Stability Conditions for Coupled Oscillators in Linear Arrays

Pablo Enrique Baldivieso Blanco
Portland State University, pbaldivieso@hotmail.com

J.J.P. Veerman
Portland State University, veerman@pdx.edu

Stability Conditions for Coupled Oscillators in Linear Arrays

Pablo E. Baldivieso*, J. J. P. Veerman†

February 20, 2019

Abstract

In this paper, we give necessary conditions for stability of flocks in $\mathbb{R}$. We focus on linear arrays with decentralized agents, where each agent interacts with only a few its neighbors. We obtain explicit expressions for necessary conditions for asymptotic stability in the case that the systems consists of a periodic arrangement of two or three different types of agents, i.e. configurations as follows: ...2-1-2-1 or ...3-2-1-3-2-1. Previous literature indicated that the (necessary) condition for stability in the case of a single agent (...1-1-1) held that the first moment of certain coefficients governing the interactions between agents has to be zero. Here, we show that that does not generalize. Instead, the (necessary) condition in the cases considered is that the first momentum *plus a nonlinear correction term* must be zero.

1 Introduction

Linear arrays of agents, or particles have been studied in many areas such as flock formations, see [13], [16] and vehicular platooning, see [3], [6], and [11]. In this paper, we direct our attention to flocks in $\mathbb{R}$, namely N cars driving on a one-lane road. These equations have the following general form

$$\frac{d}{dt}\begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} 0 & I \\ L_x & L_v \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix} , \tag{1}$$

where I is the $N \times N$ identity, L_x and L_v are $N \times N$ so-called Laplacian matrices. The symbol $\mathbf{z}$ is used for the N positions of the agents on the line. This equation is meant to express the idea that the acceleration of the kth agent depends on the *positions relative to it* of some of his neighbors — this is expressed through the matrix L_x — and on the *velocities relative to it* — expressed through L_v. Agents whose response depends only on positions and velocities *relative to them* are called *decentralized*. The fact they are decentralized implies that L_x and L_v have row-sum zero. Hence they share many charateristics with the usual Laplacian operator (for details, see [10] and [15]). Ultimately, what we want to know is the behavior of the flock when the following happens. For $t \leq 0$ the flock is in equilibrium, that is: $z_i = 0$ and $\dot{z}_i$ is constant. For $t \geq 0$, the first car changes its velocity, and the others "try" to follow.

But here a double complication arises. First, L_x and L_v do not (generally) commute, and thus we have no analytical means of solving these equations, and second, there may be non-trivial boundary conditions at the beginning and end of the flock.

This problem was partially overcome in [4] and [5]. In those papers a series of conjectures was proposed that relate solutions of the system on the real line (with non-trivial boundary conditions) to solutions of system on the circle (i.e. periodic boundary conditions). The reason this simplifies the equations is that for systems on the circle, the Laplacians L_x and L_v become *circulant* matrices. Since circulant matrices can be simultaneously diagonalized ([14]), this renders the system on the circle, at least in principle, soluble by analytical means. Note that this takes care of *both* problems just mentioned, because on the circle there is no boundary, and, hence, no dependence on boundary conditions. That is: any quantitative outcome of the

*Oregon State University - Cascades; e-mail: pablo.baldivieso@osucascades.edu
†Fariborz Maseeh Dept. of Math. and Stat., Portland State Univ.; e-mail: veerman@pdx.edu

theory will be independent of boundary conditions. Naturally, flocks with few agents may be substantially influenced by boundary conditions. So, the theory that results from using the circular flocks to understand flocks on the line is asymptotic in N, the number of agents in the system. That is to say, it gives a prediction for the trajectories of the individual agents; and the relative difference between predicted and actual trajectories should go to zero as the number of agents, N, tends to infinity.

Thus we can solve these systems on the circle. The delicate part in this, of course, is to find out how exactly to transition from solutions in the circular flock to solution of the flock on the line. This is described in the conjectures formulated in [5]. These conjectures are quite detailed, but in spirit they are akin to the traditional "periodic boundary" approach commonly used in physical systems [1]. However, physical systems such as crystals have symmetric interactions, whereas the equations we consider (generally) do not. Indeed, it is quite reasonable to allow for the possibility to react differently to a trajectory of car behind than to a car in front. As a result, the validity of the "periodic boundary" approach commonly used in physical systems does not imply validity of the conjectures in [5]. However, fairly extensive numerical testing has been performed by [5], [9], and [8], to the effect that in all simulations, the theory appears to have been confirmed.

The theory developed in [4] and [5] can also be used to develop a necessary condition for stability of the flock. Let P be the parameter space, then such a condition typically has the form $f(p) = 0$ where $f : P \to \mathbb{R}$. Let us take as example the systems studied in [4] and [5].

$$\ddot{z}_k = g_x \left(z_k + \sum_{j \neq 0} \rho_{x,j} z_{k+j} \right) + g_v \left(\dot{z}_k + \sum_{j \neq 0} \rho_{v,j} \dot{z}_{k+j} \right) . \tag{2}$$

Here, the assumption is that all agents are equal, and so each agent interacts the same way with the kth agent in front (or behind) it. Due to the Laplacian property of L_x and L_v, we have $\sum_{j\neq 0} \rho_{x,j} = \sum_{j\neq 0} \rho_{v,j} = -1$. What was proved in [4] is that *if* $\sum_{j\neq 0} \rho_{x,j} j \neq 0$*, then for large* N *the system on the circle is unstable.* The conjectures in [5] then imply that if that condition holds, then for large N the system on the line has some form of instability. This means that either the system on the line is unstable (Definition 1.1), or it is stable but has a transient that grows exponentially in N, the number of agents (Definition 1.2). This was called *flock unstable* in [5]. Both types of instabilities are undesirable if we want to have large efficient traffic flow. Thus $\sum_{j\neq 0} \rho_{x,j} j = 0$ is a necessary condition for stability (though generally not sufficient).

Thus, it seemed that there was a very general principle that *first moment of the coeffients of the spatial Laplacian* L_x to the stability of the system. This was confirmed by [9] and [8] in more detail and accompanied by extensive simulations. In looking to prove such a far-reaching statement, we, very unexpectedly, found that for more complicated systems — presented in this work — that statement is generally false. In what follows, we will show that for certain systems where we allow more than 1 type of agent, a necessary condition for stability may still be derived, but its form is more complicated than the previous papers led to expect. Corollaries 3.1 and 2.1 show that in the cases at hand, a nonlinear correction to the first moment needs to be taken into account. We also present numerical simulations to show that, in spite of this, the predictions to which the theory developed leads us, are still asymptotically (for large N) accurate.

This is of considerable importance if one studies the effect of non-symmetric interactions in these systems. Indeed, these formulas show that, surprisingly, *stability is a co-dimension one phenomenon*! Thus, without the help of these formulae, it would be nigh impossible to find stable flocks with non-symmetric interactions (in the spatial Laplacian) by experiment, and one might be tempted to conclude that there are none. On the other hand, the non-symmetric stable interactions are important, because they allow us to further optimize these systems for applications. In addition, they provide qualitatively new types of solutions (see [8]).

For future reference, we need a definition of stability. In consequence of the fact that L_v and L_x are Laplacians, we see that for arbitrary constant x_0 and v_0 (1) has a solution $z_i = x_0 + v_0 t$. This is desirable for a flock. It *does* mean, however, that the matrix associated with this linear system must have a Jordan block of dimension 2 associated to the eigenvalue 0. In this paper, we will call a flock stable if all *other* eigenvalues have strictly negative real part.

