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RM/17/015

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# Cost of transformation: a measure on matchings

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June 6, 2017

**Abstract**

This paper constructs a normative framework to quantify the difference between two matchings in roommate markets. We investigate the "cost of transformation" from one mechanism to another, based on the differences in the outputs of these mechanisms. Several conditions are introduced to ensure this cost reflects the welfare effect of the transformation on individuals. We introduce a measure called the Borda measure, which is fully characterized by these conditions. Several possible applications of this measure under different contexts interpretations are also discussed, such as measuring how unstable, how unfair, or how inefficient a matching is.

*Keywords:* Matching markets, measure function.

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# 1 Introduction

Matching theory analyzes markets where agents, e.g., buyers and sellers, hospitals and interns, high schools and students, are matched according to their preferences, and thereby conduct some transactions within the relevant context. Some of the well-known mechanisms are the deferred acceptance (introduced by Gale and Shapley (1962), characterized by Kojima and Manea (2010) and Morrill (2013a)), the Boston mechanism (characterized by Afacan (2013) and Kojima and Ünver (2014)), and the top trading cycle (introduced by Shapley and Scarf (1974) and or characterized by Morrill (2013b) and Pycia and Ünver (2017)). These mechanisms produce matchings with various normative features[1], e.g., stability, Pareto efficiency, fairness, etc. They also have different computational complexity[2].

Given two mechanisms with different features and complexity, a measure on matchings can be used to compare the outcomes of the mechanisms and hence quantify the *cost of transformation* from one mechanism to the other. Such a measure can be interpreted in various ways. For instance, it can be interpreted as the *cost of stability* if one mechanism is not stable and the other is, or as the *cost of simplicity* if one mechanism complex and the other is not, e.g., in terms of computation.

The most intuitive way to compare two matchings is by simply looking at the number of individuals who are matched differently. This measure[3] would be zero if the matchings are identical in all pairs, and would be maximal if they have nothing in common, i.e., the matchings are disjoint. However, this method neglects individuals' preferences in the market, i.e., it does not matter how individuals rank their partners in corresponding matchings. Therefore it is not sensible to use it as a measure with the interpretation of cost of transformation.

In this paper, we introduce several conditions to endogenize preferences in quantification of dissimilarity between two matchings in a roommate market[4]. The *Metric condition* requires the measure to be a metric[5] on matchings. *Betweenness* requires that if every individual ranks a matching between two other matchings in their preference, the measure must be additive on these three matchings[6]. *Monotonicity* implies that if from one market to another, the set of agents

[1] For comparisons of some of these methods, see Abdulkadiroglu and Sönmez (2003); Abdulkadiroğlu et al. (2005); Ergin and Sönmez (2006); Chen and Sönmez (2006); Erdil and Ergin (2008); Kesten (2010); Abdulkadiroğlu et al. (2011); Kesten and Ünver (2015).

[2] See Irving (1985), Irving et al. (2000), Manlove et al. (2002)

[3] See Klaus et al. (2010) for an example of such a measure in stochastic markets.

[4] A roommate market is a one-sided one-to-one matching market. However, since the roommate markets are superdomains of the marriage markets, all the results apply the to latter as well.

[5] A metric is a function which satisfies non-negativity, identity of indiscernibles, symmetry, and triangular Inequality.

[6] This is a standard strengthening of triangular inequality where the weak inequality becomes equality, e.g., when three points are on a line in the Euclidian sense (see Kemeny (1959)).

ranked between two matchings expand, the measure should also increase. *Anonymity* requires that relabelling of individuals do not effect the measure. *Independence of irrelevant newcomers*, implies that when an "irrelevant" newcomer[7] joins the market, the measure is unchanged if he or she remains single in both matchings. Finally *normalization* sets the minimal possible value for two disjoint matchings to the size of the market.

We introduce the *Borda measure* on matchings which depends on how each individual ranks their partners under those matchings. Given a market, the measure is the sum of absolute values of the differences in the Borda scores[8] of the two matchings for each individual. We show that the Borda measure is the only method that satisfies all the aforementioned conditions. We formulate our result on the domain of roommate markets since we are also interested in markets that are not necessarily solvable, i.e., markets in which there are no stable matchings. This creates richness in the way the measure can be employed to compare any two matchings. When the measure is applied on the set of stable matchings only, it can serve as a utility to find a "fair" compromise among stable matchings, e.g., between men-optimal and women-optimal stable matchings in a marriage market. Furthermore it can also be used to quantify the level of positive discrimination or favorism in the choice of stable matchings in a marriage market.

The paper proceed as follows. In Section 2, we present the notation and the definition. Section 3 introduces the proposed conditions and the Borda measure. Section 4 is devoted to the proof of our main results. Finally, Section 5 concludes the paper.

## 2 Model

We consider a countable and infinite set of potential individuals, denoted by $\mathcal{N}$, with a non-empty and finite subset $N \subsetneq \mathcal{N}$ interpreted as a set of agents[9]. For each $i \in N$, let $R_i$ denote the preference of agent $i$, that is a complete, transitive and antisymmetric binary relation over $N$, while $R \equiv (R_i)_{i\in N}$ is the preference profile. We say an agent $j$ is "at least as good as" agent $k$ for agent $i$ whenever $j\ R_i\ k$. We denote the position of agent $j$ in the preference $R_i$, by $rank(j, R_i) = |\{k \in N : k\ R_i\ j\}|$. A generic roommate market (also referred to as a roommate problem) is denoted by $P = (N, R)$, and the set of all possible roommate problems over a particular set of agents $N$ by $\mathcal{P}(N)$. We denote the domain of all roommate problems by $\mathcal{D} = \langle \mathcal{P}(N) \rangle_{N \subsetneq \mathcal{N}}$, i.e., the set of all

[7] We consider a newcomer irrelevant when he prefers being single to being matched with the incumbents, and the incumbents prefer being matched among themselves to being matched with the newcomer.

[8] The Borda score of a matching for an individual is the number of alternatives that are ranked strictly below the partner of the individual in that matching.

[9] We assume that $N$ has cardinality $n$ and the agents are labelled $1, 2, \ldots, n$.

possible roommate problem over all possible set of agents.

A *matching* $\mu$ is a permutation on $N$ such that for all $i, j \in N$, $\mu(i) = j$ if and only if $\mu(j) = i$. We refer to $j$ as the *partner* (roommate) of $\mu(i)$ at matching $\mu$, and in case $\mu(i) = i$, $i$ is said to be a *single* at matching $\mu$. A matching in which every agent is single is referred to as the *identity matching* and is denoted by $\mu^I$. We denote the set of all possible matchings on $N$ by $\mathcal{M}(N)$. Given any problem $P = (N, R)$ and any two matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$, the set of agents that are preferred between $\mu(i)$ and $\bar{\mu}(i)$ according to $R_i$ forms an *interval* denoted by $[\mu, \bar{\mu}]_{R_i}$. Formally,

$$[\mu, \bar{\mu}]_{R_i} = \{j \in N : \mu \; R_i \; j \; R_i \; \bar{\mu} \;\text{ or }\; \bar{\mu} \; R_i \; j \; R_i \; \mu\}.$$

The *length* of an interval is denoted by $|\mu, \bar{\mu}|_{R_i} = \#[\mu_1, \mu_2]_{R_i} - 1$, i.e., the cardinality of the interval minus 1 (as an example in Figure 4, $[\mu^1, \mu^3]_{R_1} = \{2, 4, 3\}$ and $|\mu^1, \mu^3|_{R_1} = 2$ ).

We say a matching $\bar{\mu}$ is *between* matchings $\mu$ and $\bar{\bar{\mu}}$, if $\bar{\mu}(i) \in [\mu, \bar{\bar{\mu}}]_{R_i}$ for all $i \in N$, i.e., $\bar{\mu}(i)$ is contained in the interval defined by $\mu$ and $\bar{\bar{\mu}}$ for all agent. Given any sequence of matching $\mu^1, \ldots, \mu^t$ in $\mathcal{M}(N)$ we say $\mu^1, \ldots, \mu^t$ are "on a line" and denote it as $[\mu^1 - \mu^2 - \cdots - \mu^t]$ if $\mu^j$ is between $\mu^i$ and $\mu^k$ for all $1 \leqslant i \leqslant j \leqslant k \leqslant t$. We say a matching $\mu$ is *weakly above* $\bar{\mu}$ whenever $\mu(i) \; R_i \; \bar{\mu}(i)$ for all $i \in N$. In addition, we say $\mu$ and $\bar{\mu}$ are *adjacent* whenever $|\mu, \bar{\mu}|_{R_i} = 1$ for all $i \in N$.

Consider problem $P = (N, R)$ and $\pi$ as a permutation over the set of agents. We denote the *permuted preference profile* by $R^\pi$ where for all $i, j, k \in N$, $j \; R_i \; k$ if and only if $\pi(j) \; R^\pi_{\pi(i)} \; \pi(k)$ and define the *permuted problem* $P^\pi = (N, R^\pi)$ accordingly. Given a matching $\mu \in \mathcal{M}(N)$, we denote the *permuted matching* by $\mu^\pi$ where for all $i, j \in N$, $\mu(i) = j$ if and only if $\mu^\pi(\pi(i)) = \pi(j)$.[10] The permutations are denoted by the cycle notation, e.g., $\pi = (123)(45)$ denotes $\pi(1) = 2$, $\pi(2) = 3$, $\pi(3) = 1$, $\pi(4) = 5$, $\pi(5) = 4$ and $\pi(i) = i$ for all $i \in N \setminus \{1, 2, 3, 4, 5\}$.

We consider particular variable population scenarios wherein a problem is extended by an *irrelevant* newcomer. Consider a problem $P = (N, R)$, and a newcomer agent $a \in \mathcal{N} \setminus N$, and an *extension of the problem* $\widehat{P} = (N \cup \{a\}, \widehat{R})$ by agent $a$, such that for all $i, j \in N$ we have,

1. $rank(a, \widehat{R}_a) = 1$, i.e., the newcomer prefers to be single,
2. $rank(j, R_i) = rank(j, \widehat{R}_i)$, i.e., preferences of the incumbents over incumbents are unchanged,
3. $rank(j, \widehat{R}_i) < rank(a, \widehat{R}_i)$, i.e., the newcomer is ranked at the bottom by all incumbents.

Similarly, we say $\bar{\mu} \in \mathcal{M}(N \cup \{a\})$, is the extension of a matching $\mu \in \mathcal{M}(N)$ by agent $a \in \mathcal{N} \setminus N$, whenever $\bar{\mu}(i) = \mu(i)$ for all $i \in N$, and $\bar{\mu}(a) = a$. Let $A = \{a_1, a_2, \ldots, a_k\}$, be a set of agents such

[10] This is a typical definition for permutations in roommate markets, as examples of this see Klaus (2017); Özkal-Sanver (2010); Sasaki and Toda (1992).

that $N \cap A = \emptyset$. Consider the sequence $P^{a_0}, P^{a_1}, P^{a_2}, \ldots, P^{a_k}$ of problems such that $P^{a_0} = P$ and, $P^{a_t}$ is an extension of $P^{a_{t-1}}$ by agent $a_t \in A$. Then we say $P^{a_k}$ is an extension of $P$ by the set of agents $A$. Similarly, we can define the extension of a matching with a set of agents.

