On Non-eliminability of the Cut Rule and the Roles of Associativity and Distributivity in Non-commutative Substructural Logics

Takeshi Ueno, Koji Nakaogawa, and Osamu Watari

Abstract. We introduce a sequent calculus **FL'**, which has at most one formula on the right side of sequent, and which excludes three structural inference rules, i.e. contraction, weakening and exchange. Our formulations of the inference rules of **FL'** are based on the results and considerations carried out in our previous papers on how to formulate Gentzen-style natural deduction for non-commutative substructural logics.

Our present formulation **FL'** of sequent system for non-commutative substructural logic, which has no structural rules, has the same proof strength as the ordinary and standard sequent calculus **FL** (Full Lambek), which is often called Full Lambek calculus, i.e., the basic sequent calculus for all other substructural logics. For the standard **FL** (Full Lambek), we use Ono's formulation.

Although our **FL'** and the standard formulation **FL** (Full Lambek) are equivalent, there is a subtle difference in the left rule of implication. In the standard formulation, two parameters Γ_1 and Γ_2 (resp.), each of which is just an finite sequence of arbitrary formulas, appear on the left and right side (resp.) of a formula appearing on the left side of the sequent on the upper left side the left rule $\supset$ (which corresponds to $\supset'$ in FL'). On the other hand, there is no such parameter on the left side of the sequent on the upper left side in the left rule for $\supset'$ of our system **FL**. In our sys-

tem **FL'**, Γ_1 is always empty, and only Γ_2 is allowed to occur in the left rule for $\supset'$ (similar differences occur in the multiplicative conjunction, additive conjunction and additive disjunction). This subtle difference between our system **FL'** and the standard system **FL** (Full Lambek) matters deeply, for we are led to a construction of proof-figures in **FL'**, which show how the associativity of multiplicative conjunction and the distributivity of multiplicative conjunction over additive disjunction are involved in the eliminations of the cut rule in those proofs. We clarify and specify how associativity and distributivity are related to the non-eliminability of an application of the cut rule in those proof-figures of **FL'**.

1 Introduction

The situation surrounding the syntactic aspects of non-commutative substructural logics does not seem to be fully clarified. In particular, the process of eliminating applications of the cut rule in a given proof-figure of intuitionistic sequent system for **FL'**, (defined below) , where **FL** stands for "Full Lambek" and is the most basic system for substructural logics, sometimes succeeds and terminates, and some other times, does not succeed and does not terminate to produce a proof-figure which contains no applications of the cut rule. Indeed, it depends on the subtlety of where one is allowed to place parameters (side formulas) in some of the inference rules of **FL'**.

In the present paper, we introduce a system of inference rules of intuitionistic ("left heavy") sequent calculus for substructural logic, **FL'**, which lacks all structural inference rules, namely, exchange, weakening, and contraction rules. Furthermore, the parameters in its inference rules are placed in such a way that their positions reflect the "natural order" of

non-cancelled hypothesis in the (Gentzen-style) natural deduction for non-commutative substructural logic. Using our system **FL'**, we will show how the associative law for multiplicative conjunction and the distributive law of multiplicative conjunction over additive disjunction are entangled in the elimination process of applications of the cut rule. Analysis the relevancy of these two rules as to the cut elimination process has become possible to us, for we fixed the positions of parameters in the inference rules of **FL'** according to our analysis of normalization procedures in Gentzen-style natural deduction for non-commutative substructural logic. (This paper does not assume the knowledge of our previous papers on Gentzen-style natural deductions for substructural logics.)

2 Language L and its Formulas

Our language L has propositional constant symbols A, B, C,···. As for logical connectives, it has the implication symbols $\supset$, $\supset'$, the negation symbols $\neg$, $\neg'$, the multiplicative conjunction symbol*, the additive conjunction symbol$\wedge$, the additive disjunction symbol$\vee$. In L, there are constant symbols t to denote the unit element for the multiplicative conjunction, **f** to denote the unit element for the multiplicative disjunction which is not introduced in this paper, $\top$ to denote the unit element for the additive conjunction, and $\bot$ to denote the unit element for the additive disjunction.

The formulas of L are defined inductively as a finite sequence of these symbols together with parenthesizes.