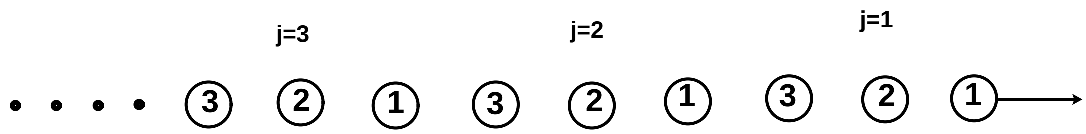

Figure 1: *Periodic arrangement of flock with three types of agents, labeled by 1,2, and 3. At time $t = 0$, the first agent start moving to the right.*

Definition 1.1. *The system* (1) *is linearly stable if it has one eigenvalue zero with geometric multiplicity one and algebraic multiplicity two, and all other eigenvalues have real part less than zero. The system is unstable if at least one eigenvalue has positive real part.*

Definition 1.2. *The system* (1) *is flock-stable if it is linearly stable* and *if transients grow less than exponentially fast in the number of agents. It is called flock-unstable if the growth is exponential.*

2 Periodic Arrangements with Nearest Neighbor Interactions.

Linear flocks in $\mathbb{R}$ of type ...1-1-1 with nearest neighbor interactions have been thoroughly studied ([5], [9]). The necessary condition for stability is that the first moment of the coefficients of the spatial Laplacian must be zero. For flocks of type ...2-1-2-1, the same is true. Details of the latter can be found in [2]. Here we will look at the arrangement ...3-2-1-3-2-1. Thus we consider of linear arrays with $3N$ (N of each type) agents in which each agent interacts with its nearest neighbors. The quantities $z_j^{(i)}$ are the deviations from the equilibrium position at a fixed distance from the leader (or "positions", for short). the quantities $\dot{z}_j^{(i)}$, $i = 1, 2, 3$, and $j = 1 \ldots N$ are their derivatives with respect to time.

The equations of motions for each type of particle are (see Figure 1):

$$\begin{aligned} \ddot{z}_j^{(1)} &= g_x^{(1)} \left(z_j^{(1)} + \rho_{x,1}^{(1)} z_j^{(2)} + \rho_{x,-1}^{(1)} z_{j-1}^{(3)} \right) + g_v^{(1)} \left(\dot{z}_j^{(1)} + \rho_{v,1}^{(1)} \dot{z}_j^{(2)} + \rho_{v,-1}^{(1)} \dot{z}_{j-1}^{(3)} \right) \\ \ddot{z}_j^{(2)} &= g_x^{(2)} \left(z_j^{(2)} + \rho_{x,1}^{(2)} z_j^{(3)} + \rho_{x,-1}^{(2)} z_j^{(1)} \right) + g_v^{(2)} \left(\dot{z}_j^{(2)} + \rho_{v,1}^{(2)} \dot{z}_j^{(3)} + \rho_{v,-1}^{(2)} \dot{z}_j^{(1)} \right) \\ \ddot{z}_j^{(3)} &= g_x^{(3)} \left(z_j^{(3)} + \rho_{x,1}^{(3)} z_{j+1}^{(1)} + \rho_{x,-1}^{(3)} z_j^{(2)} \right) + g_v^{(3)} \left(\dot{z}_j^{(3)} + \rho_{v,1}^{(3)} \dot{z}_{j+1}^{(1)} + \rho_{v,-1}^{(3)} \dot{z}_j^{(2)} \right) \end{aligned} \quad . \tag{3}$$

We assume the flocks to be *decentralized*, that is: the acceleration of an individual depends only on observation *relative* to that individual. For example, the first of the equations in equation (3), should be thought of as:

$$\ddot{z}_j^{(1)} = g_x^{(1)} \left[\rho_{x,1}^{(1)} \left(z_j^{(2)} - z_j^{(1)} \right) + \rho_{x,-1}^{(1)} \left(z_{j-1}^{(3)} - z_j^{(1)} \right) \right] + g_x^{(1)} \left[\rho_{x,1}^{(1)} \left(\dot{z}_j^{(2)} - \dot{z}_j^{(1)} \right) + \rho_{x,-1}^{(1)} \left(\dot{z}_{j-1}^{(3)} - \dot{z}_j^{(1)} \right) \right] \ .$$

This leads to the following constraints: for $i \in \{1, 2, 3\}$

$$\rho_{x,1}^{(i)} + \rho_{x,-1}^{(i)} = -1, \qquad \rho_{v,1}^{(i)} + \rho_{v,-1}^{(i)} = -1 \quad . \tag{4}$$

We will assume that $g_x^{(1)}, g_x^{(2)}, g_x^{(3)}, g_v^{(1)}, g_v^{(2)}$, and $g_v^{(3)}$ are real numbers.

According to the strategy described in the introduction, instability in the system with periodic boundary condition will imply some form of instability (Definition 1.1 or Definition 1.2) in the system on the real line if N is large. Thus our task reduces to deriving a criterion for instability for the system, given periodic boundary conditions. The system subject to periodic boundary conditions is described as follows.

$$\frac{d}{dt}\begin{pmatrix}\mathbf{z}^{(1)}\\ \mathbf{z}^{(2)}\\ \mathbf{z}^{(3)}\\ \dot{\mathbf{z}}^{(1)}\\ \dot{\mathbf{z}}^{(2)}\\ \dot{\mathbf{z}}^{(3)}\end{pmatrix} = \left(\begin{array}{c|c|c|c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}\\ \hline g_x^{(1)}\mathbf{I} & g_x^{(1)}\rho_{x,1}^{(1)}\mathbf{I} & g_x^{(1)}\rho_{x,-1}^{(1)}\mathbf{P}_- & g_v^{(1)}\mathbf{I} & g_v^{(1)}\rho_{v,1}^{(1)}\mathbf{I} & g_v^{(1)}\rho_{v,-1}^{(1)}\mathbf{P}_-\\ \hline g_x^{(2)}\rho_{x,-1}^{(2)}\mathbf{I} & g_x^{(2)}\mathbf{I} & g_x^{(2)}\rho_{x,1}^{(2)}\mathbf{I} & g_v^{(2)}\rho_{v,-1}^{(2)}\mathbf{I} & g_v^{(2)}\mathbf{I} & g_v^{(2)}\rho_{v,1}^{(2)}\mathbf{I}\\ \hline g_x^{(3)}\rho_{x,1}^{(3)}\mathbf{P}_+ & g_x^{(3)}\rho_{x,-1}^{(3)}\mathbf{I} & g_x^{(3)}\mathbf{I} & g_v^{(3)}\rho_{v,1}^{(3)}\mathbf{P}_+ & g_v^{(3)}\rho_{v,-1}^{(3)}\mathbf{I} & g_v^{(3)}\mathbf{I} \end{array}\right)\begin{pmatrix}\mathbf{z}^{(1)}\\ \mathbf{z}^{(2)}\\ \mathbf{z}^{(3)}\\ \dot{\mathbf{z}}^{(1)}\\ \dot{\mathbf{z}}^{(2)}\\ \dot{\mathbf{z}}^{(3)}\end{pmatrix}, \tag{5}$$

where $\mathbf{P}_+$ and its inverse $\mathbf{P}_-$ are $N \times N$ permutations matrices

$$\mathbf{P}_+ = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \ddots & \ddots & 0 & 1\\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \mathbf{P}_- = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1\\ 1 & 0 & 0 & \ddots & 0\\ 0 & 1 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0 & 0\\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{6}$$

We will abbreviate equation (5) simply as

$$\frac{d}{dt}\begin{pmatrix}\mathbf{z}\\ \dot{\mathbf{z}}\end{pmatrix} = \mathbf{M}\begin{pmatrix}\mathbf{z}\\ \dot{\mathbf{z}}\end{pmatrix}. \tag{7}$$

Definition 2.1. *From now on, we set $\phi_m = \frac{2\pi m}{N}$, $m \in \{0, \cdots N-1\}$. When there is no ambiguity, we will often drop the subscript from ϕ_m. We let $\mathbf{v}_m$ be the n-vector whose jth component equals $e^{ij\phi_m}$.*

Proposition 2.1. *The eigenvalues ν and associated eigenvectors $\mathbf{u}_\nu(\phi_m)$ of M satisfy*

$$\mathbf{u}_\nu(\phi_m) = \begin{pmatrix}\epsilon_1\mathbf{v}_m\\ \epsilon_2\mathbf{v}_m\\ \epsilon_3\mathbf{v}_m\\ \nu\epsilon_1\mathbf{v}_m\\ \nu\epsilon_2\mathbf{v}_m\\ \nu\epsilon_3\mathbf{v}_m\end{pmatrix}.$$

