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The Kalai-Smorodinsky Solution with Loss Aversion

Bram Driesen[*]   Andrés Perea[*]   Hans Peters[*]

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Abstract

We consider bargaining games under the assumption that bargainers are loss averse, i.e. experience disutility from obtaining an outcome lower than some reference point. We follow the approach of Shalev (2002) by imposing the *self-supporting* condition on a solution. Given a bargaining game, we say outcome z is self-supporting under a given bargaining solution, whenever transforming the game using outcome z as reference point, yields a transformed game in which the solution is z.

We show that n-player bargaining games have a unique self-supporting outcome under the Kalai-Smorodinsky (KS) solution. We define a bargaining solution, giving exactly this outcome, and characterize it by the standard axioms of *Scale Invariance* [SI], *Individual Monotonicity* [IM], and *Strong Individual Rationality* [SIR], and a novel axiom called *Proportional Concession Invariance* [PCI]. A bargaining solution satisfies PCI if moving the utopia point in the direction of the solution outcome, does not change this outcome.

1   Introduction

In his seminal paper, Nash (1950) defined the 'bargaining problem' as a proxy for real life bargaining situations. His model is described by two players attempting to find agreement on a certain feasible outcome. They are motivated by the fact that failure to cooperate results in an unfavorable outcome for all. Nash himself defined and characterized a solution for this problem, the well known Nash bargaining solution. A wide range of other solutions have been formulated since. One of the most prominent alternatives to the Nash bargaining solution is the *Kalai-Smorodinsky* solution, defined by Raiffa (1953) and characterized by Kalai and Smorodinsky (1975).

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In economics – and bargaining is no exception – risk attitudes tend to play a pervasive role on how agents behave. In the bargaining literature, much attention has been paid to the influence of risk attitudes, and in particular risk aversion, on the outcomes players finally agree on according to certain bargaining solutions. Several studies, among which Kannai (1977), Kihlstrom, Roth, and Schmeidler (1981), and Roth (1985), find that the Nash bargaining solution favors the less risk averse bargainer. Kihlstrom, Roth and Schmeidler (1981) find similar results for the Kalai-Smorodinsky solution. Köbberling and Peters (2003) also study the effect of risk aversion on the Kalai-Smorodinsky solution, but distinguish between *probabilistic* risk aversion and *utility* risk aversion. They find it is an advantage to have a more utility risk averse opponent, or a less probabilistically risk averse opponent.

In the present article we investigate the Kalai-Smorodinsky solution under a related behavioral phenomenon, called *loss aversion*. Loss aversion was introduced by Kahneman and Tversky (1979, 1984). It is based on the premise that economic agents derive utility from changes in wealth in comparison to some reference point, and that changes for the worse – i.e. losses – carry more weight than equivalent changes for the better.

To introduce this concept in the bargaining problem, we follow the approach of Shalev (2002). Each bargainer's preference is represented by a von Neumann-Morgenstern *utility function*, a nonnegative *loss aversion coefficient*, and a *reference point*; if a player's utility level is below his reference point, then he experiences a disutility equal to the size of his incurred *loss*, inflated by the loss aversion coefficient. Thus, in a bargaining context, incorporating the players' loss aversion is equivalent to applying a particular transformation to the bargaining problem. Reference points are subsequently endogenized by imposing the *self-supporting* condition. An outcome z is said to be self-supporting under a given solution, whenever transforming the bargaining game using outcome z as reference point, yields a game of which the solution is z. We may interpret a bargainer's reference point as the expectation or the aspiration of what utility payoff that bargainer may realize given a certain bargaining solution. The self-supporting condition then imposes that the bargaining solution assigns to each player exactly the (initially) aspired utility level, and as a consequence, that no losses are incurred.