3 Sequent Calculus FL' (our formulation)

The sequent of the language L have the following form

$$\Gamma \rightarrow \Delta .$$

The left hand side of a sequent may be empty. The right hand side of a sequent is either empty or consists of a single formula. To specify the element of Γ and Δ we write

$$\gamma_0, \cdots, \gamma_{m-1} \rightarrow \delta.$$

When both sides of a sequent are empty, we write

$$\rightarrow$$

Next, we introduce the sequent calculus **FL'** as follows. We say that $(\Gamma_1 \rightarrow X_1,\ \Gamma_2 \rightarrow X_2,\ \cdots,\ \Gamma_n \rightarrow X_n\ /\Gamma \rightarrow X)$ is an instance of a certain inference rule if it has the form indicated by the corresponding figure. If $(\Gamma_1 \rightarrow X_1,\ \Gamma_2 \rightarrow X_2,\ \cdots,\ \Gamma_n \rightarrow X_n\ /\Gamma \rightarrow X)$ is an instance of an inference rule α, we call $\Gamma_i \rightarrow X_i$ the i-th upper sequent of α, and $\Gamma \rightarrow X$ the lower sequent of α.

(The origin of **FL** goes back to a classical paper written by J. Lambek in 1950's.

Our presentation of **FL'** is based on Ono [4].), but ours is different from his in important places.

In our **FL'**, the positions of parameters in its inference rules are determined according to the un-cancelled hypothesis of Gentzen-style natural deduction for non-commutative substructural logic. The reader should take note that the positions of parameters in inferences rules of our **FL'** are different from those of Ono's.

– Axioms and rule for logical constants :

$$A \to A$$

$$\to \mathbf{t} \qquad \mathbf{f} \to$$

$$\Gamma \to \top \qquad \bot, \Gamma \to C$$

$$\frac{\Gamma \to C}{\mathbf{t}, \Gamma \to C}\ \mathbf{tw} \qquad \frac{\Gamma \to}{\Gamma \to \mathbf{f}}\ \mathbf{fw}$$

– Structural inference rule :

$$\frac{\Gamma_1 \to A \quad \Gamma_2, A, \Gamma_3 \to C}{\Gamma_2, \Gamma_1, \Gamma_3 \to C}\ (\text{cut})$$

– Logical inference rule :

$$\frac{\Gamma_1 \to A \quad B, \Gamma_2 \to C}{\Gamma_1, A \supset B, \Gamma_2 \to C}\ (\supset \text{ left}) \qquad \frac{A, \Gamma \to B}{\Gamma \to A \supset B}\ (\supset \text{ right})$$

$$\frac{\Gamma_1 \to A \quad B, \Gamma_2 \to C}{A \supset' B, \Gamma_1, \Gamma_2 \to C}\ (\supset' \text{ left}) \qquad \frac{\Gamma, A \to B}{\Gamma \to A \supset' B}\ (\supset' \text{ right})$$

$$\frac{\Gamma \to A}{\Gamma, \neg A \to}\ (\neg \text{ left}) \qquad \frac{A, \Gamma \to}{\Gamma \to \neg A}\ (\neg \text{ right})$$

$$\frac{\Gamma \to A}{\neg' A, \Gamma \to}\ (\neg' \text{ left}) \qquad \frac{\Gamma, A \to}{\Gamma \to \neg' A}\ (\neg' \text{ right})$$

$$\frac{A, B, \Gamma \to C}{A * B, \Gamma \to C}\ (* \text{ left}) \qquad \frac{\Gamma_1 \to A \quad \Gamma_2 \to B}{\Gamma_1, \Gamma_2 \to A * B}\ (* \text{ right})$$

$$\frac{A, \Gamma \to C}{A \wedge B, \Gamma \to C}\ (\wedge_1 \text{ left}) \qquad \frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B}\ (\wedge \text{ right})$$

$$\frac{B, \Gamma \to C}{A \wedge B, \Gamma \to C}\ (\wedge_2 \text{ left})$$

$$\frac{A, \Gamma \to C \quad B, \Gamma \to C}{A \vee B, \Gamma \to C}\ (\vee \text{ left}) \qquad \frac{\Gamma \to A}{\Gamma \to A \vee B}\ (\vee_1 \text{ right})$$

$$\frac{\Gamma \to B}{\Gamma \to A \vee B}\ (\vee_2 \text{ right})$$

4 Sequent Calculus FL (Ono's formulation)

The reader should be warned that the $\supset$ of **FL'** corresponds with $\supset'$ of Ono's, and $\supset'$ of **FL'** corresponds with $\supset$ of Ono's. $\neg$ of **FL'** corresponds with $\neg'$ of Ono's, and $\neg'$ of **FL'** corresponds with $\neg$ of Ono's.