For each $m \in \{0, \cdots N-1\}$ given, there are six eigenpairs (counting multiplicity) determined by solving the following equation for ν and ϵ_i:

$$\begin{pmatrix} g_x^{(1)} + \nu g_v^{(1)} - \nu^2 & g_x^{(1)}\rho_{x,1}^{(1)} + \nu g_v^{(1)}\rho_{v,1}^{(1)} & \left(g_x^{(1)}\rho_{x,-1}^{(1)} + \nu g_v^{(1)}\rho_{v,-1}^{(1)}\right)e^{-i\phi}\\ g_x^{(2)}\rho_{x,-1}^{(2)} + \nu g_v^{(2)}\rho_{v,-1}^{(2)} & g_x^{(2)} + \nu g_v^{(2)} - \nu^2 & g_x^{(2)}\rho_{x,1}^{(2)} + \nu g_v^{(2)}\rho_{v,1}^{(2)}\\ \left(g_x^{(3)}\rho_{x,1}^{(3)} + \nu g_v^{(3)}\rho_{v,1}^{(3)}\right)e^{i\phi} & g_x^{(3)}\rho_{x,-1}^{(3)} + \nu g_v^{(3)}\rho_{v,-1}^{(3)} & g_x^{(3)} + \nu g_v^{(3)} - \nu^2 \end{pmatrix}\begin{pmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Proof. From equations (5) and (7), we see that an eigenvector $\begin{pmatrix}\mathbf{z}\\ \dot{\mathbf{z}}\end{pmatrix}$ associated to the eigenvalue ν satisfies $\dot{\mathbf{z}} = \nu\mathbf{z}$. Now $P_+^n = I$, and so $\epsilon^{i\phi_m}$ and $\mathbf{v}_m$ are the eigenvalues and eigenvectors of P_+, and $\epsilon^{-i\phi_m}$ and v_m of P_-. Then by substituting $\mathbf{u}_\nu$ into (5), one sees that these are the eigenvectors of $\mathbf{M}$.

For the second part, we can write

$$\mathbf{M}\mathbf{u} = \nu\mathbf{u}, \tag{8}$$

substitute the form of the eigenvector just derived, and substitute that in equation (5). We obtain three non-trivial equations (from the last three lines of (5)), which can be simplified and rearranged to give the second part of the proposition. □

In short, we can find all eigenpairs by setting to zero the determinant of the matrix in Proposition 2.1. We obtain a polynomial Q of degree six in ν. In its full glory, the polynomial is more than a little cumbersome. From now on, we take superscripts g and ρ modulo 3. For example, $g_x^{(5)} = g_x^{(2)}$. This allows us to manage the expressions a little better.

Definition 2.2. *Let a, b, c, d, and t be real numbers, define*

$$\begin{aligned} D(a,b,c;t) &\equiv abc(e^{it}-1) - (1+a)(1+b)(1+c)(e^{-it}-1)\ , \\ E(a,b,c,d) &\equiv ab(1+c+cd)\ . \end{aligned}$$

The following Lemma is the result of substantial bookkeeping which we leave to the reader.

Lemma 2.1. *When $\phi = 0$, the matrix of Proposition 2.1 has determinant $Q(\nu)$ equal to*

$$\begin{array}{ll} -\nu^2 & \sum_{i=1}^3 E(g_x^{(i)}, g_x^{(i+1)}, \rho_{x,1}^{(i)}, \rho_{x,1}^{(i+1)}) \\ -\nu^3 & \sum_{i=1}^3 E(g_x^{(i)}, g_v^{(i+1)}, \rho_{x,1}^{(i)}, \rho_{v,1}^{(i+1)}) + E(g_v^{(i)}, g_x^{(i+1)}, \rho_{v,1}^{(i)}, \rho_{x,1}^{(i+1)}) \\ -\nu^4 & \sum_{i=1}^3 [g_x^{(i)} + E(g_x^{(i)}, g_x^{(i+1)}, \rho_{x,1}^{(i)}, \rho_{x,1}^{(i+1)})] \\ +\nu^5 & \sum_{i=1}^3 g_x^{(i)} \\ -\nu^6 & 1 \end{array} .$$

The full expression of the constant term of $Q(\nu)$ is $a_0(\phi)$, where

$$a_0(\phi) = g_x^{(1)} g_x^{(2)} g_x^{(3)} D(\rho_{x,1}^{(1)}, \rho_{x,1}^{(2)}, \rho_{x,1}^{(3)}; \phi)\ .$$

To simplify the statement of the main results further, we also need the following definition.

Definition 2.3. *For j and k positive, we define*

$$\alpha_{x,j}^{(k)} \equiv \rho_{x,j}^{(k)} + \rho_{x,-j}^{(k)} \quad \textit{and} \quad \beta_{x,j}^{(k)} \equiv \rho_{x,j}^{(k)} - \rho_{x,-j}^{(k)}\ .$$

Because of the constraint (4), the α's are equal to 1 in this case (but not in the next section).

Theorem 2.1. *If any of the following conditions are violated, then for large N, the system given by (3) on the circle is not stable:*
(i) $g_x^{(1)} \neq 0$, $g_x^{(2)} \neq 0$, and $g_x^{(3)} \neq 0$ and
(ii) $\sum_{i=1}^3 E(g_x^{(i)}, g_x^{(i+1)}, \rho_{x,1}^{(i)}, \rho_{x,1}^{(i+1)}) \neq 0$ and
(iii) $\displaystyle\sum_{i=1}^{3} \beta_{x,1}^{(i)} + \prod_{i=1}^{3} \beta_{x,1}^{(i)} = 0\ .$

Proof. We start with part (i). Suppose for example that $g_x^{(1)} = 0$. Then the first row of the matrix in Proposition 2.1 has a factor ν. Since the determinant is a linear function of the rows, it follows that the determinant of that matrix also has a factor ν. This implies that the zero eigenvalue has multiplicity of at least N, contradicting Definition 1.1.

Suppose (ii) is false. Then for $\phi = 0$, we get multiplicity 3 for the eigenvalue zero. This violates Definition 1.1. (In addition, we may get eigenvalues with positive real part, if Corollary 3.2 applies.)

Part (iii) must hold, because if we assume part (ii), then by Lemma 2.1, the system satisfies all conditions of Proposition 3.2, *except* the condition that $a_0'(0) \neq 0$. That proposition implies instability, and so to avoid the system from being unstable, we must have $a_0'(0) = 0$. Part (i) plus the second part of Lemma 2.1 then imply

$$\frac{\partial}{\partial \phi} D(\rho_{x,1}^{(1)}, \rho_{x,1}^{(2)}, \rho_{x,1}^{(3)}; \phi)\big|_{\phi=0} = 0\ .$$

Use Definition 2.3 and the constraints (4), to substitute

$$\rho_{x,1}^{(i)} = \frac{1}{2}\left(\beta_{x,1}^{(i)} - 1\right) \tag{9}$$

in this equation. Part (iii) follows by differentiation and setting $\phi = 0$. □

Now, the conjectures in [5] state that instability on the circle implies some form of instability on the line. It seems unlikely that the system on the line can be stable if one of the g_x's is zero. However, we have no proof of this. Thus we have to formulate the corollary for systems on the line carefully.

Corollary 2.1. *The conjectures of [5] imply the following. If $g_x^{(1)} \neq 0$, $g_x^{(2)} \neq 0$, and $g_x^{(3)} \neq 0$ and*

$$\sum_{i=1}^{3} \beta_{x,1}^{(i)} + \prod_{i=1}^{3} \beta_{x,1}^{(i)} \neq 0 \,,$$

then for large N, the system on the line given by 3 has some form of instability (Definitions 1.1 or 1.2).