The first purpose of this article is to show that n-player bargaining games, $n \geq 2$, have exactly one outcome that is self-supporting under the Kalai-Smorodinsky solution[1]. Kalai and Smorodinsky (1975) defined their solution on two-player bargaining games. In view of Roth's (1979) result that it is not generally defined on all n-player bargaining games, we consider the subclass of all n-player bargaining games, defined by Peters and Tijs (1984), which excludes the problematic cases. On these games we define a class of asymmetric n-person Kalai-Smorodinsky solutions. Consider an outcome in the Pareto set and the line segments respectively connecting that outcome to the disagreement

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[1]This was already remarked – without proof – by Shalev (2002) for the case of two-player bargaining games.

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point and the utopia point. For any two players we may then project these line segments into the plane. Our solution is defined by the unique outcome in the Pareto set such that for any two players, the slopes of these projected line segments satisfy a given proportion. We establish that for each loss aversion profile, there is a single bargaining solution in our class that yields the associated self-supporting outcome. This implies uniqueness of a self-supporting outcome under the Kalai-smorodinsky solution for n-player bargaining games. Moreover, the bargaining solution we define provides an easy method of finding this outcome.

We next provide an axiomatic characterization of this class of bargaining solutions. In particular, we show that it is fully defined by the standard properties *Strong Individual Rationality*, *Scale Invariance*, and *Individual Monotonicity*, and a new axiom named *Proportional Concession Invariance* [PCI]. The axiom of PCI says that if players make concessions with respect to their utopia values in such a way that the *new* utopia point is on the line segment connecting the solution outcome and the *original* utopia point, then the solution outcome is left unchanged.

The article proceeds as follows. Section 2 introduces the notation, and in Section 3 we show how loss aversion is incorporated in the bargaining problem. Section 4 describes the concept of monotonic curves, their associated bargaining solutions, and defines the Kalai-Smorodinsky solution as a special case. In Section 5 we define a class of bargaining solutions, and we show that every solution in this class is the self-supporting outcome under the Kalai-Smorodinsky solution for some loss aversion profile. Section 6 contains the axiomatic characterization of this class of bargaining solutions. We conclude in Section 7.

All claims made in the text are proven in the Appendix.

2   Preliminaries

The set of *players* or *bargainers* is denoted $N := \{1, \dots, n\}$, with $n \geq 2$. For $x, y \in \mathbb{R}^N$ we say $x \geq y$ if $x_i \geq y_i$ for all $i \in N$, and $x > y$ if $x_i > y_i$ for all $i \in N$. The relations '$\leq$' and '$<$' are defined similarly. We define $\mathbb{R}^N_+ := \{z \in \mathbb{R}^N \mid z \geq 0\}$ and $\mathbb{R}^N_{++} := \{z \in \mathbb{R}^N \mid z > 0\}$. For $x, y \in \mathbb{R}^N$ we denote by xy the vector $(x_1 y_1, \dots, x_n y_n)$, and for $S \subseteq \mathbb{R}^N$ we define $xS := \{xy \in \mathbb{R}^N \mid y \in S\}$. Similarly, we denote $(x_1 + y_1, \dots, x_n + y_n)$ as $x + y$, and the set $\{x + z \mid z \in S\}$ as $x + S$. The vector in $\mathbb{R}^N$ that has i-th coordinate equal to 1 and all other coordinates equal to 0 is denoted e^i. The vector e^M, $M \subseteq N$ and $M \neq \emptyset$, has all coordinates $i \in M$ equal to 1, and all coordinates $i \notin M$ equal to 0. For $x \in \mathbb{R}^N_{++}$, we denote $\left(\frac{1}{x_1}, \dots, \frac{1}{x_n} \right)$ as x^{-1}.

A *bargaining problem for N* is defined as a pair (S, d) where

- $S \subset \mathbb{R}^N$ is non-empty, closed, convex, and comprehensive,

- $d \in S$,

- there exists a $z \in S$ such that $z > d$, and

- $S_d := \{z \in S \mid z \geq d\}$ is bounded.