## 3 Conditions and measure

Given a problem $P = (N, R) \in \mathcal{D}$, and two matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$, a *distance function*, $\delta_P(\mu, \bar{\mu}) : \mathcal{M}(N) \times \mathcal{M}(N) \to R_+$ assigns a non-negative real number to each pair of matchings. We consider *measures* on matchings, i.e., collections of distance functions on all possible problems in the domain, denoted by:

$$\delta = \langle \delta_P \rangle_{P \in \mathcal{D}}.$$

**Condition 1 (Metric conditions):** $\delta$ satisfies *metric condition* if for all problems $P = (N, R) \in \mathcal{D}$ and all matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$

I. $\delta_P(\mu, \bar{\mu}) \geqslant 0$ ( non-negativity),

II. $\delta_P(\mu, \bar{\mu}) = 0$ if and only if $\mu = \bar{\mu}$ (identity of indiscernibles),

III. $\delta_P(\mu, \bar{\mu}) = \delta_P(\bar{\mu}, \mu)$ (symmetry),

IV. $\delta_P(\mu, \bar{\bar{\mu}}) \leq \delta_P(\mu, \bar{\mu}) + \delta_P(\bar{\mu}, \bar{\bar{\mu}})$ (triangular inequality).

The following condition is the betweenness condition which is an strengthening of the triangular inequality. It requires that if a matching is such that the partner of each agent is included in the interval defined by two other matching then the measure should be additive on these three matchings.

**Condition 2 (Betweenness):** $\delta$ satisfies *betweenness* if for all problems $P = (N, R) \in \mathcal{D}$ and for all matchings $\mu, \bar{\mu}, \bar{\bar{\mu}} \in \mathcal{M}(N)$ such that $\bar{\mu}$ is between $\mu, \bar{\bar{\mu}}$

$$\delta_P(\mu, \bar{\bar{\mu}}) = \delta_P(\mu, \bar{\mu}) + \delta_P(\bar{\mu}, \bar{\bar{\mu}}).$$

The anonymity condition is a well-known condition which implies that the naming of the agents are irrelevant.

**Condition 3 (Anonymity):** $\delta$ satisfies *anonymity* if for all problems $P = (N, R) \in \mathcal{D}$ and for all matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$ and permutation $\pi : N \to N$

$$\delta_P(\mu, \bar{\mu}) = \delta_{P^\pi}(\mu^\pi, \bar{\mu}^\pi).$$

Consider two problem with the same matchings such that in one of them the interval is expanded at least for one agent. Then it is natural to require that the measure assign a greater value to the problem with greater interval. Monotonicity condition reflect this idea.

**Condition 4 (Monotonicity):** $\delta$ satisfies *monotonicity* if for all problems $P = (N, R) \in \mathcal{D}$ and $\widehat{P} = (N, \widehat{R}) \in \mathcal{D}$ and all matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$ such that $[\mu, \bar{\mu}]_{R_i} \subseteq [\mu, \bar{\mu}]_{\widehat{R}_i}$ for all $i \in N$

$$\delta_P(\mu, \bar{\mu}) \leq \delta_{\widehat{P}}(\mu, \bar{\mu}).$$

**Remark 1.** *Immediate implication of monotonicity is that if for two matchings $\mu$ and $\bar{\mu}$, the intervals remain the same across two problems, then the distance should not change. Furthermore changing the relative order of $\mu$, $\bar{\mu}$ in individual preferences, does not alter the distance as long as the intervals remain the same.*

The following condition deals with the extension of the problems and the extension of matchings. That is if a problem as well as the two matchings are extended by a newcomer the measure should be unchanged.

**Condition 5 (Independence of irrelevant newcomers):** $\delta$ satisfies *Independence of irrelevant newcomers* if for all problem $P = (N, R) \in \mathcal{D}$ and any extension $\widehat{P} = (\widehat{N}, \widehat{R}) \in \mathcal{D}$ and all matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$ with the extension $\mu^*, \bar{\mu}^* \in \mathcal{M}(\widehat{N})$ by some agent $a \in \mathcal{N} \setminus N$

$$\delta_P(\mu, \bar{\mu}) = \delta_{\widehat{P}}(\mu^*, \bar{\mu}^*).$$

**Remark 2.** *Immediate implication of independence of irrelevant newcomers is that if $\widehat{P}, \mu^*, \bar{\mu}^*$ are an extension of the $P, \mu, \bar{\mu}$, by a set of agents $A$, then $\delta_P(\mu, \bar{\mu}) = \delta_{\widehat{P}}(\mu^*, \bar{\mu}^*)$.*

The normalization condition acts as an anchor point and sets the minimum distance for any two disjoint matching, i.e., matchings that have nothing in common, to be in accordance with the size of the problem. Formally, given any problem $P = (N, R) \in \mathcal{D}$, and any two matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$, we say $\mu$ and $\bar{\mu}$ are *disjoint*, whenever $\mu(i) \neq \bar{\mu}(i)$ for all $i \in N$.

**Condition 6 (Normalization):** $\delta$ satisfies *normalization* if for all disjoint matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$

$$\min_{P \in \mathcal{P}(N)} \{\delta_P(\mu, \bar{\mu})\} = |N|.$$

Finally, we define the Borda measure. The Borda measure for a given problem is the total sum of absolute value of the rank difference between each agent's partner in two given matching. This corresponds to the sum of the lengths of each individual interval formed by the two matchings. Formally,

**Borda Measure:** A measure is called the *Borda Measure*, denoted by $\delta^{Borda}$, whenever for all $P = (N, R) \in \mathcal{D}$, and for all matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$

$$\delta_P^{Borda}(\mu, \bar{\mu}) = \sum_{i \in N} \left|\mu, \bar{\mu}\right|_{R_i}.$$

The reader can verify that the Borda measure satisfies the six conditions introduced above. We will show that it is indeed the only measure which satisfies all these conditions, hence a characterization. In the following sections, we use $P = (N, R) \in \mathcal{D}$ to denote a generic problem with $|N| = n$ and $\delta_P$ denotes associated distance function with this problem which satisfies metric, anonymity, betweenness, monotonicity, independence of irrelevant newcomers and normalization.

# 4 Results

First, we propose a lemma which enables to decompose the distance between any two matchings into (possibly) several components (matchings). Then, by using this lemma in section 4.1, we show that the distance between any matching and the identity matching equals the Borda measure. Finally, in section 4.2 by using the results of section 4.1 we show that the distance between any two matchings equals the Borda measure.

To state the first lemma, let $\mu, \bar{\mu} \in \mathcal{M}(N)$ be two matchings and $S$ be a subset of agents that are matched among themselves in $\mu$ and $\bar{\mu}$. Formally, let $S \subseteq N$ denote a subset of agents such that $\mu(i), \bar{\mu}(i) \in S$ for all $i \in S$. Based on the set $S$ we define the following two matchings, say $\mu^S$ and $\mu^{\bar{S}}$

1. for all $i \in S$ let $\mu^S(i) = \mu(i)$ and for all $i \in N \setminus S$ let $\mu^S(i) = \bar{\mu}(i)$,
2. for all $i \in S$ let $\mu^{\bar{S}}(i) = \bar{\mu}(i)$ and for all $i \in N \setminus S$ let $\mu^{\bar{S}}(i) = \mu(i)$.

In the following lemma we show that the distance between $\mu, \bar{\mu}$ can be decomposed into the sum of the distances from $\mu^S$ and $\mu^{\bar{S}}$ to $\mu$ (or $\bar{\mu}$). Figure 1 shows a demonstration of this decomposition.

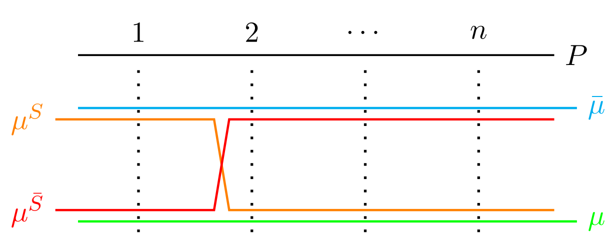


Figure 1: The general view of the Decomposition Lemma.

**Lemma 1** (Decomposition Lemma)**.** *Let $\mu, \bar{\mu} \in \mathcal{M}(N)$. Then, for all $S \subseteq N$ such that $\mu(i), \bar{\mu}(i) \in S$ for all $i \in S$, we have*

$$\delta_P(\mu, \bar{\mu}) = \delta_P(\mu, \mu^{\bar{S}}) + \delta_P(\mu, \mu^S) = \delta_P(\mu^{\bar{S}}, \bar{\mu}) + \delta_P(\mu^S, \bar{\mu}).$$

*Proof.* By definition of $\mu^{\bar{S}}$ and $\mu^S$, both are between $\mu$ and $\bar{\mu}$, hence betweenness yields

$$\delta_P(\mu, \bar{\mu}) = \delta_P(\mu, \mu^{\bar{S}}) + \delta_P(\mu^{\bar{S}}, \bar{\mu}) \text{ and,} \tag{1}$$

$$\delta_P(\mu, \bar{\mu}) = \delta_P(\mu, \mu^S) + \delta_P(\mu^S, \bar{\mu}). \tag{2}$$

Since $\mu$ and $\bar{\mu}$ are both between $\mu^S$ and $\mu^{\bar{S}}$ betweenness results in

$$\delta_P(\mu^S, \mu^{\bar{S}}) = \delta_P(\mu^S, \mu) + \delta_P(\mu, \mu^{\bar{S}}) \text{ and,}$$
$$\delta_P(\mu^S, \mu^{\bar{S}}) = \delta_P(\mu^S, \bar{\mu}) + \delta_P(\bar{\mu}, \mu^{\bar{S}}).$$

The four above equations yield

$$\delta_P(\mu, \mu^{\bar{S}}) + \delta_P(\mu^{\bar{S}}, \bar{\mu}) = \delta_P(\mu, \mu^S) + \delta_P(\mu^S, \bar{\mu}) \text{ and,}$$
$$\delta_P(\mu^S, \mu) + \delta_P(\mu, \mu^{\bar{S}}) = \delta_P(\mu^S, \bar{\mu}) + \delta_P(\bar{\mu}, \mu^{\bar{S}}).$$

by subtracting these from each other $\delta_P(\mu^{\bar{S}}, \bar{\mu}) - \delta_P(\mu^S, \mu) = \delta_P(\mu, \mu^S) - \delta_P(\bar{\mu}, \mu^{\bar{S}})$. Hence,

$$\delta_P(\mu^{\bar{S}}, \bar{\mu}) = \delta_P(\mu^S, \mu). \tag{3}$$

Plugging Equation 3 into Equation 1 yields $\delta_P(\mu, \bar{\mu}) = \delta_P(\mu, \mu^{\bar{S}}) + \delta_P(\mu, \mu^S)$, and plugging Equation 3 into Equation 2 results in $\delta_P(\mu, \bar{\mu}) = \delta_P(\mu^{\bar{S}}, \bar{\mu}) + \delta_P(\mu^S, \bar{\mu})$. ■

## 4.1 Comparing any matching with the identity matching

In this section we focus on the distance between any matching and the identity matching. By Remark 1, as long as the intervals between two matchings remain the same the distance will be unchanged. Therefore without loss of generality, we draw the identity matching below the one-couple matchings. Furthermore, in order to keep figures simple, we often denote the matchings as straight lines whenever possible.

Consider a matching in which only two agents, say $\mu(i) = j$ with $i \neq j$, are matched and the rest of the agents are single. We call such a matching a *one-couple matching* and denote it by $\mu^{ij}$. Given a problem $P = (N, R)$, we say a one-couple matching $\mu^{ij}$ is of length $(x, y)$ whenever $(|\mu^{ij}(i), i|_{R_i}, |\mu^{ij}(j), j|_{R_j}) = (x, y)$. Note that for a one-couple matching of length $(x, y)$, $\delta_P^{Borda}(\mu^{ij}, \mu^I) = x + y$.