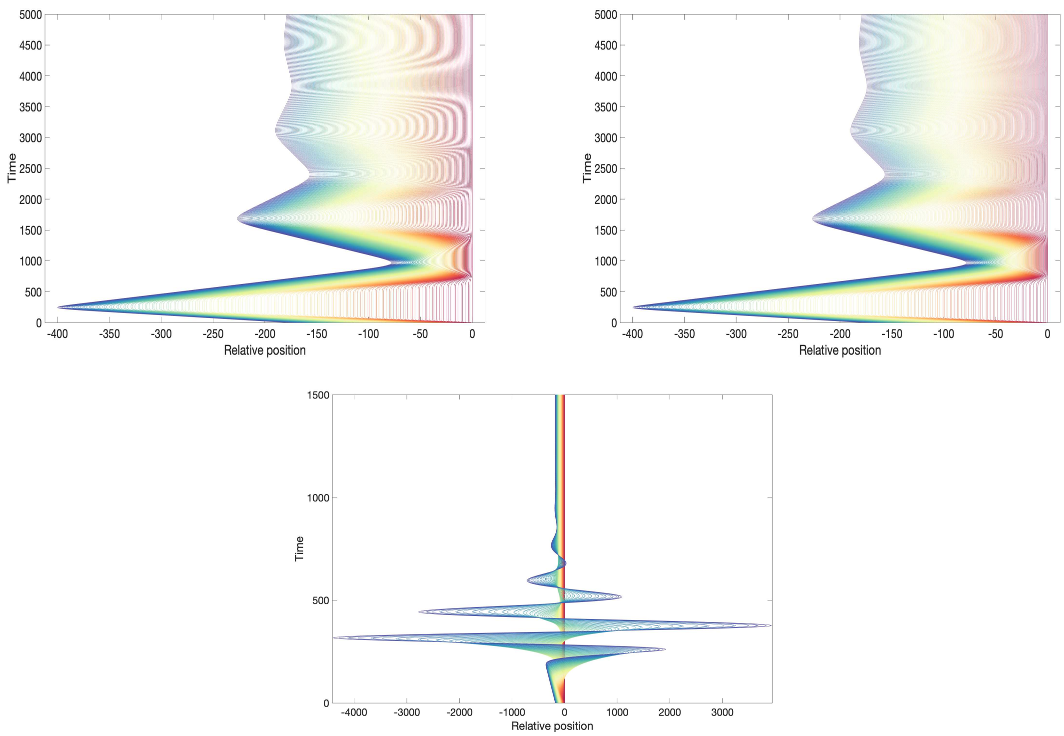

Figure 2: *a) Boundary Condition Type I. Maximum amplitude of -221.0 at $t = 244.6$. b) Boundary Condition Type II. Maximum amplitude of -220.8 at $t = -244.4$. c) Dynamics of a flock unstable system.*

The surprise is that the first moment $\sum_{i=1}^{3} \beta_{x,1}^{(i)}$ apparently needs an unexpected cubic correction, $\prod_{i=1}^{3} \beta_{x,1}^{(i)}$. We perform simulations to see if this conclusion is borne out by simulations on the real line (independent of reasonable boundary conditions).

Similar to what was done in [8], we consider two sets of boundary conditions. We will call them Type I and Type II boundary conditions. Since we want to maintain the centralized character of the systems, both sets of boundary conditions must maintain the "Laplacian" property, namely that row-sums of each Laplacian are zero. Type I adjusts the central coefficients $\rho_{x,0}^{(i)}$, and $\rho_{v,0}^{(i)}$ on the boundaries as follows:

$$\begin{aligned}
\ddot{z}_1^{(1)} &= 0 \\
\ddot{z}_N^{(3)} &= g_x^{(3)}\left(-\rho_{x,-1}^{(3)} z_N^{(3)} + \rho_{x,-1}^{(3)} z_N^{(2)}\right) + g_v^{(3)}\left(-\rho_{v,-1}^{(3)} \dot{z}_N^{(3)} + \rho_{v,-1}^{(3)} \dot{z}_N^{(2)}\right)
\end{aligned}$$

In Type II boundary conditions, we keep the central coefficients $\rho_{x,0}^{(i)}$, and $\rho_{v,0}^{(i)}$ equal to 1 and we adjust the remaining coefficients accordingly:

$$\begin{aligned}
\ddot{z}_1^{(1)} &= 0 \\
\ddot{z}_N^{(3)} &= g_x^{(3)}\left(z_N^{(3)} - z_N^{(2)}\right) + g_v^{(3)}\left(\dot{z}_N^{(3)} - \dot{z}_N^{(2)}\right)
\end{aligned}$$

We run simulations of the system in $\mathbb{R}$ considering these two boundary conditions with initial condition:

$$z_k^{(i)}(0) = \dot{z}_k^{(i)}(0) = 0 \quad \underline{\text{except}} \quad \dot{z}_1^{(1)}(0) = 1\ .$$

Figure 2a) and 2b) are a numerical simulations on the line with parameters satisfying Corollary 2.1:

$$\begin{gathered}
N = 60 \text{ (of each type)},\ g_x^{(1)} = g_x^{(2)} = g_x^{(3)} = -1 \\
\rho_{x,1}^{(1)} = -0.6,\ \rho_{x,1}^{(2)} = -0.8,\ \rho_{x,1}^{(3)} = -0.142857...,\ \rho_{x,1}^{(1)} = \rho_{v,1}^{(2)} = \rho_{v,1}^{(3)} = -0.3
\end{gathered}$$

Thus $\sum_{i=1}^{3} \beta_{x,1}^{(i)} = -0.0858$, while $\sum_{i=1}^{3} \beta_{x,1}^{(i)} + \prod_{i=1}^{3} \beta_{x,1}^{(i)} = 0$. So it is far from satisfying the first, but satisfies the stability condition derived in this section. From the figures, it is apparent that the system is stable, and that the outcome is largely independent of the type of boundary condition.

On the other hand, Figure 2c) shows the dynamics of a *flock unstable* system. Here, the parameters

$$\begin{gathered}
N = 60 \text{ (of each type)},\ g_x^{(1)} = g_x^{(2)} = g_x^{(3)} = -1 \\
\rho_{x,1}^{(1)} = -0.6,\ \rho_{x,1}^{(2)} = -0.8,\ \rho_{x,1}^{(3)} = -0.1,\ \rho_{x,1}^{(1)} = \rho_{v,1}^{(2)} = \rho_{v,1}^{(3)} = -0.3
\end{gathered}$$

satisfy $\sum_{i=1}^{3} \beta_{x,1}^{(i)} = 0$, but not the condition derived in this section.

3 Periodic Arrangements with Next Nearest Neighbor Interactions

Next nearest neighbor interaction means that an agent can see up to two agents in front and behind it. Although such systems with identical agents were included in [4], they were more thoroughly studied in [8], where it was shown that for certain parameter values, these systems can generate so-called reflectionless waves. In this section, we consider the stability problem for the more complicated case of flocks of type ...2-1-2-1 with next nearest neighbor interaction (see Figure 3).

As before, we formulate the system with periodic boundary conditions and analyze the condition for instability. With the same notation as before, the relevant equations of motion become

$$\begin{aligned}
\ddot{z}_k^{(1)} &= g_x^{(1)}(z_k^{(1)} + \rho_{v,1}^{(1)} z_k^{(2)} + \rho_{v,-1}^{(1)} z_{k-1}^{(2)} + \rho_{v,2}^{(1)} z_{k+1}^{(1)} + \rho_{v,-2}^{(1)} z_{k-1}^{(1)}) \\
&\quad + g_v^{(1)}(\dot{z}_k^{(1)} + \rho_{v,1}^{(1)} \dot{z}_k^{(2)} + \rho_{v,-1}^{(1)} \dot{z}_{k-1}^{(2)} + \rho_{v,2}^{(1)} \dot{z}_{k+1}^{(1)} + \rho_{v,-2}^{(1)} \dot{z}_{k-1}^{(1)}) \\
\ddot{z}_k^{(2)} &= g_x^{(2)}(z_k^{(2)} + \rho_{x,-1}^{(2)} z_k^{(1)} + \rho_{x,1}^{(2)} z_{k+1}^{(1)} + \rho_{x,-2}^{(2)} z_{k-1}^{(2)} + \rho_{x,2}^{(2)} z_{k+1}^{(2)}) \\
&\quad + g_v^{(2)}(\dot{z}_k^{(2)} + \rho_{v,-1}^{(2)} \dot{z}_k^{(1)} + \rho_{v,1}^{(2)} \dot{z}_{k+1}^{(1)} + \rho_{v,-2}^{(2)} \dot{z}_{k-1}^{(2)} + \rho_{v,2}^{(2)} \dot{z}_{k+1}^{(2)})
\end{aligned} \quad . \tag{10}$$