A set S is *comprehensive* if $x \in S$ and $x \geq y$ together imply $y \in S$. In a bargaining context, this assumption can be interpreted as free disposal of utility in the sense that any player can choose a lower utility payoff without it leading to an infeasible outcome. Bargainers seek agreement on an outcome z in S where each player $i \in N$ obtains utility z_i. In case no agreement is reached the *disagreement outcome d* results.

The set of all bargaining problems is denoted B^N. For $(S,d) \in B^N$ and each $i \in N$, we define

$$u_i(S,d) := \max\{z_i \mid z \in S_d\}.$$

This represents the highest possible utility payoff bargainer i can attain in the bargaining problem (S,d), given that no bargainer $j \in N$, $j \neq i$, obtains a utility payoff lower than d_j. The vector $u(S,d) := (u_1(S,d), \ldots, u_n(S,d))$ is termed the *utopia point* of (S,d). For all $(S,d) \in B^N$ we define the *Pareto set* of (S,d) as

$$P(S) := \{z \in S \mid \text{for all } x \in \mathbb{R}^N \text{ if } x \geq z \text{ and } x \neq z, \text{ then } x \notin S\}.$$

A *bargaining solution* or in short, a *solution*, is a map $\varphi : B^N \to \mathbb{R}^N$ that assigns to any bargaining problem $(S,d) \in B^N$ a single point $\varphi(S,d) \in S$.

3 Bargaining with Loss Aversion

Shalev (2000, 2002) introduced a transformation that models bargainers' loss aversion. Each bargainer $i \in N$ has a non-negative *loss aversion coefficient* λ_i and a *reference point* r_i. We denote the vector $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^N_+$ by λ. Similarly, $r := (r_1, \ldots, r_n) \in \mathbb{R}^N$. Each bargainer i evaluates a utility payoff $z_i \in \mathbb{R}$ by the transformation w_i, defined as

$$w_i(z_i, \lambda_i, r_i) := \begin{cases} z_i & \text{if } z_i \geq r_i \\ z_i - \lambda_i(r_i - z_i) & \text{if } z_i < r_i. \end{cases}$$

Thus, a bargainer i who incurs a loss – i.e. obtains a utility payoff z_i below his reference point r_i, experiences a disutility that is equal to his loss $r_i - z_i$, inflated by the loss aversion coefficient λ_i. Payoffs above the reference point are left unchanged. Note that we may also write

$$w_i(z_i, \lambda_i, r_i) = (1 + \lambda_i)z_i - \lambda_i \max\{r_i, z_i\}.$$

For utility outcomes $z \in \mathbb{R}^N$ we define

$$w(z, \lambda, r) := (w_1(z_1, \lambda_1, r_1), \ldots, w_n(z_n, \lambda_n, r_n)).$$

For sets $T \subset \mathbb{R}^N$ we write $w(T, \lambda, r) := \{w(z, \lambda, r) \mid z \in T\}$. Henceforth, the transformation $w : \mathbb{R}^N \times \mathbb{R}^N_+ \times \mathbb{R}^N \to \mathbb{R}^N$ is referred to as the *Shalev transformation*.

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For bargaining problems $(S, d) \in B^N$ we define

$$w((S, d), \lambda, r) := (w(S, \lambda, r), w(d, \lambda, r)).$$

We then have the following.

Lemma 3.1 *Let $(S, d) \in B^N$, $\lambda \in \mathbb{R}_+^N$, and $r \in \mathbb{R}^N$. Then $w((S, d), \lambda, r) \in B^N$.*

Let $\varphi : B^N \to \mathbb{R}^N$ be some bargaining solution, and let (S, d) be some problem in B^N. Then by Lemma 3.1, $\varphi(w(S, d), \lambda, r)$ is well-defined for all $\lambda \in \mathbb{R}_+^N$ and $r \in \mathbb{R}^N$. A point $z \in S$ is called a *self-supporting outcome under φ* if $z = \varphi(w((S, d), \lambda, z))$. The set of all such points is defined as

$$Self^\varphi((S, d), \lambda) := \{z \in S \mid z = \varphi(w((S, d), \lambda, z))\}.$$

Then $Self^\varphi : B^N \times \mathbb{R}_+^N \to \mathbb{R}^N$ is a correspondence that assigns to each bargaining problem $(S, d) \in B^N$ and loss aversion profile $\lambda \geq 0$ a possibly empty subset of S.