**Remark 3.** *Consider any matching $\mu$ with $k$ distinct couples. Then, by Decomposition Lemma, and letting $S = \{i, j\}$ and $\bar{S} = N \setminus S$ for each couple of $\mu$, the distance between $\mu$ and $\mu^I$ can be decomposed as the sum of distances of each of these $k$ one-couple matchings, and the identity matching.*

According to Remark 3, to compute the distance between any matching and the identity matching, we only need to focus on the distance between a one-couple matching and the identity matching. Then the total distance equals to the sum of each of these one-couple matchings. In the sequel, we will show that the distance between a one-couple matching and identity matching is the same for all problems whenever the interval lengths are the same. In Lemma 2, we show this for the case where the interval length is $(x, 1)$, see Figure 2. Then in Lemma 3, we extend this to any interval length $(x, y)$, see Figure 3.

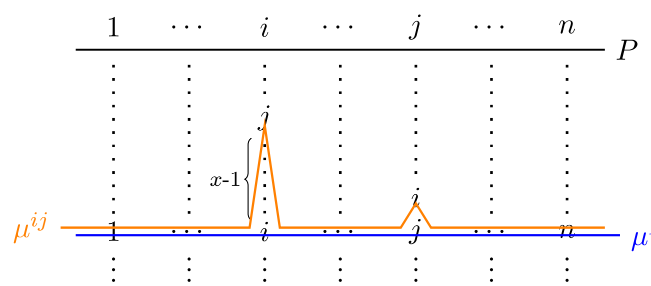


Figure 2: A one-couple matching $\mu^{ij}$ with interval length $(x, 1)$.

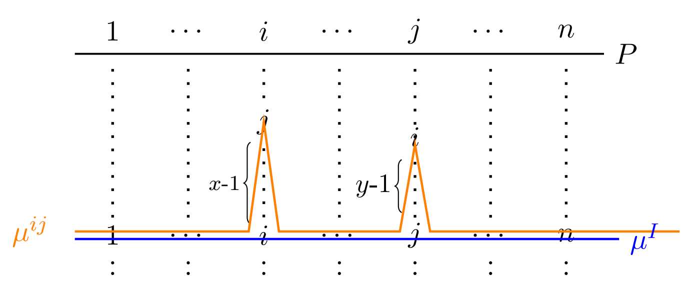


Figure 3: A one-couple matching $\mu^{ij}$ with interval length $(x, y)$.

**Lemma 2.** *Consider any $N, N' \subsetneq \mathcal{N}$. Let $\mu^{ij} \in \mathcal{M}(N)$ ba a one-couple matching with interval length $(x, 1)$ in any problem $P \in \mathcal{P}(N)$, and $\mu^{i'j'} \in \mathcal{M}(N')$ ba a one-couple matching with interval length $(x, 1)$ in any problem $P' \in \mathcal{P}(N')$, where $x$ is a strictly positive integer. Let $\mu^I$ and $\mu^{I'}$ denote the identity matchings in corresponding problems, then*

$$\delta_P(\mu^{ij}, \mu^I) = \delta_{P'}(\mu^{i'j'}, \mu^{I'}).$$

*Proof.* See Appendix A.1. ■

Lemma 2, shows that the distance between the identity matching and all one-couple matchings with interval length $(x, 1)$, are equal to each other across all the problems. To simplify notation we denote this distance by $\alpha_{x1}$. The next lemma extends Lemma 2 to any one-couple matching with interval length $(x, y)$.

**Lemma 3.** *Consider any set of agents and any problem $N \subsetneq \mathcal{N}$, and any problem $P \in \mathcal{P}(N)$. Let $\mu^{ij} \in \mathcal{M}(N)$ be a one-couple matching with interval length $(x, y)$, where $x$ and $y$ are strictly positive integers. Then*

$$\delta_P(\mu^{ij}, \mu^I) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}.$$

*Proof.* See Appendix A.2. ■

Lemma 3, shows that the distance between the identity matching and all one-couple matchings with interval length $(x, y)$, are equal to each other across all the problems. To simplify notation we denote this distance by $\alpha_{xy}$. Next as a particular case of Lemma 3, we show that for any strictly positive integer $x$, $\alpha_{xx} = 2x$, i.e., the distance between a one-couple matching with interval length $(x, x)$ and the identity matching must equal $2x$.

**Lemma 4.** *Let $\mu^{ij}$ be a one-couple matching in $\mathcal{M}(N)$ with interval length $(x, x)$ where $x$ is a strictly positive integer. Then $\delta_P(\mu^{ij}, \mu^I) = 2x$.*

*Proof.* See Appendix A.3. ■

Now we propose our first result, Theorem 1. When all six conditions in Section 2 are imposed on a measure $\delta$, Theorem 1 states that the distance between the identity matching and any other matching must equal the Borda measure.

**Theorem 1.** *For any problem $P = (N, R)$ and any $\mu \in \mathcal{M}(N)$ we have*

$$\delta_P(\mu, \mu^I) = \delta_P^{Borda}(\mu, \mu^I).$$

*Proof.* By Decomposition Lemma, the distance between $\mu$ and the identity matching, is the sum of $k$ distinct one-couple matchings. By Lemma 3, the distance between any one-couple matching with interval length $(x, x)$ and the identity matching is $\alpha_{xx} = 2\alpha_{x1} - \alpha_{11}$. By Lemma 4, $\alpha_{xx} = 2x$ and $\alpha_{11} = 2$, hence $\alpha_{x1} = \frac{2x+2}{2} = x + 1$. Using Lemma 3 the distance between any one-couple matching with interval length $(x, y)$ and the identity matching is $\alpha_{xy} = \alpha_{x1} + \alpha_{y1} - \alpha_{11}$. Therefore,

$\alpha_{xy} = (x+1) + (y+1) - 2 = x + y$, which is the same as the Borda measure for any one-couple matching with the interval length $(x, y)$. ■

## 4.2 Comparing any two non-identity matching

In this section, we generalize Theorem 1 to any two matchings. To do so, first we porpose two propositions for four agents problems, and use these two propositions as the blocks to prove Theorem 2.

**Proposition 1.** *Consider a problem $P$ over four agents with the preference profile and the matchings shown in Figure 4. Note that one singleton is nested between $\mu^1$ and $\mu^2$ and one is nested between $\mu^2$ and $\mu^3$. In such specific cases, $\delta_P(\mu^1, \mu^2) = \delta_P(\mu^2, \mu^3) = \alpha_{21} + \alpha_{11}$, i.e., they are equal the Borda measure.*

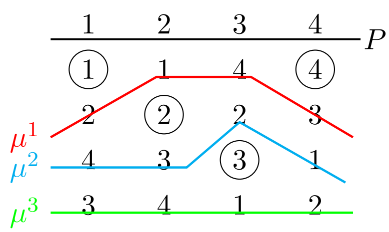


Figure 4: A problem over four agent with one singleton agent between the matchings.

*Proof.* See Appendix B.1. ■

**Proposition 2.** *Consider a problem $P$ over four agents with the preference profile and the matchings shown in Figure 5. Note that two singletons are nested between $\mu^1$ and $\mu^2$ and two are nested between $\mu^2$ and $\mu^3$. In such specific cases, $\delta_P(\mu^1, \mu^2) = \delta_P(\mu^2, \mu^3) = \alpha_{22} + \alpha_{11}$, i.e., they are equal to Borda measure.*

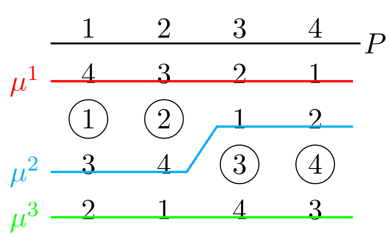


Figure 5: Problem $P$ over four agents with two singleton agents between the matchings.

*Proof.* See Appendix B.2. ■

Finally we propose our main result, Theorem 2. Note that the previous theorem, Theorem 1, shows the measure to be equal to the Borda measure when comparing a matching only with the identity matching. Theorem 2 extends this result to comparing a matching with any other. When all six conditions in Section 2 are imposed on a measure $\delta$, Theorem 2 states that the distance between any two matching must equal the Borda measure.

**Theorem 2.** *For any problem $P = (N, R)$ and $\mu, \tilde{\mu} \in \mathcal{M}(N)$, we have*

$$\delta_P(\mu, \tilde{\mu}) = \delta_P^{Borda}(\mu, \tilde{\mu}).$$

*Proof.* Take any $N = \{1, 2, \ldots, n\}$ as the set of agents and consider any $P \in \mathcal{P}(N)$. If $\mu = \tilde{\mu}$, then by metric conditions $\delta_P(\mu, \tilde{\mu}) = 0 = \sum\limits_{i \in N} |\mu, \tilde{\mu}|_{R_i}$ and it equals to $\delta_P^{Borda}(\mu, \tilde{\mu})$. If $\mu = \mu^I$ (or $\tilde{\mu} = \mu^I$), then by Theorem 1, $\delta_P(\mu, \tilde{\mu}) = \delta_P^{Borda}(\mu, \tilde{\mu})$. Therefore consider any $\mu, \tilde{\mu} \in \mathcal{M}(N) \setminus \{\mu^I\}$ such that $\mu \neq \tilde{\mu}$.

In what follows, if the number of agents is odd, we can use extensions of $P, \mu, \tilde{\mu}$ by one irrelevant newcomer. By independence of irrelevant newcomers, the distance is unchanged. So without loss of generality we can assume that the number of agents to be even.

By Remark 1, we assume that $\mu$ is weakly above $\tilde{\mu}$. Let $N' = \{1', 2', \ldots, n'\}$, be a set of agents such that $|N| = |N'|$ and $N \cap N' = \emptyset$. Let $\bar{N} = N \cup N'$. Let $P^*, \mu^*, \tilde{\mu}^*$ be an extension of $P, \mu, \tilde{\mu}$ by the set $N'$. By Remark 2, $\delta_P(\mu, \tilde{\mu}) = \delta_{P^*}(\mu^*, \tilde{\mu}^*)$. For simplicity we abuse the notation and write $P$, $\mu$ and $\tilde{\mu}$ instead of writing $P^*, \mu^*$ and $\tilde{\mu}^*$, respectively. Let us define two additional matchings $\mu^B, \mu^T \in \mathcal{M}(\bar{N})$ such that:

- for all $i \in N$, $\mu^B(i) = i' \in N'$,
- for all odd $i \in N$, $\mu^T(i) = (i+1)' \in N'$ and for all even $i \in N$, $\mu^T(i) = (i-1)' \in N'$.

Next we construct the following problem $\bar{P} = (\bar{N}, \bar{R})$ on the same set of agents, $\bar{N}$ such that

1. $[\mu, \tilde{\mu}]_{R_i} = [\mu, \tilde{\mu}]_{\bar{R}_i}$ for all $i \in \bar{N}$, i.e., the intervals of $\mu$ and $\tilde{\mu}$ in $\bar{P}$ are the same as those in $P$,
2. $\mu^T$ is weakly above $\mu^B$, $\mu^B$ is weakly above $\mu$ (and they are adjacent), and $\mu$ is weakly above $\tilde{\mu}$,
3. if $i \in [\mu, \tilde{\mu}]_{\bar{R}_i}$ then $[\mu^T, \mu^B]_{\bar{R}_i} = \{\mu^T(i), \mu^B(i)\}$, i.e., if $i$ is nested between $\mu(i)$ and $\tilde{\mu}(i)$ then no agent is nested between $\mu^T(i)$ and $\mu^B(i)$,
4. if $i \notin [\mu, \tilde{\mu}]_{\bar{R}_i}$ then $[\mu^T, \mu^B]_{\bar{R}_i} = \{\mu^T(i), i, \mu^B(i)\}$, i.e., if $i$ is not nested between $\mu(i)$ and $\tilde{\mu}(i)$ then only $i$ is nested between $\mu^T(i)$ and $\mu^B(i)$.