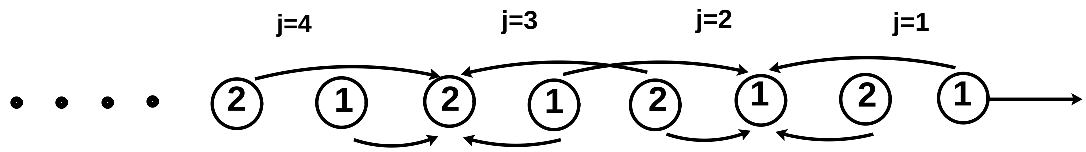

Figure 3: *Periodic arrangement of flock with two types of agents, labeled by 1 and 2. Each agent uses information from four others; the arrows indicate information flow. At time $t = 0$, the first agent start moving to the right.*

Because we assume the equations are decentralized, we get the constraints:

$$\sum_{j=-2, j\neq 0}^{2} \rho_{x,j}^{(i)} = -1 \, , \qquad \sum_{j=-2, j\neq 0}^{2} \rho_{v,j}^{(i)} = -1 \, . \tag{11}$$

As before, the system can be written more compactly as (7). But now $\mathbf{M}$ is given by

$$\left(\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \hline g_x^{(1)}\mathbf{B}_x^{(1)} & g_x^{(1)}\mathbf{A}_x^{(1)} & g_v^{(1)}\mathbf{B}_v^{(1)} & g_v^{(1)}\mathbf{A}_v^{(1)} \\ \hline g_x^{(2)}\mathbf{A}_x^{(2)} & g_x^{(2)}\mathbf{B}_x^{(2)} & g_v^{(2)}\mathbf{A}_v^{(2)} & g_v^{(2)}\mathbf{B}_v^{(2)} \end{array}\right) \, . \tag{12}$$

The $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$ are defined below in terms of the permutation matrices $\mathbf{P}_\pm$ of (6). All matrices $\mathbf{A}$ and $\mathbf{B}$ are circulant $N \times N$ matrices. Thus in the basis $\mathbf{v}_m$ given in Definition 2.1 is an eigenbasis for all, and the eigenvalues are trivial to compute. We list all matrices and their eigenvalues here.

$$\begin{array}{llll}
\mathbf{A}_x^{(1)} & = \rho_{x,1}^{(1)}\mathbf{I} + \rho_{x,-1}^{(1)}\mathbf{P}_- \; ; & \lambda_x^{(1)}(\phi) & = \rho_{x,1}^{(1)} + \rho_{x,-1}^{(1)} e^{-i\phi} \, . \\
\mathbf{A}_x^{(2)} & = \rho_{x,-1}^{(2)}\mathbf{I} + \rho_{x,1}^{(2)}\mathbf{P}_+ \; ; & \lambda_x^{(2)}(\phi) & = \rho_{x,-1}^{(2)} + \rho_{x,1}^{(2)} e^{i\phi} \, . \\
\mathbf{A}_v^{(1)} & = \rho_{v,1}^{(1)}\mathbf{I} + \rho_{v,-1}^{(1)}\mathbf{P}_- \; ; & \lambda_v^{(1)}(\phi) & = \rho_{v,1}^{(1)} + \rho_{v,-1}^{(1)} e^{-i\phi} \, . \\
\mathbf{A}_v^{(2)} & = \rho_{v,-1}^{(2)}\mathbf{I} + \rho_{v,1}^{(2)}\mathbf{P}_+ \; ; & \lambda_v^{(2)}(\phi) & = \rho_{v,-1}^{(2)} + \rho_{v,1}^{(2)} e^{i\phi} \, . \\
\mathbf{B}_x^{(1)} & = \mathbf{I} + \rho_{x,-2}^{(1)}\mathbf{P}_- + \rho_{x,2}^{(1)}\mathbf{P}_+ \; ; & \mu_x^{(1)}(\phi) & = 1 + \rho_{x,2}^{(1)} e^{i\phi} + \rho_{x,-2}^{(1)} e^{-i\phi} \, . \\
\mathbf{B}_x^{(2)} & = \mathbf{I} + \rho_{x,-2}^{(2)}\mathbf{P}_- + \rho_{x,2}^{(2)}\mathbf{P}_+ \; ; & \mu_x^{(2)}(\phi) & = 1 + \rho_{x,2}^{(2)} e^{i\phi} + \rho_{x,-2}^{(2)} e^{-i\phi} \, . \\
\mathbf{B}_v^{(1)} & = \mathbf{I} + \rho_{v,-2}^{(1)}\mathbf{P}_- + \rho_{v,2}^{(1)}\mathbf{P}_+ \; ; & \mu_v^{(1)}(\phi) & = 1 + \rho_{v,2}^{(1)} e^{i\phi} + \rho_{v,-2}^{(1)} e^{-i\phi} \, . \\
\mathbf{B}_v^{(2)} & = \mathbf{I} + \rho_{v,-2}^{(2)}\mathbf{P}_- + \rho_{v,2}^{(2)}\mathbf{P}_+ \; ; & \mu_v^{(2)}(\phi) & = 1 + \rho_{v,2}^{(2)} e^{i\phi} + \rho_{v,-2}^{(2)} e^{-i\phi} \, .
\end{array} \tag{13}$$

The following proposition is derived in the same way as the analogous proposition in the previous Section.

Proposition 3.1. *The eigenvalues ν and associated eigenvectors $\mathbf{u}_\nu(\phi_m)$ of M satisfy*

$$\mathbf{u}_\nu(\phi_m) = \begin{pmatrix} \epsilon_1 \mathbf{v}_m \\ \epsilon_2 \mathbf{v}_m \\ \nu\epsilon_1 \mathbf{v}_m \\ \nu\epsilon_2 \mathbf{v}_m \end{pmatrix} \, .$$

For each $m \in \{0, \cdots N-1\}$ given, there are four eigenpairs (counting multiplicity) determined by solving the following equation for ν and ϵ_i (we dropped the argument ϕ):

$$\begin{pmatrix} g_x^{(1)}\mu_x^{(1)} + \nu g_v^{(1)}\mu_v^{(1)} - \nu^2 & g_x^{(1)}\lambda_x^{(1)} + \nu g_v^{(1)}\lambda_v^{(1)} \\ g_x^{(2)}\lambda_x^{(2)} + \nu g_v^{(2)}\lambda_v^{(2)} & g_x^{(2)}\mu_x^{(2)} + \nu g_v^{(2)}\mu_v^{(2)} - \nu^2 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \, .$$

Lemma 3.1. *When $\phi = 0$, the matrix of Proposition 3.1 has determinant $Q(\nu)$ equal to*

$$\nu^2 \left[\nu^2 + \nu\left(g_v^{(1)}\alpha_{v,1}^{(1)} + g_v^{(2)}\alpha_{v,1}^{(2)}\right) + \left(g_x^{(1)}\alpha_{x,1}^{(1)} + g_x^{(2)}\alpha_{x,1}^{(2)}\right)\right] .$$

The full expression of the constant term of $Q(\nu)$ is $a_0(\phi)$, where

$$a_0(\phi) = g_x^{(1)} g_x^{(2)} \left(\mu_x^{(1)}(\phi)\mu_x^{(2)}(\phi) - \lambda_x^{(1)}(\phi)\lambda_x^{(2)}(\phi)\right) .$$