For two-player bargaining games (S, d), Shalev (2002) characterized the set of self-supporting outcomes under the Nash bargaining solution. Specifically, he showed that it is a closed, connected subset of the Pareto set $P(S)$.

4   The Kalai-Smorodinsky Solution

Raiffa (1953), and Kalai and Smorodinsky (1975) defined and characterized a solution – the Kalai-Smorodinsky solution (KS) – for bargaining problems in $B^{\{1,2\}}$. It was shown by Roth (1979) that the n-player extension of the KS solution, $n > 2$, is not defined on all bargaining problems in B^N. Therefore, Peters and Tijs (1984) introduced the rather large subclass I^N of bargaining problems in B^N for which this non-existence result does not hold. Consider the following condition.

For all $x \in S, x \geq d$, and $i \in N$ we have: $\qquad\qquad\qquad\qquad$ (1)

$x \notin P(S)$ and $x_i < u_i(S, d) \Rightarrow$ there is an $\varepsilon > 0$ such that $x + \varepsilon e^i \in S$

Then

$$I^N := \{(S, d) \in B^N \mid (S, d) \text{ satifies condition (1)}\}.$$

Condition (1) says that if a feasible outcome x is not Pareto optimal, then for any bargainer who does not get his utopia payoff there is a feasible outcome that this bargainer strictly prefers over x. The class of bargaining problems $(S, 0) \in I^N$ is denoted I_0^N; for bargaining problems in I_0^N we henceforth omit the disagreement point, i.e. we denote $(S, 0) \in I_0^N$ as S.

Peters and Tijs (1984) defined the n-player extension of the KS solution by making use of monotonic curves. Specifically, a *monotonic curve* for N is a map

$$\vartheta : [1, n] \to \left\{x \in \mathbb{R}_+^n \mid x_i \leq 1 \text{ for all } i \in N, \text{ and } 1 \leq \sum_{i \in N} x_i\right\},$$

such that for all $1 \leq s \leq t \leq n$ we have $\vartheta(s) \leq \vartheta(t)$ and $\sum_{i \in N} \vartheta_i(s) = s$. The set of all monotonic curves for N is denoted Θ^N.

Lemma 4.1 (Peters and Tijs, 1984) *For each $\vartheta \in \Theta^N$ and $S \in I_0^N$ with $u(S) = e^N$, the set*

$$P(S) \cap \{\vartheta(t) \mid t \in [1, n]\}$$

contains exactly one point.

Let ϑ be some monotonic curve in Θ^N. In view of Lemma 4.1 we can define $\rho^\vartheta : I^N \to \mathbb{R}^N$, *the solution associated with ϑ*. Let $S \in I_0^N$; if $u(S) = e^N$, then

$$\left\{\rho^\vartheta(S)\right\} := P(S) \cap \{\vartheta(t) \mid t \in [1, n]\},$$

and if $u(S) = \beta$, then $\rho^\vartheta(S) := \beta \rho^\vartheta(\beta^{-1}S)$. For $(S, d) \in I^N$, we define $\rho^\vartheta(S, d) := d + \rho^\vartheta(S - d)$. The class of all solutions associated with a monotonic curve in Θ^N is referred to as the class of *individually monotonic bargaining solutions*. The KS solution is an element of this class. Specifically, it is defined as

$$\rho^{\hat\vartheta} \text{ where } \hat\vartheta(t) := \frac{te^N}{n} \text{ for } t \in [1, n].$$

Observe that $\hat\vartheta$ defines a straight line in $\mathbb{R}^N$, which for bargaining games $S \in I_0^N$ where $u(S) = e^N$, is concurrent with the line connecting the disagreement point 0 and the utopia point e^N. For general bargaining problems $(S, d) \in I^N$, the KS solution is the intersection of the Pareto set $P(S)$ and the straight line that connects the disagreement point d and the utopia point $u(S, d)$. To ease notation, we also refer to $\rho^{\hat\vartheta}$ as KS, i.e. $KS \equiv \rho^{\hat\vartheta}$.