Figure 6 shows a general view of the structure of problem $\bar{P}$.

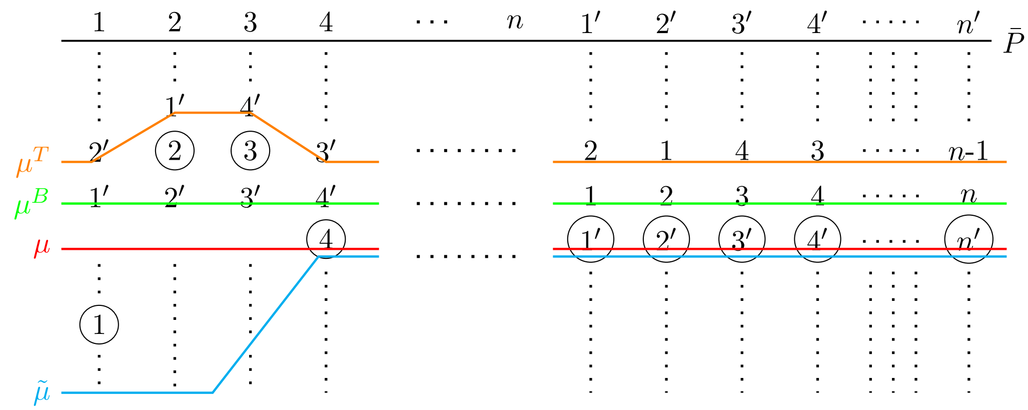


Figure 6: General view.

Note that by construction $[\mu, \tilde{\mu}]_{R_i} = [\mu, \tilde{\mu}]_{\bar{R}_i}$ for all $i \in \bar{N}$ therefore by monotonicity $\delta_P(\mu, \tilde{\mu}) = \delta_{\bar{P}}(\mu, \tilde{\mu})$. Since $[\mu^T - \mu^B - \mu - \tilde{\mu}]$ are on a line, then by betweenness $\delta_{\bar{P}}(\mu^T, \tilde{\mu}) = \delta_{\bar{P}}(\mu^T, \mu^B) + \delta_{\bar{P}}(\mu^B, \mu) + \delta_{\bar{P}}(\mu, \tilde{\mu})$ and hence

$$\delta_{\bar{P}}(\mu, \tilde{\mu}) = \delta_{\bar{P}}(\mu^T, \tilde{\mu}) - \delta_{\bar{P}}(\mu^T, \mu^B) - \delta_{\bar{P}}(\mu^B, \mu). \tag{4}$$

In the next three steps we show that the distance between each of the three pairs of matchings on the right hand side of Equation 4 equals the Borda measure. This in return implies $\delta_{\bar{P}}(\mu, \tilde{\mu}) = \delta_{\bar{P}}^{Borda}(\mu, \tilde{\mu})$.

**Step 1.** (Showing that $\delta_{\bar{P}}(\mu^T, \tilde{\mu})$ equals Borda measure.) By construction of $\bar{P}$, $[\mu^T - \mu^I - \tilde{\mu}]$ are on a line. Then by betweenness and Theorem 1, $\delta_{\bar{P}}(\mu^T, \tilde{\mu}) = \delta_{\bar{P}}^{Borda}(\mu^T, \tilde{\mu})$.

**Step 2.** (Showing that $\delta_{\bar{P}}(\mu^T, \mu^B)$ equals Borda measure.) Consider the following partition of $\bar{N}$ into sets of agents each, $T_1 = \{1, 2, 1', 2'\}$, $T_2 = \{3, 4, 3', 4'\}$, $\ldots$, $T_{\frac{n}{2}} = \{n-1, n, (n-1)', n'\}$ where $\bar{N} = \bigcup_{l=1}^{\frac{n}{2}} T_l$. Let $\mu^{T_l}$ denote a matching where $\mu^{T_l}(i) = \mu^T(i)$ for all $i \in T_l$, and $\mu^{T_l}(i) = \mu^B(i)$ for all $i \in \bar{N} \setminus T_l$. By construction for all $l \in \{1, \ldots, \frac{n}{2}\}$, $\mu^{T_l}$ is between $\mu^T$ and $\mu^B$. By Decomposition Lemma

$$\delta_{\bar{P}}(\mu^T, \mu^B) = \sum_{l=1}^{\frac{n}{2}} \delta_{\bar{P}}(\mu^{T_l}, \mu^B).$$

In the squeal, we denote a generic $\mu^{T_l}$ by simply $\mu^S$. Based on the construction of $\mu^T$ and $\mu^B$, each of these matchings $\mu^S$ will have one of the following three structures,

1. No singleton is nested between $\mu^S$ and $\mu^B$ (see Figure 7),

2. Only one singleton is nested between $\mu^S$ and $\mu^B$ (see Figure 8 and 11),

3. Two singleton is nested between $\mu^S$ and $\mu^B$ (see Figure 12).

In each case we show that the distance equals the Borda measure.

- **Case 1. (no singleton)** Consider the case that no singleton is nested between $\mu^S$ and $\mu^B$ (Figure 7).

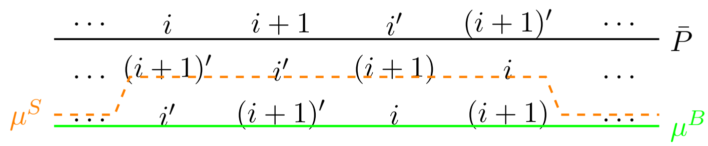


Figure 7: The no singleton structure.

By construction of $\bar{P}$ we can consider any problem $\bar{\bar{P}}$ where $[\mu^S - \mu^B - \mu^I]$ are on a line, and the intervals of $\mu^S$ and $\mu^B$ are unchanged, i.e., $[\mu^S, \mu^B]_{\bar{R}_i} = [\mu^S, \mu^B]_{\bar{\bar{R}}_i}$ for all $i \in \bar{N}$. By Remark 1, the distance is unchanged, therefore by betweenness, and Theorem 1, $\delta_{\bar{P}}(\mu^S, \mu^B)$ also equals the Borda measure.

- **Case 2. (one singleton)** Consider the case that exactly one singleton is nested between $\mu^S$ and $\mu^B$. By construction of $\mu^T$ and $\mu^B$ the singleton is either $i$ or $i+1$. Therefore, two situations are plausible,

I. $i$ is the singleton nested (see Figure 8).

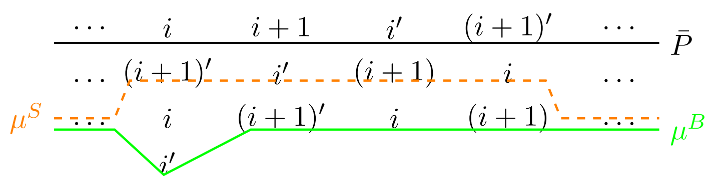


Figure 8: The one singleton structure with $i$ as the singleton.

Consider the four agent problem $P$ in Proposition 1, and rename the agents as $2 = i, 4 = i+1, 3 = i'$ and $1 = (i+1)'$. Let $\widehat{P}$ be an extension of this problem $P$, by the set of agents $A = \bar{N} \setminus \{i, i', (i+1), (i+1)'\}$, and $\widehat{\mu}^1$ and $\widehat{\mu}^2$ be the extension of $\mu^1$ and $\mu^2$ by the set $A$, respectively (see Figure 9). By Remark 2,

$$\delta_P(\mu^1, \mu^2) = \delta_{\widehat{P}}(\widehat{\mu}^1, \widehat{\mu}^2). \tag{5}$$

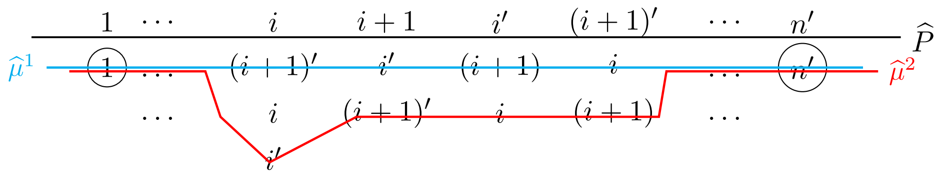


Figure 9: The four agents problem $P$ of Proposition 1 after adding the set of agents $A$ as irrelevant newcomers.

Now, consider another problem $P'$ shown in Figure 10. Note that by monotonicity

$$\delta_{\widehat{P}}(\widehat{\mu}^1, \widehat{\mu}^2) = \delta_{P'}(\widehat{\mu}^1, \widehat{\mu}^2). \tag{6}$$

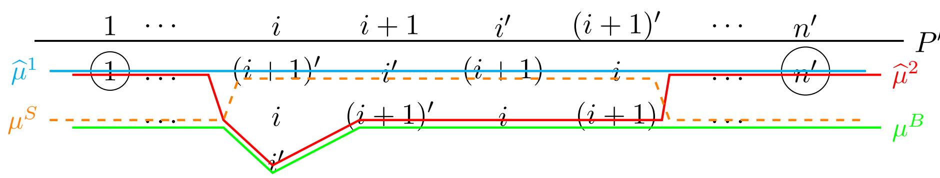


Figure 10: The problem $P'$.

Note that the structure of the four matchings, $\widehat{\mu}^1, \widehat{\mu}^2, \mu^S\mu$, in problem $P'$ above corresponds to the four matchings in Figure 1 (to $\bar{\mu}, \mu^{\bar{S}}, \mu^S, \mu^B$ respectively). Therefore by Equation 3 in Decomposition Lemma we have

$$\delta_{P'}(\widehat{\mu}^1, \widehat{\mu}^2) = \delta_{P'}(\mu^S, \mu^B). \tag{7}$$

Putting together Equations 5, 6 and 7 results in $\delta_P(\mu^1, \mu^2) = \delta_{P'}(\mu^S, \mu^B)$. As in Proposition 1, we showed that $\delta_P(\mu^1, \mu^2)$ equals the Borda measure then $\delta_{P'}(\mu^S, \mu^B)$ also equals the Borda measure. Furthermore by monotonicity, $\delta_{P'}(\mu^S, \mu^B) = \delta_{\bar{P}}(\mu^S, \mu^B)$, hence $\delta_{\bar{P}}(\mu^S, \mu^B)$ equals the Borda measure.

II. $(i+1)$ is the singleton nested (see Figure 11).

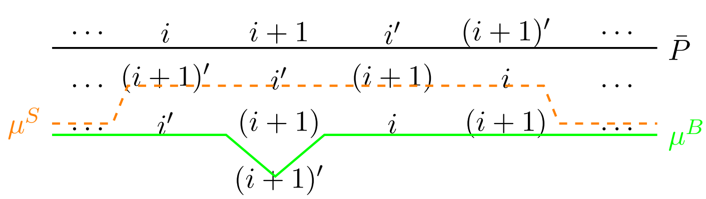


Figure 11: The one singleton structure with $i+1$ as the singleton.

By Proposition 1, and renaming the agents as $4 = i, 2 = i+1, 1 = i'$ and $3 = (i+1)'$ and a similar argument to aforementioned, we have $\delta_{\bar{P}}(\mu^S, \mu^B)$ equals the Borda measure.

- **Case 3. (two singleton)** Consider the case that exactly two singletons are nested between $\mu^S$ and $\mu^B$. By the construction of $\mu^T$ and $\mu^B$ only $i$ and $i+1$ can be the singletons (see Figure 12).