Proof. The full determinant of the matrix in Proposition 3.1 is equal to

$$\begin{array}{ll} 1 & g_x^{(1)} g_x^{(2)} \left(\mu_x^{(1)}\mu_x^{(2)} - \lambda_x^{(1)}\lambda_x^{(2)}\right) \\ +\nu & \left(g_x^{(1)} g_v^{(2)} \left(\mu_x^{(1)}\mu_v^{(2)} - \lambda_x^{(1)}\lambda_v^{(2)}\right) + g_v^{(1)} g_x^{(2)} \left(\mu_v^{(1)}\mu_x^{(2)} - \lambda_v^{(1)}\lambda_x^{(2)}\right)\right) \\ +\nu^2 & \left(-g_x^{(1)}\mu_x^{(1)} - g_x^{(2)}\mu_x^{(2)} + g_v^{(1)} g_v^{(2)} \left(\mu_v^{(1)}\mu_v^{(2)} - \lambda_v^{(2)}\lambda_v^{(2)}\right)\right) \\ +\nu^3 & \left(-g_v^{(1)}\mu_v^{(1)} - g_v^{(2)}\mu_v^{(2)}\right) \\ +\nu^4 & 1 \end{array} .$$

Now set $\phi = 0$. From (13) and recalling Definition 2.3, we see that for $r \in \{x, v\}$ and $i \in \{1, 2\}$:

$$\mu_r^{(i)}(0) = 1 + \alpha_{r,2}^{(i)} \quad \text{and} \quad \lambda_r^{(i)}(0) = \alpha_{r,1}^{(i)} .$$

Note that the constraint (11) gives for $r \in \{x, v\}$

$$1 + \alpha_{r,1}^{(i)} + \alpha_{r,2}^{(i)} = 0 \quad \Longrightarrow \quad -\mu_r^{(i)}(0) = \lambda_r^{(i)}(0) = \alpha_{r,1}^{(i)} .$$

Substituting this, and some algebra, yields the Lemma. □

The next results are entirely analogous to the ones in the previous section, and we mention them almost without comment or proof.

Theorem 3.1. *Let $g_x^{(1)}$ and $g_x^{(2)}$ be real numbers. Then necessary conditions for stability of* (10) *are*
(i) $g_x^{(1)} \neq 0$ and $g_x^{(2)} \neq 0$, and
(ii) $g_x^{(1)}\alpha_{x,1}^{(1)} + g_x^{(2)}\alpha_{x,1}^{(2)} > 0$, and $g_v^{(1)}\alpha_{v,1}^{(1)} + g_v^{(2)}\alpha_{v,1}^{(2)} > 0$ and
(iii) $\alpha_{x,1}^{(2)}\left(\beta_{x,1}^{(1)} + 2\beta_{x,2}^{(1)}\right) + \alpha_{x,1}^{(1)}\left(\beta_{x,1}^{(2)} + 2\beta_{x,2}^{(2)}\right) = 0$.

Proof. By and large, this proof is very similar to that of Theorem 2.1. Part (ii) is now more easily derived by explicitly solving for the zero in $Q(\nu)$ when $\phi = 0$ (see Lemma 3.1). In (iii), it is best to differentiate the formula in the second part of Lemma 3.1 directly. The derivatives of the λ's and μ's are easily expressed directly in the α's and β's. □

Corollary 3.1. *The conjectures of [5] imply the following. If $g_x^{(1)} \neq 0$ and $g_x^{(2)} \neq 0$ and*

$$\alpha_{x,1}^{(2)}\left(\beta_{x,1}^{(1)} + 2\beta_{x,2}^{(1)}\right) + \alpha_{x,1}^{(1)}\left(\beta_{x,1}^{(2)} + 2\beta_{x,2}^{(2)}\right) \neq 0 ,$$

then for large N, the system on the line given by (10) *has some form of instability (Definitions 1.1 or 1.2).*

Denoting the first moment of the coefficients $\rho_{x,j}^{(i)}$ of agent of type i by $M^{(i)}$, we can reformulate this condition as:

$$M^{(1)} + M^{(2)} - \frac{\alpha_{x,1}^{(1)}}{\alpha_{x,1}^{(1)} + \alpha_{x,2}^{(1)}} M^{(1)} - \frac{\alpha_{x,1}^{(2)}}{\alpha_{x,1}^{(1)} + \alpha_{x,2}^{(1)}} M^{(1)} \neq 0 .$$

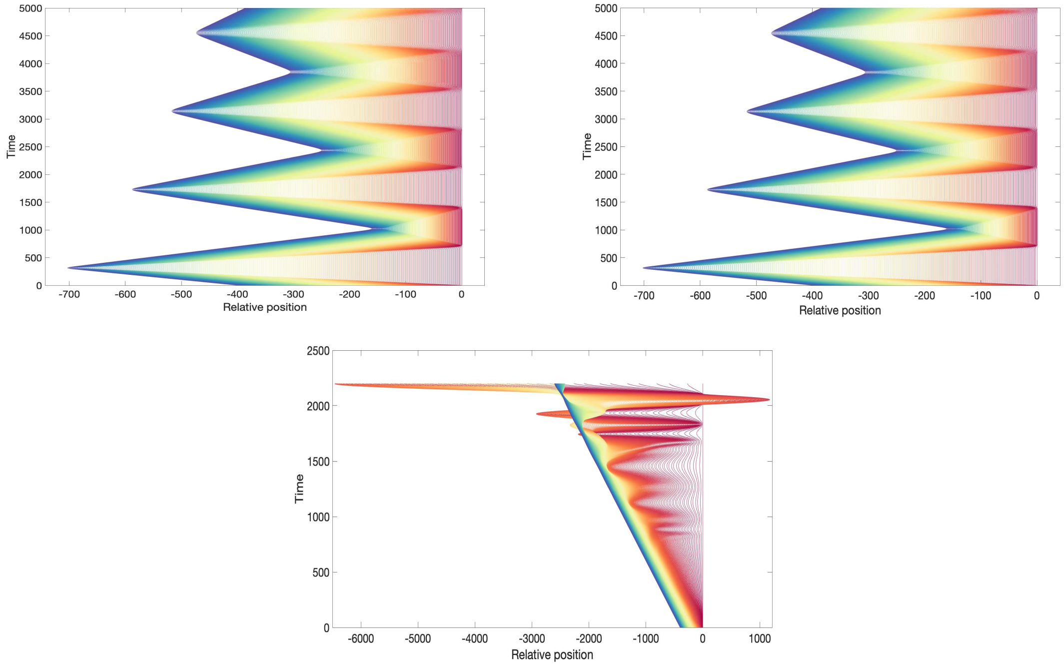

Figure 4: *a) Boundary Condition Type I. Maximum amplitude of* -303.9 *at* $t = 314.7$*. b) Boundary Condition Type II. Maximum amplitude of* -303.1 *at* $t = 314.3$*. c) Dynamics of an unstable system.*

Thus, we see that the first moment apparently needs a quadratic correction. Again, we do some experiments to make sure that this phenomenon actually occurs in the simulations.