– Axioms and rules for logical constants:

$$A \to A$$

$$\to \mathbf{t} \qquad\qquad \mathbf{f} \to$$

$$\Gamma \to \top \qquad\qquad \Gamma_1, \bot, \Gamma_2 \to C$$

$$\frac{\Gamma_1, \Gamma_2 \to C}{\Gamma_1, \mathbf{t}, \Gamma_2 \to C}\ \mathbf{t}\mathrm{w} \qquad\qquad \frac{\Gamma \to}{\Gamma \to \mathbf{f}}\ \mathbf{f}\mathrm{w}$$

– Structural inference rules:

$$\frac{\Gamma_1 \to A \quad \Gamma_2, A, \Gamma_3 \to C}{\Gamma_2, \Gamma_1, \Gamma_3 \to C}\ (\mathrm{cut})$$

– Logical inference rules:

$$\frac{\Gamma_1 \to A \quad \Gamma_2, B, \Gamma_3 \to C}{\Gamma_2, \Gamma_1, A \supset' B, \Gamma_3 \to C}\ (\supset' \text{ left}) \qquad \frac{A, \Gamma \to B}{\Gamma \to A \supset' B}\ (\supset' \text{ right})$$

$$\frac{\Gamma_1 \to A \quad \Gamma_2, B, \Gamma_3 \to C}{\Gamma_2, A \supset B, \Gamma_1, \Gamma_3 \to C}\ (\supset \text{ left}) \qquad \frac{\Gamma, A \to B}{\Gamma \to A \supset B}\ (\supset \text{ right})$$

$$\frac{\Gamma \to A}{\Gamma, \neg' A \to}\ (\neg' \text{ left}) \qquad \frac{A, \Gamma \to}{\Gamma \to \neg' A}\ (\neg' \text{ right})$$

$$\frac{\Gamma \to A}{\neg A, \Gamma \to}\ (\neg \text{ left}) \qquad \frac{\Gamma, A \to}{\Gamma \to \neg A}\ (\neg \text{ right})$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \to C}{\Gamma_1, A * B, \Gamma_2 \to C}\ (* \text{ left}) \qquad \frac{\Gamma_1 \to A \quad \Gamma_2 \to B}{\Gamma_1, \Gamma_2 \to A * B}\ (* \text{ right})$$

$$\frac{\Gamma_1, A, \Gamma_2 \to C}{\Gamma_1 A \wedge B, \Gamma_2 \to C}\ (\wedge_1 \text{ left}) \qquad \frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B}\ (\wedge \text{ right})$$

$$\frac{\Gamma_1, B, \Gamma_2 \to C}{\Gamma_1, A \wedge B, \Gamma_2, \to C}\ (\wedge_2 \text{ left})$$

$$\frac{\Gamma_1, A, \Gamma_2 \to C \quad \Gamma_1, B, \Gamma_2, \to C}{\Gamma_1, A \vee B, \Gamma_2 \to C} \ (\vee \text{ left}) \qquad \frac{\Gamma \to A}{\Gamma \to A \vee B} \ (\vee_1 \text{ right}) \quad \frac{\Gamma \to B}{\Gamma \to A \vee B} \ (\vee_2 \text{ right})$$

5 Equivalence of FL and FL'

Theorem 1 (Equivalence of FL and FL').

Let ϕ be a formula of the language L. Let Γ be a list of formulas of L. Then, the sequent $\Gamma \to \phi$ is provable in **FL** *if and only if the sequent $\Gamma \to \phi$ is provable in* **FL'**.

Proof. First, we prove that if sequent $\Gamma \to \phi$ is provable with proof Π in **FL**, then sequent $\Gamma \to \phi$ is provable with a proof Σ in **FL'**. To prove this direction, we use induction on the number $\sharp(\Pi)$ of the applications of inference rules in the proof Π.

If $\sharp(\Pi)$ is zero, $\Gamma \to \phi$ must be an axiom of **FL**.

The axiom $\Gamma \to \top$, Γ_1, $\bot$, $\Gamma_2 \to C$ of **FL** is provable in **FL'**. This is shown by the following proof figure of **FL'**.

$$\dfrac{\bot, \Gamma_2 \to \alpha_1 * \alpha_2 \supset C \qquad \dfrac{\dfrac{\alpha_1 \to \alpha_1 \quad \alpha_2 \to \alpha_2}{\alpha_1, \alpha_2 \to \alpha_1 * \alpha_2} \ (* \text{ right}) \qquad C \to C}{\alpha_1, \alpha_2, \alpha_1 * \alpha_2 \supset C} \ (\supset \text{ left})}{\alpha_1, \alpha_2, \bot, \Gamma_2 \to C} \ (\text{cut})$$

The following proof figure shows that the axiom (rule) **tw** of **FL** is provable in **FL'**.