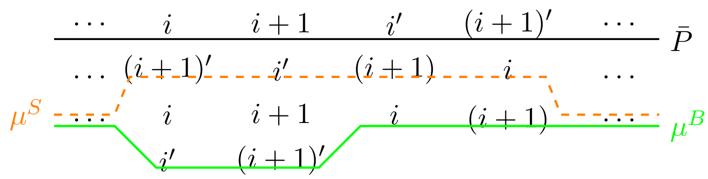


Figure 12: The two singleton structure with both $i$ and $i+1$ as the singleton agents.

Renaming the agents in Proposition 2, as $1 = i, 2 = i+1, 3 = i', 4 = (i+1)'$ and by a similar argument to the situation where $i$ single, we have $\delta_{\bar{P}}(\mu^S, \mu^B)$ equals the Borda measure.

**Step 3.** (Showing that $\delta_{\bar{P}}(\mu^B, \mu)$ equals the Borda measure.) By construction of $\bar{P}$ we can consider any problem $\bar{\bar{P}}$ where $[\mu^B - \mu - \mu^I]$ are on a line, and the intervals of $\mu^B$ and $\mu$ are unchanged, i.e., $[\mu^B, \mu]_{\bar{R}_i} = [\mu^B, \mu]_{\bar{\bar{R}}_i}$ for all $i \in \bar{N}$. By Remark 1, the distance is unchanged, therefore by betweenness, and Theorem 1, $\delta_{\bar{P}}(\mu^B, \mu)$ also equals the Borda measure.

All in all, putting together the above three steps and Equation 4, completes the proof. ■

## 5 Conclusion

Different mechanisms exhibit various desirable (or undesirable) features. In case a social planner decides to transform the design of a system by changing the mechanism employed, the question is how much change this will mean for individuals. This paper proposes a way to quantify this difference based on the outputs mechanisms produce, i.e., matchings. We interpret this difference as the cost of transformation in general. We quantify the cost of transformation by introducing normative conditions on functions. These conditions mostly address the effect of the transformation on individuals, instead of merely looking at the number of disjoint matches. We introduced the Borda measure and showed this to be the only one satisfying these conditions on the domain of roommate markets.

The Borda measure is an attempt to answer “how much?”. Depending on the context and the matchings compared, the measure is interpreted as a parameter that quantifies different things. In the sequel we briefly contemplate about possible applications of this measure and interpretations of it. We also dwell upon some open questions and directions for future research.

## 5.1 Possible applications

As we have discussed, different mechanisms have different computational requirements and different properties. Therefore, there is a trade-off between gains and losses caused by switching between two mechanisms. In such situations a measure can be used as a tool which quantifies the trade-off between these mechanisms, and hence gives the cost of transformation for changing one mechanism to another. It can be considered as the cost of changing an efficient output with a stable one (such an argument works for the school choice market). It can also be considered as the cost of sacrificing some desirable feature, like strategy proofness, in order to get some other. Another application is quantifying the cost of changing an unstable output to a stable one, i.e. cost of stability. One can similarly utilize the measure as parameters of cost of efficiency, cost of strategy proofness, and cost of fairness, etc.

Consider cases when a designer needs a refinement from a set of matchings, perhaps induced by a solution concept for a market. Note that the Borda measure can be used to compare a restricted set of matchings only, e.g., the core of matching markets which contains multiple stable matchings. Borda measure can act as a tool to refine this set or make a choice among the stable matchings. In marriage markets the interpretation is very exciting. The core of marriage markets forms a lattice structure with men-optimal and women-optimal matchings as two extremes. It is not difficult to see that, within the core, these two matchings are the farthest pair (due to the lattice structure). The Borda measure can pick a stable matching which has the minimal total distance to all other stable matchings, acting as a tool to find the "median stable matching" in the lattice. Furthermore, given any choice among the set of stable matchings, one can immediately measure, how "close" this outcome is to the men-optimal (or women-optimal) stable matchings. Namely, the measure quantifies the "gender-bias", or "favorism". In case the outcome was of a deliberate policy, it can be utilized as a measure of positive-discrimination towards a particular side of the market.

## 5.2 Directions for further research

Our framework assumes no indifference in preferences, i.e., there are no ties. The measure could possibly be extended to address the cases where ties are allowed. In such domains, however, there can be situations where there are two disjoint matchings and every agent is indifferent between their partners in these matchings. In such situations, one approach is to assign zero value to the distance of the matchings. This, however, violates the metric condition, in particular the identity of indiscernibles. So metric condition must be modified into a pseudometric[11] condition for that approach. This allows two disjoint matchings to also admit a value equal to zero.

The framework we developed for the Borda measure is applicable to one-to-one matching markets only. An immediate extension is to many-to-one situations where various interesting real life applications exists, hospital interns to hospitals, students to schools etc. One additional complication in such markets is the introduction of quotas. An approach to utilize the the Borda measure in these markets can be creating clones of each agent corresponding to their quotas, e.g., treating a hospital with quota 10 would be considering 10 hospitals with the same preference instead. A natural modification of the Borda measure then can be utilized in this setup considering best (and worst) case scenarios.

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[11] A pseudometric is a function where all metric conditions are satisfied accept the identity of indiscernibles. Instead a weaker version is applied, i.e., for all $\mu, \bar{\mu}$, if $\mu = \bar{\mu}$, then $\delta(\mu, \bar{\mu}) = 0$.

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# Appendix

## A Proofs of Section 4.1

### A.1 Proof of Lemma 2

*Proof.* Consider an extension $\bar{P}$ of $P$ and the extension $\bar{\mu}^{ij}$ and $\bar{\mu}^I$ of matchings $\mu^{ij}$ and $\mu^I$ by the set of agents $N' \setminus N$, respectively. By Remark 2, $\delta_P(\mu^{ij}, \mu^I) = \delta_{\bar{P}}(\bar{\mu}^{ij}, \bar{\mu}^I)$. Therefore, with abuse of notation we write $P$, $\mu^{ij}$ and $\mu^I$ instead of $\bar{P}$, $\bar{\mu}^{ij}$ and $\bar{\mu}^I$, respectively. Also, consider an extension $\bar{P}'$ of $P'$ and the extension $\bar{\mu}^{i'j'}$ and $\bar{\mu}^{I'}$ of matchings $\mu^{i'j'}$ and $\mu^{I'}$ by the set of agents $N \setminus N'$, respectively. By Remark 2, $\delta_P(\mu^{i'j'}, \mu^{I'}) = \delta_{\bar{P}}(\bar{\mu}^{i'j'}, \bar{\mu}^{I'})$. Therefore, with abuse of notation we write $P'$, $\mu^{i'j'}$ and $\mu^{I'}$ instead of $\bar{P}'$, $\bar{\mu}^{i'j'}$ and $\bar{\mu}^{I'}$, respectively. Note that now both $P$ and $P'$ (as well as the matchings) are defined on the same set of agents $\bar{N} = N' \cup N$. We consider two case either $x = 1$ or $x > 1$.

**Case 1.** $x = 1$: That is no agent is nested between $j$ and $i$ in $\mu^{ij}$ and no agent is nested between $j'$ and $i'$ in $\mu^{i'j'}$. Consider permutation $\pi = (ii')(jj')$. Applying this permutation on $P$, and using anonymity condition yields $\delta_P(\mu^{ij}, \mu^I) = \delta_{P^\pi}((\mu^{ij})^\pi, (\mu^I)^\pi)$. Since by this permutation, $(\mu^{ij})^\pi = \mu^{i'j'}$ and $(\mu^I)^\pi = \mu^{I'}$, then $\delta_{P^\pi}((\mu^{ij})^\pi, (\mu^I)^\pi) = \delta_{P^\pi}(\mu^{i'j'}, \mu^{I'})$. As both problems are defined on the same set of agents monotonicity implies $\delta_{P^\pi}(\mu^{i'j'}, \mu^{I'}) = \delta_{P'}(\mu^{i'j'}, \mu^{I'})$. Therefore, $\delta_P(\mu^{ij}, \mu^I) = \delta_{P'}(\mu^{i'j'}, \mu^{I'})$.

**Case 2.** $x>1$: Let the set of agents nested between $j$ and $i$ in $R_i$ be $Z = \{z_1, \ldots, z_{x-1}\}$, and the agents nested between $j'$ and $i'$ in $R'_{i'}$ be $Z' = \{z'_1, \ldots, z'_{x-1}\}$. We consider two situation either $Z = Z'$ or $Z \neq Z'$.

I. $Z = Z'$. Consider permutation $\pi = (ii')(jj')$. Applying this permutation on $P$, and using anonymity condition yields $\delta_P(\mu^{ij}, \mu^I) = \delta_{P^\pi}((\mu^{ij})^\pi, (\mu^I)^\pi)$. Since by this permutation, $(\mu^{ij})^\pi = \mu^{i'j'}$ and $(\mu^I)^\pi = \mu^{I'}$, then $\delta_{P^\pi}((\mu^{ij})^\pi, (\mu^I)^\pi) = \delta_{P^\pi}(\mu^{i'j'}, \mu^{I'})$, and (since $Z = Z'$ and both problems are defined on the same set of agents) monotonicity implies $\delta_{P^\pi}(\mu^{i'j'}, \mu^{I'}) = \delta_{P'}(\mu^{i'j'}, \mu^{I'})$. Therefore, $\delta_P(\mu^{ij}, \mu^I) = \delta_{P'}(\mu^{i'j'}, \mu^{I'})$.

II. $Z \neq Z'$. In this case we add the same set of irrelevant newcomers to both problems $P$ and $P'$, and map the agents in $Z$ and $Z'$ to these newcomers so that the set of agents that are nested between the two matchings in these two problems become the same, then part $I$ implies the result. Formally, let $A = \{a_1, \ldots, a_{x-1}\}$, be a set of agents such that $\bar{N} \cap A = \emptyset$. Next, let $\widehat{P}$ and $\widehat{P}'$ be an extension of $P$ and $P'$ by the set of agents $A$, respectively. Also, let $\widehat{\mu}^{ij}$

and $\widehat{\mu}^I$ be the extension of $\mu^{ij}$ and $\mu^I$, and $\widehat{\mu}^{i'j'}$ and $\widehat{\mu}^{I'}$ be the extension of $\mu^{i'j'}$ and $\mu^{I'}$, respectively all by the same set of agents $A$. By Remark 2, $\delta_P(\mu^{ij}, \mu^I) = \delta_{\widehat{P}}(\widehat{\mu}^{ij}, \widehat{\mu}^I)$ and $\delta_{P'}(\mu^{i'j'}, \mu^{I'}) = \delta_{\widehat{P}'}(\widehat{\mu}^{i'j'}, \widehat{\mu}^{I'})$. With abuse of notation, we write $P$, $\mu^{ij}$ and $\mu^I$ instead of $\widehat{P}$, $\widehat{\mu}^{ij}$ and $\widehat{\mu}^I$, and we write $P'$, $\mu^{i'j'}$ and $\mu^{I'}$ instead of $\widehat{P}'$, $\widehat{\mu}^{i'j'}$ and $\widehat{\mu}^{I'}$, respectively.

Consider permutation $\pi = (z_t a_t)$ for all $t \in \{1, \dots, x-1\}$. Applying this permutation on $P$ causes the agents nested between $j$ and $i$ in $R_i$ to become the set of agents $A$. Also, applying permutation $\pi' = (z'_t a_t)$ for all $t \in \{1, \dots, x-1\}$ on problem $P'$ causes the agents nested between $j'$ and $i'$ in $R'_{i'}$ to become the set of agents $A$. In both problems anonymity results the distance to be unchanged. Now as the set of agents nested between the two matchings in $P$ and $P'$ are the identical, part $I$ yields $\delta_P(\mu^{ij}, \mu^I) = \delta_{P'}(\mu^{i'j'}, \mu^{I'})$.