As before, we also check for dependence on (non-trivial) boundary condition. In type I boundary conditions, the central coefficients $\rho_{x,0}^{(i)}$, and $\rho_{v,0}^{(i)}$ are adjusted.

$$
\begin{aligned}
\ddot{z}_1^{(1)} &= 0\\
\ddot{z}_N^{(1)} &= g_x^{(1)}\left(-(\rho_{x,1}^{(1)} + \rho_{x,-1}^{(1)} + \rho_{x,-2}^{(1)})z_N^{(1)} + \rho_{x,1}^{(1)}z_N^{(2)} + \rho_{x,-1}^{(1)}z_{N-1}^{(2)} + \rho_{x,-2}^{(1)}z_{N-1}^{(1)}\right)\\
&\quad + g_v^{(1)}\left(-(\rho_{v,1}^{(1)} + \rho_{v,-1}^{(1)} + \rho_{v,-2}^{(1)})\dot{z}_N^{(1)} + \rho_{v,1}^{(1)}\dot{z}_N^{(2)} + \rho_{v,-1}^{(1)}\dot{z}_{N-1}^{(2)} + \rho_{v,-2}^{(1)}\dot{z}_{N-1}^{(1)}\right)\\
\ddot{z}_1^{(2)} &= g_x^{(2)}\left(-(\rho_{x,-1}^{(2)} + \rho_{x,1}^{(2)} + \rho_{x,2}^{(2)})z_1^{(2)} + \rho_{x,-1}^{(2)}z_1^{(1)} + \rho_{x,1}^{(2)}z_2^{(1)} + \rho_{x,2}^{(2)}z_2^{(2)}\right)\\
&\quad + g_v^{(2)}\left(-(\rho_{v,-1}^{(2)} + \rho_{v,1}^{(2)} + \rho_{v,2}^{(2)})\dot{z}_1^{(2)} + \rho_{v,-1}^{(2)}\dot{z}_1^{(1)} + \rho_{v,1}^{(2)}\dot{z}_2^{(1)} + \rho_{v,2}^{(2)}\dot{z}_2^{(2)}\right)\\
\ddot{z}_N^{(2)} &= g_x^{(2)}\left(-(\rho_{x,-1}^{(2)} + \rho_{x,-2}^{(2)})z_N^{(2)} + \rho_{x,-1}^{(2)}z_N^{(1)} + \rho_{x,-2}^{(2)}z_{N-1}^{(2)}\right)\\
&\quad + g_v^{(2)}\left(-(\rho_{v,-1}^{(2)} + \rho_{v,-2}^{(2)})\dot{z}_N^{(2)} + \rho_{v,-1}^{(2)}\dot{z}_N^{(1)} + \rho_{v,-2}^{(2)}\dot{z}_{N-1}^{(2)}\right)
\end{aligned}
\quad .
$$

For Type II BC, we keep the central coefficients $\rho_{x,0}^{(i)}$, and $\rho_{v,0}^{(i)}$ equal to 1 and we adjust the remaining coefficients accordingly such that the sum of coefficients is zero as follows:

$$\begin{aligned}
\ddot{z}_1^{(1)} &= 0 \\
\ddot{z}_N^{(1)} &= g_x^{(1)} \left(z_N^{(1)} + \rho_{x,1}^{(1)} z_N^{(2)} + \rho_{x,-1}^{(1)} z_{N-1}^{(2)} - (1 + \rho_{x,1}^{(1)} + \rho_{x,-1}^{(1)}) z_{N-1}^{(1)} \right) \\
&\quad + g_v^{(1)} \left(\dot{z}_N^{(1)} + \rho_{v,1}^{(1)} \dot{z}_N^{(2)} + \rho_{v,-1}^{(1)} \dot{z}_{N-1}^{(2)} - (1 + \rho_{z,1}^{(1)} + \rho_{v,-1}^{(1)}) \dot{z}_{N-1}^{(1)} \right) \\
\ddot{z}_1^{(2)} &= g_x^{(2)} \left(z_1^{(2)} + \rho_{x,-1}^{(2)} z_1^{(1)} + \rho_{x,1}^{(2)} z_2^{(1)} - (1 + \rho_{x,1}^{(2)} + \rho_{x,-1}^{(2)}) z_2^{(2)} \right) \\
&\quad + g_v^{(2)} \left(\dot{z}_1^{(2)} + \rho_{v,-1}^{(2)} \dot{z}_1^{(1)} + \rho_{v,1}^{(2)} \dot{z}_2^{(1)} - (1 + \rho_{v,1}^{(2)} + \rho_{v,-1}^{(2)}) \dot{z}_2^{(2)} \right) \\
\ddot{z}_N^{(2)} &= g_x^{(2)} \left(z_N^{(2)} + (\rho_{x,1}^{(2)} + \rho_{x,-1}^{(2)}) z_N^{(1)} + (\rho_{x,2}^{(2)} + \rho_{x,-2}^{(2)}) z_{N-1}^{(2)} \right) \\
&\quad + g_v^{(2)} \left(\dot{z}_N^{(2)} + (\rho_{v,1}^{(2)} + \rho_{v,-1}^{(2)}) \dot{z}_N^{(1)} + (\rho_{v,2}^{(2)} + \rho_{v,-2}^{(2)}) \dot{z}_{N-1}^{(2)} \right)
\end{aligned}$$

We run simulations of the system in $\mathbb{R}$ considering these two boundary conditions with initial condition:

$$z_k^{(i)}(0) = \dot{z}_k^{(i)}(0) = 0 \quad \underline{\text{except}} \quad \dot{z}_1^{(1)}(0) = 1 \; .$$

Figures 4a) and 4b) show the dynamics of a system with next nearest neighbor interactions and boundary conditions Type I and Type II respectively. The parameters were chosen to satisfy Theorem 3.1 as follows:

$$\begin{aligned}
&\rho_{x,1}^{(1)} = -0.0833\cdots, \rho_{x,-1}^{(1)} = -0.25, \rho_{x,2}^{(1)} = -0.333\cdots, \rho_{x,-2}^{(1)} = -0.333\cdots \\
&\rho_{x,1}^{(2)} = -0.45, \rho_{x,-1}^{(2)} = -0.15, \rho_{x,2}^{(2)} = -0.20, \rho_{x,-2}^{(2)} = -0.20 \\
&\rho_{v,1}^{(1)} = -0.30, \rho_{v,-1}^{(1)} = -0.70, \rho_{v,2}^{(1)} = 0, \rho_{v,-2}^{(1)} = 0 \\
&\rho_{v,1}^{(2)} = -0.30, \rho_{v,-1}^{(2)} = -0.70, \rho_{v,2}^{(2)} = 0, \rho_{v,-2}^{(2)} = 0 \\
&N = 200 \text{ (of each type)}, g_x^{(1)} = g_x^{(2)} = g_v^{(1)} = g_v^{(2)} = -1
\end{aligned} \tag{14}$$

On the other hand, Figure 4c) shows the dynamics of a system in $\mathbb{R}$ with an evident instability of some type. All parameters are the same as in the previous simulation just above *except*

$$\begin{aligned}
&\rho_{x,1}^{(1)} = -0.30, \rho_{x,-1}^{(1)} = -0.25, \rho_{x,2}^{(1)} = -0.25, \rho_{x,-2}^{(1)} = -0.20 \\
&\rho_{x,1}^{(2)} = -0.30, \rho_{x,-1}^{(2)} = -0.55, \rho_{x,2}^{(2)} = -0.10, \rho_{x,-2}^{(2)} = -0.05 \; .
\end{aligned} \tag{15}$$

These were chosen to satisfy $\sum_{i\in\{1,2\}} \beta_{x,1}^{(i)} + 2\beta_{x,2}^{(i)} = 0$, but not the condition of Corollary 3.1.

Appendix

Proposition 3.2. *For $n \geq 2$, define Q_n as follows:*

$$Q_n(z) = \sum_{i=2}^{n} a_i(t) z^i + 2a_1(t) z + a_0(t) \; ,$$

where the a_i are analytic functions on $\mathbb{R}$ modulo 2π. Assume further that

$$a_0(0) = a_1(0) = 0 \quad \textit{and} \quad a_2(0) \neq 0 \quad \textit{and} \quad a_0'(0) \neq 0 \; .$$

Then there is a neighborhood N of the origin and an $\epsilon > 0$ in which the zeros of $\{Q_n(t)\}_{t\in(-\epsilon,\epsilon)}$ form two differentiable curves intersecting orthogonally at the origin.

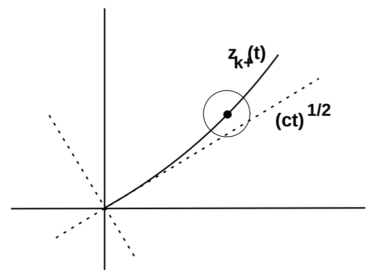

Figure 5: *The curve* γ_L *around* $z_{k,+}(t)$ *(solid) which itself is on a curve tangent to* $\sqrt{ct}$ *(dashed).*

In particular, it follows that near the origin, the solutions form a perpendicular cross and thus at least one on the arms of the cross extends into the right half-plane.