5 The Solution Class $\mathcal{D}^N$

In this section we show that self-supporting outcomes under the KS solution are well-defined, and that each game in I^N has exactly one such outcome. Peters and Tijs (1984) show that $I^{\{1,2\}} = B^{\{1,2\}}$, which implies that our result generalizes Shalev's (2002) remark about the uniqueness of a self-supporting outcome under the KS solution for two-player bargaining games. Furthermore, we introduce a class $\mathcal{D}^N$ of bargaining solutions on I^N, where for any $\lambda \in \mathbb{R}_+^N$ there is a single $\varphi \in \mathcal{D}^N$ such that $\varphi(S, d)$ is the unique self-supporting outcome of the game (S, d) under the KS solution.

Observe that from Lemma 3.1 and the fact that the Shalev transformation preserves the ordering of payoffs, we obtain that $(S, d) \in I^N$ implies $w((S, d), \lambda, r) \in I^N$ for all $\lambda \in \mathbb{R}_+^N$ and $r \in \mathbb{R}^N$. Therefore, $Self^{KS}((S, d), \lambda)$, the set of self-supporting outcomes under the KS solution, is well-defined.

We now introduce the class $\mathcal{D}^N$ of bargaining solutions. Denote by $\bar{N}$ the player set without player n – i.e. $\bar{N} := N \setminus \{n\}$, and define the correspondence

$D^k : I^N \to \mathbb{R}^N$ for all $k \in \mathbb{R}_{++}^{\bar{N}}$ as

$$D^k(S,d) := \{z \in P(S) \mid \text{for all } i \in \bar{N} \text{ we have:}$$

$$(u_n(S,d) - z_n)(z_i - d_i) = k_i(u_i(S,d) - z_i)(z_n - d_n)\}. \qquad (2)$$

It is not hard to verify that $D^k \neq D^{k'}$ whenever $k, k' \in \mathbb{R}_{++}^{\bar{N}}$ with $k \neq k'$. The set $\mathcal{D}^N := \left\{ D^k \mid k \in \mathbb{R}_{++}^{\bar{N}} \right\}$ is defined as the set that contains all such correspondences. For any $k \in \mathbb{R}_{++}^{\bar{N}}$, define

$$\vartheta^k(t) := \left\{ z \in G^k \mid \sum_{i=1}^{n} z_i = t \right\}, \qquad (3)$$

where $t \in [1,n]$, and $G^k := \{z \in \mathbb{R}_+^N \mid (1 - z_n)z_i = k_i(1 - z_i)z_n \text{ for all } i \in \bar{N}\}$. In Lemma A.1, which can be found in the Appendix, we show that any correspondence ϑ^k is a monotonic curve in Θ^N.

Theorem 5.1 *For all $k \in \mathbb{R}_{++}^{\bar{N}}$ and $(S,d) \in I^N$, we have $D^k(S,d) = \left\{ \rho^{\vartheta^k}(S,d) \right\}$.*

It follows from Theorem 5.1 that the set $\mathcal{D}^N := \left\{ D^k \mid k \in \mathbb{R}_{++}^{\bar{N}} \right\}$ is a subset of the class of individually monotonic bargaining solutions. The following theorem establishes the equivalence between $\{Self^{KS}(\, . \, , \lambda) \mid \lambda \in \mathbb{R}_+^N\}$ and $\mathcal{D}^N$.

