\omega_2;u) = \frac{(e^{2\pi i\frac{u}{\omega_2}};\boldsymbol{q})_\infty}{(\widetilde{\boldsymbol{q}}e^{2\pi i\frac{u}{\omega_1}};\widetilde{\boldsymbol{q}})_\infty},$$

where $\widetilde{\boldsymbol{q}}{:=}e^{-2\pi i\frac{\omega_2}{\omega_1}}$. Then we remark that the integral representation of $S(\omega_1,\omega_2;u)$ (4.1) is valid if the parameter $\boldsymbol{q}$ is on the unit circle $|\boldsymbol{q}|=1$. More on the double sine function and some related topics, see Narukawa [Naru], Kurokawa [K].

Due to Bytsko-Teschner [BT], and Frenkel-Ip [FI], and Ip [Ip], modular doubles have the following properties:

- Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of simply-laced type over $\mathbb{C}$ and $\mathfrak{g}_{\mathbb{R}}$ be the real form. Then the modular double of the quantum group $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ is defined by

$$\mathcal{U}_{q\widetilde{q}^{-1}}(\mathfrak{g}_{\mathbb{R}}) := \mathcal{U}_q(\mathfrak{g}_{\mathbb{R}}) \otimes \mathcal{U}_{\widetilde{q}^{-1}}(\mathfrak{g}_{\mathbb{R}}),$$

  where for parameters $q = e^{2\pi i\omega}$ ($\omega \in \mathbb{C}$), set $\widetilde{q} := e^{-2\pi i/\omega}$. Then representations of $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ and $\mathcal{U}_{\widetilde{q}^{-1}}(\mathfrak{g}_{\mathbb{R}})$ commute with each other.

- The universal R operator of $\mathcal{U}_{q\widetilde{q}^{-1}}(\mathfrak{g}_{\mathbb{R}})$ is written by the double sine function.

For more details including modular doubles of non-simply-laced type, see Ip [Ip]. Relations between the modular double of $U_q(sl_2(\mathbb{R}))$ and the quantum Teichmüller theory are studied in Nidaiev-Teschner [NT].

Next let us consider how to construct the modular double of the Ding-Iohara-Miki algebra. First by comparing universal R operators of $U_q(sl_2(\mathbb{C}))$ and of the modular double of $U_q(sl_2(\mathbb{R}))$, the following substitution is essential:

$$(x;q)_\infty \xrightarrow[\text{modular analog}]{} S(\omega_1,\omega_2;u).$$

Second the Ding-Iohara-Miki algebra $\mathcal{U}(q,t)$ is discovered from the kernel function for the Macdonald operator which has the form as

$$\Pi_{MN}(q,t)(x,y) = \prod_{\substack{1\le i\le M\\ 1\le j\le N}} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}.$$

Then we may have a question:

If we define the kernel function by using the double sine function as

$$\Pi_{MN}(\omega_1,\omega_2,\sigma)(u,v) := \prod_{\substack{1\le i\le M\\ 1\le j\le N}} \frac{S(\omega_1,\omega_2;u_i+v_j+\sigma)}{S(\omega_1,\omega_2;u_i+v_j)} \quad (\sigma\in\mathbb{C}), \tag{4.2}$$

what kind of quantum groups can we obtain ?

At present, by using bosons obtained from the kernel function (4.2), the author have checked the emergence of an algebra looks like

$$\mathcal{U}(e(\omega_1/\omega_2), e(\sigma/\omega_2)) \otimes \mathcal{U}(e(\omega_2/\omega_1), e(\sigma/\omega_1)) \quad (e(u) := e^{2\pi i u}, u \in \mathbb{C})$$

which has the modular double-like properties. It seems that the algebra would be the modular double of the Ding-Iohara-Miki algebra $\mathcal{U}(q,t)$, however we need more researches.

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