Proof. We start with $n = 2$. In this case, we can write out the solutions:

$$z_\pm(t) = \frac{-a_1 \pm \sqrt{-a_0 a_2 + a_1^2}}{a_2} = \pm\sqrt{-\frac{a_0}{a_2}}\,\sqrt{1 - \frac{a_1^2}{a_0 a_2}} \;-\frac{a_1}{a_2}\;.$$

Let us define a curve $\delta(t)$ to be tangent to a curve $\eta(t)$ at the origin for $t = 0$ if $\delta(0) = \eta(0) = 0$ and

$$\lim_{t\to 0} \frac{|\delta(t) - \eta(t)|}{|\eta(t)|} = 0\;.$$

One checks that we need all the assumptions on the coefficients a_i, $i \in \{0, 1, 2\}$, to show that $z_\pm(t)$ is tangent to $\pm\sqrt{-\frac{a_0'(0)}{a_2(0)}\,t}$.

We proceed by doing $n-2$ induction steps. Given Q_n, we form all the intermediate polynomials $\{Q_k\}_{k=2}^n$. Consider $t \in N_\epsilon = (-\epsilon, \epsilon)$ for ϵ small. We wish to prove that $t \in N_\epsilon$, the solutions of Q_k form two curves $z_{k,\pm}(t)$ tangent at the origin to $\pm\sqrt{-\frac{a_0'(0)}{a_2(0)}\,t}$ which we will from now one denote by $\pm\sqrt{ct}$. See Figure 5.

We proved the statement holds for $n = 2$. The induction hypothesis is that the above statement holds for some fixed $k \in \{2, \cdots n-1\}$. Fix an arbitrarily large L. Then fix $\epsilon > 0$ small enough, so that the conditions in the following hold. Q_k has no other zeros in an $2\sqrt{|c\epsilon|}$ neighborhood. So in the $\sqrt{|c\epsilon|}$ neighborhood, Q_k can be written as $(z - z_{k,+})(z - z_{k,-})\tilde{Q}_k(t,z)$, where $|\tilde{Q}_k(t,z)| \geq \frac{1}{2}|\tilde{Q}_k(0,0)| \neq 0$. Let $\gamma_L(s)$ be the curve $z_{k,+}(t) + \frac{|z_{k,+}(t)|}{L}\,e^{is}$. By the induction hypothesis, $z_{k,\pm}(t)$ are tangent to $\pm\sqrt{ct}$. Now we have for ϵ small enough

$$\begin{aligned}
|a_{k+1}(t)\gamma_L^{k+1}| &\leq |a_{k+1}(t)|\,|z_{k,+}(t)|^{k+1}\,|1 + L^{-1}|^{k+1} \\
&\leq 2|a_{k+1}(0)|\,|2ct|^{\frac{k+1}{2}}\;. \\
|Q_k(\gamma_L)| &= |\gamma_L - z_{k,+}|\,|\gamma_L - z_{k,-}|\,|\tilde{Q}_k(t,\gamma_L)| \quad \text{where} \quad \tilde{Q}_k(0,0) \neq 0 \\
&= \tfrac{|z_{k,+}(t)|}{L}\,|z_{k,+}(t)(1 + \tfrac{1}{L}\,e^{is}) - z_{k,-}(t)|\,|\tilde{Q}_k(t,z)| \\
&\geq \frac{\sqrt{|ct|}}{2L}\,\frac{\sqrt{|ct|}}{2}\,\frac{|\tilde{Q}_k(0,0)|}{2}\;.
\end{aligned}$$

Thus we can choose t small enough so that, on γ_L, $|a_{k+1}(t)z^{k+1}|$ is smaller than $|Q_k(z)|$. Since neither function has poles, Rouché's theorem [12] implies that $a_{k+1}(t)z^{k+1} + Q_k(z)$ has the same number of zeros inside γ_L as does $Q_k(z)$, namely one. Thus $Q_{k+1}(z)$ has a unique zero within γ_L. Since we can do this for any value of L (at the price of making ϵ small enough), it follows that $z_{k+1,+}(t)$ is tangent to $z_{k,+}(t)$

and hence to $\sqrt{ct}$. Since we need only finitely many induction steps to get to $z_{n,+}(t)$, the statement of the proposition follows. ☐

We also need the following Corollary. The proof is similar, except that the base case now is not quadratic anymore. This complicates the first step of the proof. Since our main results do not depend on it, the details of its proof will appear elsewhere.

Corollary 3.2. *For $n > k \geq 2$, define Q_n as follows:*

$$Q_n(z) = \sum_{i=k}^{n} a_i(t) z^i + \sum_{i=0}^{k-1} a_i(t) z^i \ ,$$

where the a_i are analytic functions on $\mathbb{R}$ modulo 2π. Assume further that

$$\text{For } i \in \{0 \cdots k-1\} : \ a_i(0) = 0 \ , \ \text{and} \quad a_k(0) \neq 0 \quad \text{and} \quad a_0'(0) \neq 0 \ .$$

Then there is a neighborhood N of the origin in which the zeros of Q_n form k differentiable curves intersecting at equal angles at the origin.

References

[1] N. Ashcroft, D. Mermin, *Solid State Physics*, Harcourt, 1976. 430

[2] Pablo E. Baldivieso, *Coupled Systems of Linear Differential Equations*, PhD Dissertation, Portland State University, 2019.

[3] B. Bamieh, F. Paganini, and M. Dahleh. Distributed control of spatially invariant systems. IEEE Trans. Autom. Control, 47:1091-1107, 2002.

[4] Carlos E. Cantos, J. J. P. Veerman, D. K. Hammond, Signal Velocities in Oscillator Networks, *European Physical Journal, Special Topics*, 225, 1115-1126, Springer, 2016.

[5] Carlos E. Cantos, D. K. Hammond, J. J. P. Veerman, Transients in the Synchronization of Oscillator Arrays, *European Physical Journal, Special Topics* 225, 1199-1209, Springer, 2016.

[6] M. Defoort, F. Floquet, A. Kokosy, and W. Perruquetti. Sliding-mode formation control for cooperative autonomous mobile robots. IEEE Trans. Ind. Ele., 55:3944-3953, 2008.

[7] F. R. Gantmacher. *The Theory of Matrices*, volume 2. American Mathematical Society, Providence, USA, 2000. 172

[8] J. Herbrych, A. G. Hazirakis, N. Christakis, J. J. P. Veerman, Dynamics of Locally Coupled Oscillators with Next-Nearest-Neighbor Interaction, Differential Equations and Dynamical Systems, Accepted, 2017. https://doi.org/10.1007/s12591-017-0377-3

[9] I. Herman, D. Martinec, and J. J. P. Veerman, Transients of Platoons with Asymmetric and Different Laplacians, Systems & Control Letters, Vol. 91, 28-35, May 2016.

[10] G. Lafferriere, A. Williams, John S. Caughman, J. J. P. Veerman, *Decentralized Control of Vehicle Formations*, *Systems & Control Letters*, 54, 899-910, 2005.

[11] F. Lin, M. Fardad, and M. R. Jovanovic. Optimal control of vehicular formations with nearest neighbor interactions. IEEE Trans. Autom. Control, 57:2203-2218, 2012.

[12] J. E. Marsden, M. J. Hoffman, *Basic Complex Analysis, Third Edition*, Freeman and Company, New York , 1999.

[13] Okubo A (1986) Dynamical aspects of animal grouping: Swarms, schools, flocks and herds. Adv Biophys 22: 194.

[14] I. Kra, S. R. Simanca, On Circulant Matrices, *Notices AMS*, March 2012. 368-377

[15] J. J. P. Veerman, G. Lafferriere, John S. Caughman, A. Williams, *Flocks and Formations*, *Jourmal of Statistical Physics* 121, Vol 5-6, 901-936, 2005.

[16] Young GF, Scardovi L, Cavagna A, Giardina I, Leonard NE (2013) Starling Flock Networks Manage Uncertainty in Consensus at Low Cost. PLoS Comput Biol 9(1): e1002894. https://doi.org/10.1371/journal.pcbi.1002894.