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha_1, \alpha_2, \Gamma_2 \to C}{\alpha_1 * \alpha_2, \Gamma_2 \to C} \ (\supset \text{ left})}{\Gamma_2 \to \alpha_1 * \alpha_2 \supset C} \ (\supset \text{ right})}{\mathbf{t}, \Gamma_2 \to \alpha_1 * \alpha_2 \supset C} \ \mathbf{tw} \qquad \dfrac{\dfrac{\alpha_1 \to \alpha_1 \quad \alpha_2 \to \alpha_2}{\alpha_1, \alpha_2 \to \alpha_1 * \alpha_2} \ (* \text{ right}) \qquad C \to C}{\alpha_1, \alpha_2, \alpha_1 * \alpha_2 \supset C} \ (\supset \text{ left})}{\alpha_1, \alpha_2, \mathbf{t}, \Gamma_2 \to C} \ (\text{cut})$$

Now, we assume the theorem for $\sharp(\Pi) < n$, and prove it for $\sharp(\Pi) = n$.

Our proof is divided into cases, depending on which inference rule is used as the "bottom" inference rule in Π.

Without loss of generality, we assume that Γ_1 consits of just α_1, and α_2.

Case 1 The bottom inference rule in Π is (*left)

$$\text{i.e.}\quad \frac{\alpha_1, \alpha_2, A, B, \Gamma_2 \to C}{\alpha_1, \alpha_2, A * B, \Gamma_2 \to C}\ (* \text{ left})$$

Then we can construct a proof of α_1, α_2, $A * B$, $\Gamma_2 \to C$ in **FL'** as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha_1, \alpha_2, A, B, \Gamma_2 \to C}{\alpha_1 * \alpha_2, A, B, \Gamma_2 \to C}\ (\supset \text{ left})}{A, B, \Gamma_2 \to \alpha_1 * \alpha_2 \supset C}\ (\supset \text{ right})}{A * B, \Gamma_2 \to \alpha_1 * \alpha_2 \supset C}\ (* \text{ left}) \qquad \dfrac{\dfrac{\alpha_1 \to \alpha_1 \quad \alpha_2 \to \alpha_2}{\alpha_1, \alpha_2 \to \alpha_1 * \alpha_2}\ (* \text{ right}) \qquad C \to C}{\alpha_1, \alpha_2, \alpha_1 * \alpha_2 \supset C}\ (\supset \text{ left})}{\alpha_1, \alpha_2, A * B, \Gamma_2 \to C}\ (\text{cut})$$

or

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha_1, \alpha_2, A, B, \Gamma_2 \to C}{\alpha_1 * \alpha_2, A, B, \Gamma_2 \to C}\ (\supset \text{ left})}{A, B, \Gamma_2 \to \alpha_1 * \alpha_2 \supset C}\ (\supset \text{ right})}{A * B, \Gamma_2 \to \alpha_1 * \alpha_2 \supset C}\ (* \text{ left}) \qquad \dfrac{\dfrac{\dfrac{\alpha_1 \to \alpha_1 \quad C \to C}{\alpha_1, \alpha_1 \supset C \to C}\ (\supset \text{ left})}{\alpha_2 \to \alpha_2 \quad \alpha_1 \supset C \to \alpha_1 \supset C}\ (\supset \text{ right}) \qquad \dfrac{\alpha_1 \to \alpha_1 \quad C \to C}{\alpha_1, \alpha_1 \supset C \to C}\ (* \text{ left})}{\alpha_2, \alpha_1 \supset \alpha_1 \supset C \to \alpha_1 \supset C}\ (\supset \text{ left})}{\alpha_1, \alpha_2, \alpha_2 \supset \alpha_1 \supset C \to C}\ (\text{cut})}{\alpha_1, \alpha_2, A * B, \Gamma_2 \to C}\ (\text{cut})$$

Case 2 The bottom inference rule in Π is ($\vee$left)

$$\text{i.e.}\quad \frac{\alpha_1, \alpha_2, A, \Gamma_2 \to C \quad \Gamma_1, B, \Gamma_2 \to C}{\alpha_1, \alpha_2, A \vee B, \Gamma_2 \to C}\ (\vee \text{ left})$$