■

## A.2 Proof of Lemma 3

*Proof.* Consider an extension $\bar{P} = (N \cup \{a,b\}, \bar{R})$ of $P$ and extensions $\bar{\mu}^{ij}, \bar{\mu}^I \in \mathcal{M}(N \cup \{a,b\})$ of $\mu^{ij}, \mu^I \in \mathcal{M}(N)$, respectively, by the set of agents $A = \{a,b\}$. By Remark 2, $\delta_P(\mu^{ij}, \mu^I) = \delta_{\bar{P}}(\bar{\mu}^{ij}, \bar{\mu}^I)$. Therefore, with abuse of notation we write $P$, $\mu^{ij}$ and $\mu^I$ instead of $\bar{P}$, $\bar{\mu}^{ij}$ and $\bar{\mu}^I$, respectively.

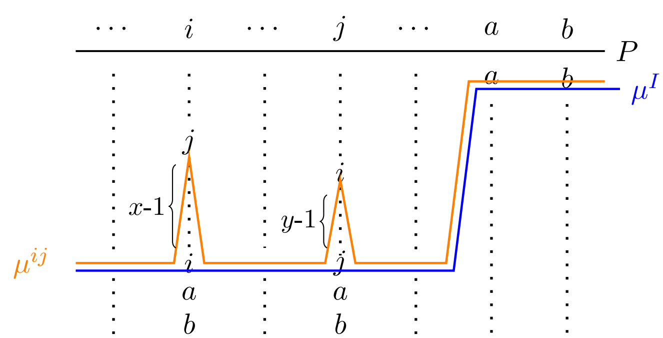


Figure 13: Problem $P$ after adding the two newcomers $a$ and $b$.

Consider any problem $P^1 = (N^1, R^1)$, shown in Figure 14, with $N^1 = N$ and $R^1$ such that

- $rank(a, R^1_i) = 1$ and $[\mu^{ij}, \mu^I]_{R_i} = [\mu^{ij}, \mu^I]_{R^1_i}$,
- $rank(b, R^1_j) = 1$ and $[\mu^{ij}, \mu^I]_{R_j} = [\mu^{ij}, \mu^I]_{R^1_j}$,
- $rank(i, R^1_a) = 1$, $rank(b, R^1_a) = n+1$, and $rank(a, R^1_a) = n+2$,

- $rank(j, R_b^1) = 1$, $rank(a, R_b^1) = n + 1$, and $rank(b, R_b^1) = n + 2$.

By monotonicity we have

$$\delta_P(\mu^{ij}, \mu^I) = \delta_{P^1}(\mu^{ij}, \mu^I). \tag{8}$$

Therefore it is sufficient to prove that $\delta_{P^1}(\mu^{ij}, \mu^I) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}$.

Consider, in $P^1$, the matchings $\mu \in \mathcal{M}(N^1)$ with $\mu(i) = j$, $\mu(a) = b$ and $\mu(t) = t$ for any other agent $t$ and $\mu^T \in \mathcal{M}(N^1)$ with $\mu^T(i) = a$, $\mu^T(j) = b$ and $\mu^T(t) = t$ for any other agent $t$.

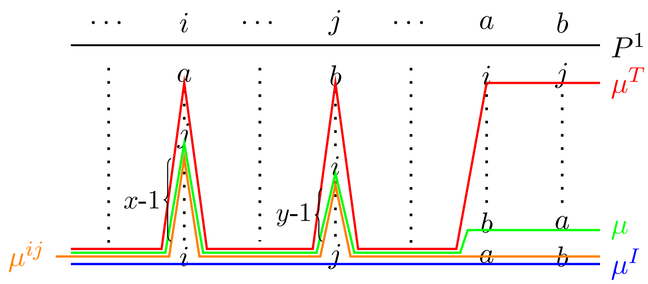


Figure 14: Problem $P^1 = (N^1, R^1)$.

**Claim.** $\delta_{P^1}(\mu, \mu^I) = \alpha_{x1} + \alpha_{y1}$.

Consider a new problem $P^2$ shown in Figure 15. Problem $P^2$ is the permuted problem of $P^1$ with $\pi = (aj)$. With this permutation the identity matching remains the same, hence in $P^2$ we write $\mu^I$ instead of $\mu_I^\pi$.

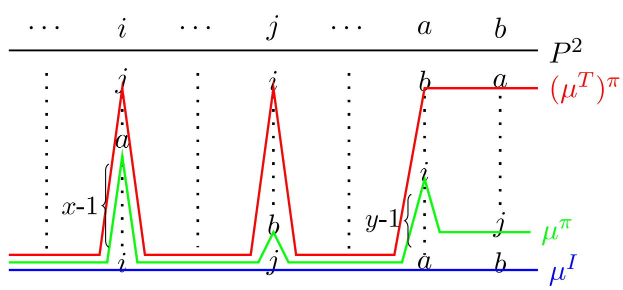


Figure 15: Problem $P^2$ after permuting problem $P^1$ in Figure 14 with $\pi = (aj)$.

Using the anonymity condition the following equations holds,

$$\delta_{P^1}(\mu^T, \mu) = \delta_{P^2}((\mu^T)^\pi, \mu^\pi). \tag{9}$$

Consider a new problem $P^3$ shown in Figure 16. Problem $P^3$ is almost identical to problem $P^2$ except that the position of the partners of each agent in $(\mu^T)^\pi$ and $\mu^\pi$ are swapped. By monotonicity condition for $P^2$ and $P^3$, $\delta_{P^2}((\mu^T)^\pi, \mu^\pi) = \delta_{P^3}((\mu^T)^\pi, \mu^\pi)$. Plugging this into Equation 9 we have,

$$\delta_{P^1}(\mu^T, \mu) = \delta_{P^3}((\mu^T)^\pi, \mu^\pi). \tag{10}$$

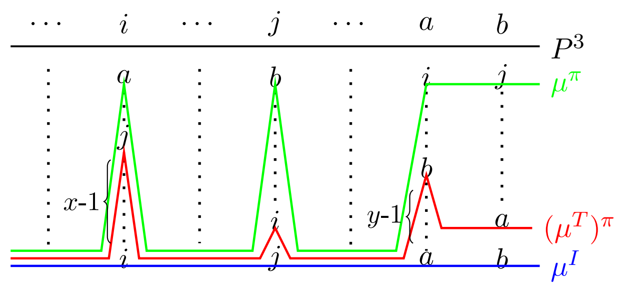


Figure 16: Problem $P^3$, after swapping the positions of $\mu^\pi$ and $(\mu^T)^\pi$ in problem $P^2$ in Figure 15.

Since $\mu$ is between $\mu^T$ and $\mu^I$ in problem $P^1$, and $(\mu^T)^\pi$ is between $\mu^\pi$ and $\mu^I$ in problem $P^3$, betweenness yields

$$\delta_{P^1}(\mu^T, \mu^I) = \delta_{P^1}(\mu^T, \mu) + \delta_{P^1}(\mu, \mu^I) \text{ and,} \tag{11}$$

$$\delta_{P^3}(\mu^\pi, \mu^I) = \delta_{P^3}(\mu^\pi, (\mu^T)^\pi) + \delta_{P^3}((\mu^T)^\pi, \mu^I). \tag{12}$$

Note that by permutation $\pi$, $\mu^\pi = \mu^T$ hence $\delta_{P^3}(\mu^T, \mu^I) = \delta_{P^3}(\mu^\pi, \mu^I)$. Considering this and the monotonicity condition for problems $P^1$ and $P^3$, $\delta_{P^1}(\mu^T, \mu^I) = \delta_{P^3}(\mu^T, \mu^I)$. Therefore the left hand sides of Equations 11 and 12 are equal which yields

$$\delta_{P^1}(\mu^T, \mu) + \delta_{P^1}(\mu, \mu^I) = \delta_{P^3}(\mu^\pi, (\mu^T)^\pi) + \delta_{P^3}((\mu^T)^\pi, \mu^I).$$

Combining this with Equation 10 results in $\delta_{P^1}(\mu, \mu^I) = \delta_{P^3}((\mu^T)^\pi, \mu^I)$. Finally, by Decomposition Lemma on $P^3$ and Lemma 2, $\delta_{P^3}((\mu^T)^\pi, \mu^I) = \alpha_{x1} + \alpha_{y1}$. Hence $\delta_{P^1}(\mu, \mu^I) = \alpha_{x1} + \alpha_{y1}$, which concludes the claim.

By Decomposition Lemma for matching $\mu$ in problem $P^1$ we have $\delta_{P^1}(\mu, \mu^I) = \delta_{P^1}(\mu^{ij}, \mu^I) + \delta_{P^1}(\mu^{ab}, \mu^I)$. By the above claim $\delta_{P^1}(\mu, \mu^I) = \alpha_{x1} + \alpha_{y1}$ and by Lemma 2 $\delta_{P^1}(\mu^{ab}, \mu^I) = \alpha_{11}$. So, $\delta_{P^1}(\mu^{ij}, \mu^I) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}$. By Equation 8, $\delta_P(\mu^{ij}, \mu^I) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}$. Finally, by Lemma 2, the right hand side of this equation is independent of the set of agents $N \subsetneq \mathcal{N}$. Therefore, for all $N \subsetneq \mathcal{N}$, for all problems $P \in \mathcal{P}(N)$ and for all one-couple matchings $\mu^{ij}$ with the interval length $(x, y)$,

$$\delta_P(\mu^{ij}, \mu^I) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}.$$

■

### A.3 Proof of Lemma 4

*Proof.* To show this, we first consider problem $\bar{P} = (\bar{N}, \bar{R})$ with $\bar{N} = \{1, 2, \ldots, 2x\}$ shown in Figure 17. Let $\bar{\mu}^{x,x+1}$ be the one-couple matching with interval length $(x, x)$. Note that the structure of matchings $\bar{\mu}^x, \ldots, \bar{\mu}^1$ in problem $\bar{P}$ is such that

- $\bar{\mu}^x(1) = 2x$, $\bar{\mu}^x(2) = 2x - 1$, so on and so forth,
- for all $k \in \{2, \ldots, x\}$, and for all $i \in \bar{N}$, $\bar{\mu}^{k-1}(i) = \bar{\mu}^k((i + 2) \ mod \ (2x))$, e.g., $\bar{\mu}^{x-1}(2x - 1) = \bar{\mu}^k(1) = 2x$,
- for all $k \in \{1, \ldots, x\}$, $\bar{\mu}^k$ and $\bar{\mu}^{k-1}$ are adjacent.

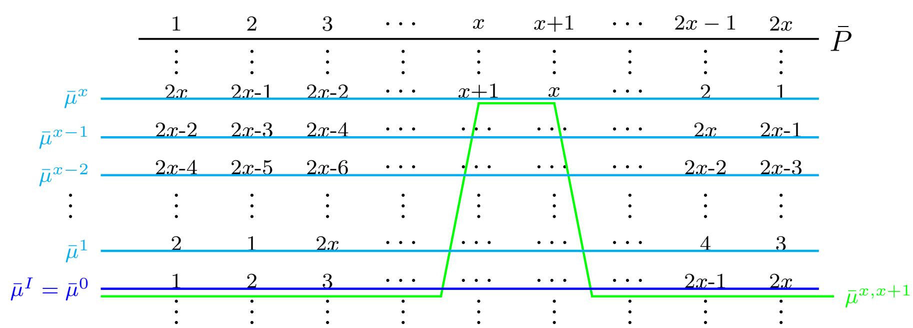


Figure 17: Problem $\bar{P} = (\bar{N}, \bar{R})$.