Then we can construct a proof of $\alpha_1, \alpha_2, A \vee B, \Gamma_2 \rightarrow C$ in **FL'** as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha_1, \alpha_2, A, \Gamma_2 \rightarrow C}{\alpha_1 * \alpha_2, A, \Gamma_2 \rightarrow C}\,(* \text{ left})}{A, \Gamma_2 \rightarrow \alpha_1 * \alpha_2 \supset C}\,(\supset \text{ right}) \quad \dfrac{\dfrac{\alpha_1, \alpha_2, B, \Gamma_2 \rightarrow C}{\alpha_1 * \alpha_2, B, \Gamma_2 \rightarrow C}\,(* \text{ left})}{B, \Gamma_2 \rightarrow \alpha_1 * \alpha_2 \supset C}\,(\supset \text{ right})}{A \vee B, \Gamma_2 \rightarrow \alpha_1 * \alpha_2 \supset C}\,(\vee \text{ left}) \quad \dfrac{\dfrac{\alpha_1 \rightarrow \alpha_1 \quad \alpha_2 \rightarrow \alpha_2}{\alpha_1, \alpha_2 \rightarrow \alpha_1 * \alpha_2}\,(* \text{ right}) \quad C \rightarrow C}{\alpha_1, \alpha_2, \alpha_1 * \alpha_2 \supset C}\,(\supset \text{ left})}{\alpha_1, \alpha_2, A * B, \Gamma_2 \rightarrow C}\,(\text{cut})$$

or

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha_1, \alpha_2, A, \Gamma_2 \rightarrow C}{\alpha_2, A, \Gamma_2 \rightarrow \alpha_1 \supset C}\,(\supset \text{ right})}{A, \Gamma_2 \rightarrow \alpha_2 \supset \alpha_1 \supset C}\,(\supset \text{ right}) \quad \dfrac{\dfrac{\alpha_1, \alpha_2, B, \Gamma_2 \rightarrow C}{\alpha_2, B, \Gamma_2 \rightarrow \alpha_1 \supset C}\,(\supset \text{ right})}{B, \Gamma_2 \rightarrow \alpha_2 \supset \alpha_1 \supset C}\,(\supset \text{ right})}{A \vee B, \Gamma_2 \rightarrow \alpha_2 \supset \alpha_1 \supset C}\,(\vee \text{ left}) \quad \dfrac{\dfrac{\dfrac{\alpha_1 \rightarrow \alpha_1 \quad C \rightarrow C}{\alpha_1, \alpha_1 \supset C \rightarrow C}\,(\supset \text{ left})}{\alpha_2, \alpha_1 \supset C \rightarrow \alpha_1 \supset C}\,(\supset \text{ right}) \quad \dfrac{\alpha_2 \rightarrow \alpha_2}{\alpha_2, \alpha_1 \supset \alpha_1 \supset C \rightarrow \alpha_1 \supset C}\,(\supset \text{ left})}{\alpha_1, \alpha_2, \alpha_2 \supset \alpha_1 \supset C \rightarrow C}\,(\text{cut}) \quad \dfrac{\alpha_1 \rightarrow \alpha_1 \quad C \rightarrow C}{\alpha_1, \alpha_1 \supset C \rightarrow C}\,(* \text{ left})}{\alpha_1, \alpha_2, A \vee B, \Gamma_2 \rightarrow C}\,(\text{cut})$$

The other inference rules are handled in a similar way.

The other direction is clear since each axiom and inference rule of our **FL'** is a particular case of those of Ono's **FL**.

6 Cut, Parameter in inference rules, associativity and distributivity

In this section, we present some examples of proof-figures of **FL'** which show how associative law and distributive law are involved with the cut-elimination process, and show how difficult it is to eliminate the applications of the cut rule in these proof-figure.

Associativity: $A * (B * C) \rightarrow (A * B) * C$

First of all, we present the following proof figure in **FL'**, which contains an application of the cut rule.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A \quad B \rightarrow B}{A, B \rightarrow A * B}\,(* \text{ right}) \quad C \rightarrow C}{A, B, C \rightarrow (A * B) * C}\,(* \text{ right})}{B, C \rightarrow A \supset (A * B) * C}\,(\supset \text{ right})}{B * C \rightarrow A \supset (A * B) * C}\,(* \text{ left}) \quad \dfrac{A \rightarrow A \quad A \supset (A * B) * C \rightarrow A \supset (A * B) * C}{A, A \supset (A * B) * C \rightarrow A \supset (A * B) * C}\,(\supset \text{ left})}{A, B * C \rightarrow (A * B) * C}\,(\text{cut})}{A * (B * C) \rightarrow (A * B) * C}\,(* \text{ left})$$