**Claim.** $\delta_{\bar{P}}(\bar{\mu}^{x,x+1}, \bar{\mu}^I) = 2x$.

**Proof.** To ease the notation in this problem we denote the identity matching by $\bar{\mu}^0$. Note that by construction of $\bar{P}$, for all $t \in \{0, \ldots, x - 1\}$, we have $\bar{\mu}^t$ and $\bar{\mu}^{t+1}$ are disjoint. By normalization, for each $\bar{\mu}^t, \bar{\mu}^{t+1}$, there exists a problem $P^t \in \mathcal{P}(\bar{N})$ such that $\delta_{P^t}(\bar{\mu}^t, \bar{\mu}^{t+1}) = |\bar{N}|$ and is minimal. Note that for all $t \in \{0, \ldots, x-1\}$ and for all $i \in \bar{N}$, as $\bar{\mu}^t, \bar{\mu}^{t+1}$ have the minimal possible intervals, we have $[\bar{\mu}^t, \bar{\mu}^{t+1}]_{\bar{R}_i} \subseteq [\bar{\mu}^t, \bar{\mu}^{t+1}]_{R^t_i}$. Therefore monotonicity implies that $\delta_{\bar{P}}(\bar{\mu}^t, \bar{\mu}^{t+1}) = |\bar{N}|$. As $[\bar{\mu}^x - \bar{\mu}^{x+1} - \cdots - \bar{\mu}^1 - \bar{\mu}^0]$ are on a line then betweenness yields

$$\delta_{\bar{P}}(\bar{\mu}^x, \bar{\mu}^0) = \sum_{t=0}^{x-1} \delta_{\bar{P}}(\bar{\mu}^t, \bar{\mu}^{t+1}) = x|\bar{N}|. \tag{13}$$

On the other hand, by Decomposition Lemma, the distance between $\bar{\mu}^x$ and $\bar{\mu}^0$ can be decomposed as the sum of $\frac{|\bar{N}|}{2}$ one-couple matching each with the same interval length $(x, x)$. Hence $\delta_{\bar{P}}(\bar{\mu}^x, \bar{\mu}^0) = \frac{|\bar{N}|}{2}\alpha_{xx}$. Together with Equation 13, $\alpha_{xx} = 2x$ which completes the proof of the claim.

Now consider the original problem $P = (N, R)$ and the aforementioned problem $\bar{P} = (\bar{N}, \bar{R})$. Without loss of generality let $N = \{1, \dots, n\}$ and $\bar{N} = \{1, \dots, 2x\}$. Based based on $n$ and $2x$ three cases are possible,

**Case 1.** $n = 2x$. By Lemma 3, the distance between any one-couple matching with interval length $(x, x)$, and the identity matching is the same and independent of the identity of agents in the intervals. As $\mu^{ij}$ and $\bar{\mu}^{x,x+1}$ are defined on the same set of agents and same interval length $(x, x)$, they should have the same distance, hence $\delta_P(\mu^{ij}, \mu^I) = \delta_{\bar{P}}(\bar{\mu}^{x,x+1}, \bar{\mu}^I)$. As proven in the claim the latter equals $2x$. Therefore, $\delta_P(\mu^{ij}, \mu^I) = 2x$.

**Case 2.** $n < 2x$. Let $\widehat{P}, \widehat{\mu}^{ij}, \widehat{\mu}^I$ be an extension of $P, \mu^{ij}, \mu^I$ by the set of agents $\bar{N} \setminus N$ as irrelevant newcomers. By Remark 2 we have $\delta_P(\mu^{ij}, \mu^I) = \delta_{\widehat{P}}(\widehat{\mu}^{ij}, \widehat{\mu}^I)$. Now as both $\widehat{P}$ and $\bar{P}$ are defined on the same set of agents, Case 1, yields $\delta_{\widehat{P}}(\widehat{\mu}^{ij}, \widehat{\mu}^I) = \delta_{\bar{P}}(\bar{\mu}^{x,x+1}, \bar{\mu}^I)$. The two equations together imply $\delta_P(\mu^{ij}, \mu^I) = \delta_{\bar{P}}(\bar{\mu}^{x,x+1}, \bar{\mu}^I)$. As proven in the claim the latter equals $2x$. Therefore, $\delta_P(\mu^{ij}, \mu^I) = 2x$.

**Case 3.** $n > 2x$. Let $\widehat{P}, \widehat{\mu}^{x,x+1}, \widehat{\mu}^I$ be an extension of $\bar{P}, \bar{\mu}^{x,x+1}, \bar{\mu}^I$ by the set of agents $N \setminus \bar{N}$ as irrelevant newcomers. By Remark 2 we have $\delta_{\bar{P}}(\bar{\mu}^{x,x+1}, \bar{\mu}^I) = \delta_{\widehat{P}}(\widehat{\mu}^{x,x+1}, \widehat{\mu}^I)$. Now as both $\widehat{P}$ and $P$ are defined on the same set of agents, Case 1, yields $\delta_{\widehat{P}}(\widehat{\mu}^{x,x+1}, \widehat{\mu}^I) = \delta_P(\mu^{ij}, \mu^I)$. The two equations together imply $\delta_P(\mu^{ij}, \mu^I) = \delta_{\bar{P}}(\bar{\mu}^{x,x+1}, \bar{\mu}^I)$. As proven in the claim the latter equals $2x$. Therefore, $\delta_P(\mu^{ij}, \mu^I) = 2x$. ■

# B Proofs of Section 4.2

## B.1 Proof of Proposition 1

$$\delta_P(\mu^1, \mu^2) = \delta_P(\mu^2, \mu^3) = \alpha_{21} + \alpha_{11}.$$

where $P$ is the problem of Figure 4.

*Proof.* To prove this, first we show $\delta_P(\mu^1, \mu^2) = \delta_P(\mu^2, \mu^3)$. To do this, consider permutation $\pi = (23)$. Applying this permutation on $P$ results in problem $P^\pi$ shown on the right hand side of Figure 18.

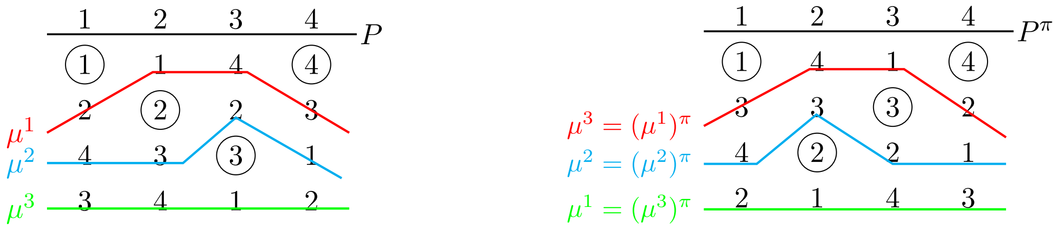


Figure 18: The original problem $P$ of Proposition 1 (on the left) and the permuted problem $P^\pi$ (on the right) after permuting with $\pi = (23)$.

By the anonymity condition

$$\delta_P(\mu^2, \mu^1) = \delta_{P^\pi}((\mu^2)^\pi, (\mu^1)^\pi).$$

As under permutation $\pi$, $(\mu^1)^\pi = \mu^3$ and $(\mu^2)^\pi = \mu^2$ monotonicity condition results in

$$\delta_{P^\pi}((\mu^2)^\pi, (\mu^1)^\pi) = \delta_P(\mu^3, \mu^2).$$

The above two equations yield,

$$\delta_P(\mu^2, \mu^1) = \delta_P(\mu^3, \mu^2).$$

By betweenness of $\mu^2$ in problem $P$, $\delta_P(\mu^1, \mu^3) = \delta_P(\mu^1, \mu^2) + \delta_P(\mu^2, \mu^3)$, and using the above equality

$$\delta_P(\mu^1, \mu^2) = \delta_P(\mu^2, \mu^3) = \frac{\delta_P(\mu^1, \mu^3)}{2}. \tag{14}$$

Hence, it is sufficient to show that $\delta_P(\mu^1, \mu^3)$ equals the Borda measure. To show this, let $\bar{P}, \bar{\mu}^1, \bar{\mu}^2$ and $\bar{\mu}^3$ be an extension of problem $P, \mu^1, \mu^2$ and $\mu^3$, respectively, by the set of agents $A = \{1', 2', 3', 4'\}$ (see Figure 19). By Remark 2, $\delta_P(\mu^1, \mu^2) = \delta_{\bar{P}}(\bar{\mu}^1, \bar{\mu}^2)$ and $\delta_P(\mu^2, \mu^3) = \delta_{\bar{P}}(\bar{\mu}^2, \bar{\mu}^3)$. Hence, with abuse of notation we write $P, \mu^1, \mu^2$ and $\mu^3$ instead of $\bar{P}, \bar{\mu}^1, \bar{\mu}^2$ and $\bar{\mu}^3$, respectively.

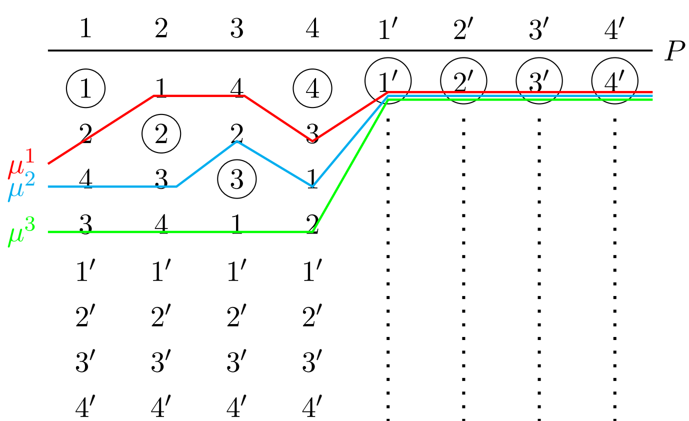


Figure 19: An extension of problem $P$ in Figure 4 by the set of agents $A = \{1', 2', 3', 4'\}$.

To show that $\delta_P(\mu^1,\mu^3)$ equals the Borda measure, consider problem $\widehat{P}$ shown in Figure 20. In this problem, $[\mu^4-\mu^1-\mu^2-\mu^3]$ are on a line hence,

$$\delta_{\widehat{P}}(\mu^4,\mu^3)=\delta_{\widehat{P}}(\mu^4,\mu^1)+\delta_{\widehat{P}}(\mu^1,\mu^2)+\delta_{\widehat{P}}(\mu^2,\mu^3). \tag{15}$$

Via monotonicity condition $\delta_P(\mu^1,\mu^2)=\delta_{\widehat{P}}(\mu^1,\mu^2)$ and $\delta_P(\mu^2,\mu^3)=\delta_{\widehat{P}}(\mu^2,\mu^3)$. Plugging this into Equation 14, $\delta_{\widehat{P}}(\mu^1,\mu^2)=\delta_{\widehat{P}}(\mu^2,\mu^3)$. Therefore Equation 15 yields

$$\delta_{\widehat{P}}(\mu^1,\mu^2)=\delta_{\widehat{P}}(\mu^2,\mu^3)=\frac{\delta_{\widehat{P}}(\mu^4,\mu^3)-\delta_{\widehat{P}}(\mu^4,\mu^1)}{2}. \tag{16}$$

Hence, it is sufficient to show that $\delta_{\widehat{P}}(\mu^4,\mu^3)$ and $\delta_{\widehat{P}}(\mu^4,\mu^1)$ both equal the Borda measure.