This application of the cut rule becomes eliminable in **FL** as the next proof figure shows:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A \quad B \to B}{A, B \to A * B}\,(* \text{ right}) \qquad C \to C}{A, B, C \to (A * B) * C}\,(* \text{ right})}{A, B * C \to (A * B) * C}\,(* \text{ left})}{A * (B * C) \to (A * B) * C}\,(* \text{ left})$$

Associativity: $(A * B) * C \to A * (B * C)$

This direction of associativity can be proved in both **FL'** and **FL**, as the next proof figure shows:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A \qquad \dfrac{B \to B \quad C \to C}{B, C \to B * C}\,(* \text{ right})}{A, B, C \to A * (B * C)}\,(* \text{ left})}{A * B, C \to A * (B * C)}\,(* \text{ left})}{(A * B) * C \to A * (B * C)}\,(* \text{ left})$$

Distributivity: $A * (B \vee C) \to (A * B) \vee (A * C)$

In **FL'**, we need an application of the cut rule to prove this direction of distributivity, as the following proof figure shows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A \quad B \to B}{A, B \to A * B}\,(* \text{ right})}{A, B \to (A * B) \vee (A * C)}\,(\vee \text{ right})}{B \to A \supset ((A * B) \vee (A * C))}\,(\supset \text{ right}) \qquad \dfrac{\dfrac{\dfrac{A \to A \quad C \to C}{A, C \to A * C}\,(* \text{ right})}{A, C \to (A * B) \vee (A * C)}\,(\vee \text{ right})}{C \to A \supset ((A * B) \vee (A * C))}\,(\supset \text{ right})}{B \vee C \to A \supset ((A * B) \vee (A * C))}\,(\vee \text{ left}) \qquad \dfrac{A \to A \quad (A * B) \vee (A * C) \to (A * B) \vee (A * C)}{A, A \supset ((A * B) \vee (A * C)) \to (A * B) \vee (A * C)}\,(\supset \text{ right})}{A, B \vee C \to (A * B) \vee (A * C)}\,(\text{cut})}{A * (B \vee C) \to (A * B) \vee (A * C)}\,(* \text{ left})$$

This application of the cut rule becomes eliminable in **FL** as the next proof figure shows:

$$\dfrac{\dfrac{\dfrac{A \to A \quad B \to B}{A, B \to A * B}\,(* \text{ right})}{A, B \to (A * B) \vee (A * C)}\,(\vee \text{ right}) \qquad \dfrac{\dfrac{A \to A \quad C \to C}{A, C \to A * C}\,(* \text{ right})}{A, C \to (A * B) \vee (A * C)}\,(\vee \text{ right})}{\dfrac{A, B \vee C \to (A * B) \vee (A * C)}{A * (B \vee C) \to (A * B) \vee (A * C)}\,(* \text{ left})}\,(\vee \text{ right})$$

Distributivity : $(A * B) \vee (A * C) \to A * (B \vee C)$

This direction of distributivity is provable in both **FL'** and **FL**.

$$\dfrac{\dfrac{\dfrac{A \to A \quad \dfrac{B \to B}{B \to B \vee C}\,(\vee \text{ right})}{A, B \to A * (B \vee C)}\,(* \text{ right})}{A * B \to A * (B \vee C)}\,(* \text{ left}) \qquad \dfrac{\dfrac{A \to A \quad \dfrac{C \to C}{C \to B \vee C}\,(\vee \text{ right})}{A, C \to A * (B \vee C)}\,(* \text{ right})}{A * C \to A * (B \vee C)}\,(* \text{ left})}{(A * B) \vee (A * C) \to A * (B \vee C)}\,(\vee \text{ left})$$

The above proofs indicate the way how associativity and distributivity matter for the non-eliminability of applications of the cut rule in a given proof of non-commutative substructural logic. Indeed, the above proof figures, showing how associativity and distributivity are related to the cut rule, are obtained through the analysis of the (unsuccessful) reduction process for a non-normalizable proof in Gentzen style natural deduction for non-commutative substructural logic. In other words, the role of associativity and distributivity (in the process of "reduction") becomes clearer in the places where cut elimination fails.

It is an open problem whether cut elimination holds for **FL'** if we add associativity and distributivity to **FL'**.

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