As in $\widehat{P}$ the identity matching is between $\mu^4$ and $\mu^3$, betweenness condition and Theorem 1 results that $\delta_{\widehat{P}}(\mu^4,\mu^3)$ equals the Borda measure. The following claim shows, $\delta_{\widehat{P}}(\mu^4,\mu^1)$ also equals the Borda measure.

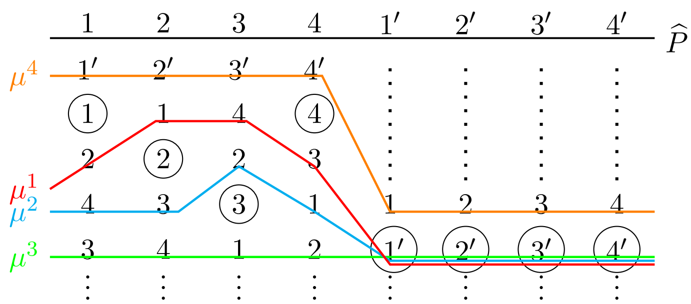


Figure 20: Problem $\widehat{P}$.

**Claim.** $\delta_{\widehat{P}}(\mu^4,\mu^1)=2(\alpha_{21}+\alpha_{11})$.

*Proof.* Consider the problem $\widetilde{P}$ shown in Figure 21, by monotonicity condition $\delta_{\widehat{P}}(\mu^4,\mu^1)=\delta_{\widetilde{P}}(\mu^4,\mu^1)$. To show that $\delta_{\widetilde{P}}(\mu^4,\mu^1)=2(\alpha_{21}+\alpha_{11})$ first we show that $\delta_{\widetilde{P}}(\mu^4,\mu^1)=\delta_{\widetilde{P}}(\mu^1,\mu^5)$. In order to do this, applying permutation $\pi=(12)(34)$, on $\widetilde{P}$ results in problem $\widetilde{P}^\pi$ which is shown in Figure 22.

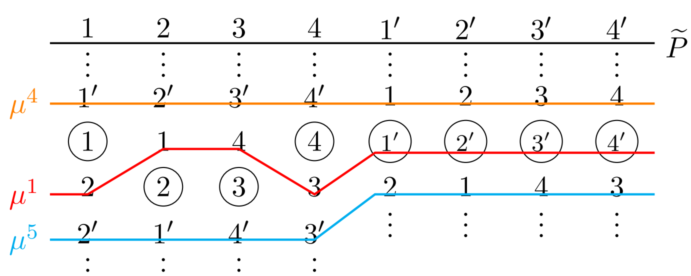


Figure 21: Problem $\widetilde{P}$.

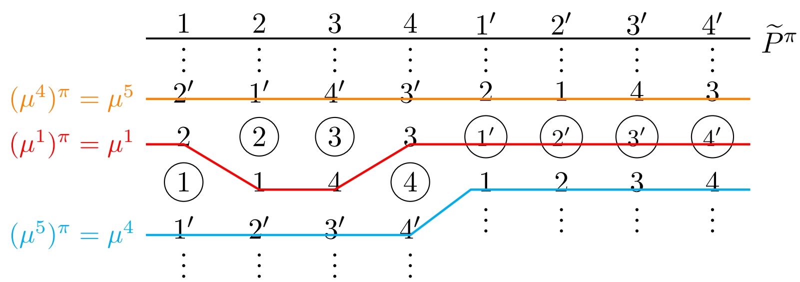


Figure 22: Problem $\widetilde{P}^{\pi}$ after permuting $\widetilde{P}$ in Figure 23 with $\pi = (12)(34)$.

By the anonymity condition $\delta_{\widetilde{P}}(\mu^1, \mu^4) = \delta_{\widetilde{P}^\pi}((\mu^1)^\pi, (\mu^4)^\pi)$. Since $(\mu^1)^\pi = \mu^1$ and $(\mu^4)^\pi = \mu^5$, by monotonicity condition we have $\delta_{\widetilde{P}}(\mu^1, \mu^5) = \delta_{\widetilde{P}^\pi}((\mu^1)^\pi, (\mu^4)^\pi)$, which shows $\delta_{\widetilde{P}}(\mu^1, \mu^4) = \delta_{\widetilde{P}}(\mu^1, \mu^5)$. Considering this and as in problem $\widetilde{P}$ matching $\mu^1$ is between $\mu^4$ and $\mu^5$, we have

$$\delta_{\widetilde{P}}(\mu^4, \mu^1) = \delta_{\widetilde{P}}(\mu^1, \mu^5) = \frac{\delta_{\widetilde{P}}(\mu^4, \mu^5)}{2}.$$

On the other hand, betweenness of $\mu^I$ in problem $\widetilde{P}$ yields $\delta_{\widetilde{P}}(\mu^4, \mu^5) = \delta_{\widetilde{P}}(\mu^4, \mu^I) + \delta_{\widetilde{P}}(\mu^I, \mu^5)$. Using Decomposition Lemma and lemma 3, $\delta_{\widetilde{P}}(\mu^4, \mu^5) = 4(\alpha_{21} + \alpha_{11})$. Therefore, $\delta_{\widetilde{P}}(\mu^4, \mu^1) = \delta_{\widetilde{P}}(\mu^1, \mu^5) = 2(\alpha_{21} + \alpha_{11})$. Now, monotonicity condition for problems $\widehat{P}$ and $\widetilde{P}$ yields $\delta_{\widehat{P}}(\mu^4, \mu^1) = \delta_{\widetilde{P}}(\mu^4, \mu^1)$, which completes the proof of the Claim. ∎

Therefore, with respect to Equation 16, $\delta_{\widehat{P}}(\mu^1, \mu^2) = \delta_{\widehat{P}}(\mu^2, \mu^3) = \alpha_{21} + \alpha_{11}$. As monotonicity condition results in $\delta_{\widehat{P}}(\mu^1, \mu^2) = \delta_P(\mu^1, \mu^2)$ and $\delta_{\widehat{P}}(\mu^2, \mu^3) = \delta_P(\mu^2, \mu^3)$, this complete the proof of Proposition 2. ∎

### B.2 Proof of Proposition 2

$$\delta_P(\mu^1, \mu^2) = \delta_P(\mu^2, \mu^3) = \alpha_{22} + \alpha_{11}.$$

where $P$ is the problem of Figure 5.

*Proof.* To prove this, first consider problem $\bar{P}$ shown on the right hand side of Figure 23. The problem is almost identical to the one in Figure 5 (the problem shown on the left). The only difference is that, the position of each agent's partner is swapped in $\mu^2$ and $\mu^3$. We claim that $\delta_{\bar{P}}(\mu^1, \mu^3) = \delta_{\bar{P}}(\mu^3, \mu^2) = \alpha_{22} + \alpha_{11}$.

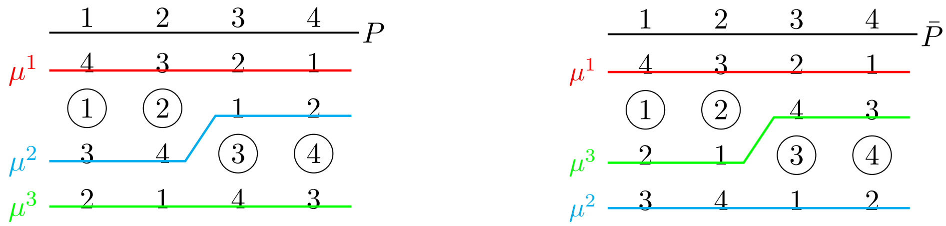


Figure 23: The original problem $P$ of Proposition 2 (on the left) and problem $\bar{P}$ (on the right) after swapping the positions of $\mu^2$ and $\mu^3$ in problem $P$.

**Claim.** $\delta_{\bar{P}}(\mu^1, \mu^3) = \delta_{\bar{P}}(\mu^3, \mu^2) = \alpha_{22} + \alpha_{11}$.

*Proof of claim.* To prove the claim, first we show that $\delta_{\bar{P}}(\mu^1, \mu^3) = \delta_{\bar{P}}(\mu^3, \mu^2)$. To do this, consider permutation $\pi = (1324)$, applying this permutation on $\bar{P}$ results in problem $\bar{P}^\pi$, which is shown in Figure 24. By the anonymity condition

$$\delta_{\bar{P}}(\mu^1, \mu^3) = \delta_{\bar{P}^\pi}((\mu^1)^\pi, (\mu^3)^\pi).$$

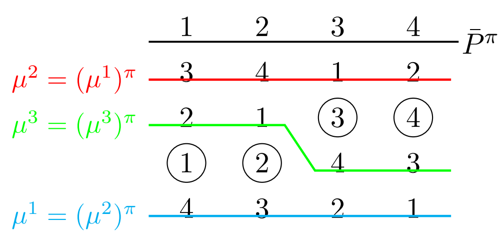


Figure 24: Permuted problem $\bar{P}^\pi$ after permuting problem $\bar{P}$ of Figure 23 with $\pi = (1324)$.

Note that $(\mu^1)^\pi = \mu^2$, $(\mu^2)^\pi = \mu^1$ and $(\mu^3)^\pi = \mu^3$. As the intervals between $(\mu^1)^\pi$ and $(\mu^3)^\pi$ in problem $\bar{P}^\pi$ and the intervals between $\mu^2$ and $\mu^3$ in problem $\bar{P}$ are the same, by monotonicity

$$\delta_{\bar{P}^\pi}((\mu^1)^\pi, (\mu^3)^\pi) = \delta_{\bar{P}}(\mu^3, \mu^2).$$

The above two equations yields

$$\delta_{\bar{P}}(\mu^1, \mu^3) = \delta_{\bar{P}}(\mu^3, \mu^2).$$

Now, as in problem $\bar{P}$ the identity matching is between $\mu^1$ and $\mu^2$, by Decomposition Lemma, Lemma 3 and betweenness condition, $\delta_{\bar{P}}(\mu^1, \mu^2) = 2(\alpha_{22}+\alpha_{11})$. As in problem $\bar{P}$, $\mu^3$ is also between

$\mu^1$ and $\mu^2$ we have

$$\begin{aligned} 2(\alpha_{22} + \alpha_{11}) &= \delta_{\bar{P}}(\mu^1, \mu^2) \\ &= \delta_{\bar{P}}(\mu^1, \mu^3) + \delta_{\bar{P}}(\mu^3, \mu^2) \\ &= 2\delta_{\bar{P}}(\mu^1, \mu^3) \\ &\Rightarrow \delta_{\bar{P}}(\mu^1, \mu^3) = \delta_{\bar{P}}(\mu^3, \mu^2) = \alpha_{22} + \alpha_{11} \end{aligned}$$

which completes the proof of the claim. ■

Now going back to problem $P$ of Figure 5, by betweenness of $\mu^I$ in $P$, and Decomposition Lemma and Lemma 3, $\delta_P(\mu^1, \mu^3) = 2(\alpha_{22} + \alpha_{11})$. Monotonicity condition for problem $P$ and $\bar{P}$ results $\delta_P(\mu^2, \mu^3) = \delta_{\bar{P}}(\mu^2, \mu^3) = \alpha_{22} + \alpha_{11}$. Now, since $\mu^2$ is between $\mu^1$ and $\mu^3$ in problem $P$, betweenness yields $\delta_P(\mu^1, \mu^3) = \delta_P(\mu^1, \mu^2) + \delta_P(\mu^2, \mu^3)$. Therefore, $\delta_P(\mu^1, \mu^2) = \alpha_{22} + \alpha_{11}$. ■