Theorem 5.2 *For all $(S,d) \in I^N$ we have*

$$Self^{KS}((S,d), \lambda) = D^k(S,d)$$

where $k := (k_1, \ldots, k_{n-1})$, and $k_i := \frac{1+\lambda_n}{1+\lambda_i}$ for all $i \in \bar{N}$. That is,

$$Self^{KS}((S,d), \lambda) := \{z \in P(S) \mid \text{for all } i \in \bar{N} \text{ we have:}$$

$$(1 + \lambda_i)(u_n(S,d) - z_n)(z_i - d_i) = (1 + \lambda_n)(u_i(S,d) - z_i)(z_n - d_n)\}.$$

With a slight abuse of notation, we henceforth consider $D^k(S,d)$ to be an outcome. That is, if $D^k(S,d) = \{z\}$, then we write $D^k(S,d) = z$. Then also

$$D^k(S,d) = \left(D_1^k(S,d), \ldots, D_n^k(S,d) \right).$$

From Theorem 5.2 it follows that for each loss aversion profile $\lambda \in \mathbb{R}_+^N$, we may look at $Self^{KS}(\, . \, , \lambda)$ as an asymmetric n-player Kalai-Smorodinsky solution where the asymmetry is fully defined by the bargainers' degrees of loss aversion. In the following section we provide an axiomatic characterization of these solutions.

6   An Axiomatic Characterization of $\mathcal{D}^N$

From Theorem 5.1 it follows that $\mathcal{D}^N$ is a subclass of the individual monotonic bargaining solutions, defined and characterized by Peters and Tijs (1984). Of their axioms we retain *Scale Invariance* [SI] and *Individual Monotonicity* [IM].

(SI) $\varphi : B^N \to \mathbb{R}^n$ satisfies *Scale Invariance* if $t(\varphi(S,d)) = \varphi(t(S), t(d))$, where $t : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation $t(x) := \alpha + \beta x$, with $\alpha \in \mathbb{R}^N$, $\beta \in \mathbb{R}^n_{++}$, and $t(S) := \alpha + \beta S$ for $S \subset \mathbb{R}^N$.

(IM) $\varphi : B^N \to \mathbb{R}^n$ satisfies *Individual Monotonicity* if $\varphi_i(S,d) \leq \varphi_i(T,d)$ for all $(S,d), (T,d) \in B^N$ and $i \in N$ with $S \subseteq T$ and $u_j(S) = u_j(T)$ for all $j \in N \setminus \{i\}$.

The axiom of SI is consistent with the premise that bargainers' preferences are representable by von Neumann-Morgenstern utility functions. Kalai and Smorodinsky (1975) introduced IM as a possible alternative to Nash's (1950) *Independence of Irrelevant Alternatives* after this axiom had been criticized. A third well-known property we require is *Strong Individual Rationality* [SIR].

(SIR) $\varphi : B^N \to \mathbb{R}^n$ satisfies *Strong Individual Rationality* if $\varphi(S,d) > d$ for all $(S,d) \in B^N$.

We now introduce a novel axiom called *Proportional Concession Invariance* [PCI]. One can consider a solution outcome $\varphi(S,d)$ as a representation of concessions that bargainers make with respect to their utopia point values $u(S,d)$. The PCI axiom then says that if we replace $u(S,d)$ by a point $\hat{u}$ on the line segment connecting $\varphi(S,d)$ and $u(S,d)$, and shrink the bargaining set accordingly, then the solution outcome should not change. In other words, if the players' utopia values are reduced in such a way that their concessions with respect to the original solution outcome change proportionally, then this solution outcome should not change.

(PCI) $\varphi : B^N \to \mathbb{R}^n$ satisfies *Proportional Concession Invariance* if for a bargaining problem $(S,d) \in B^N$ with solution $\varphi(S,d)$, and a bargaining problem $(\hat{S}, d)$ with
$$\hat{S} := \{z \in S \mid z \leq \hat{u}\},$$
where $\hat{u} = \alpha \varphi(S,d) + (1-\alpha)u(S,d)$ for some $\alpha \in [0,1]$, we have $\varphi(\hat{S}, d) = \varphi(S,d)$.

The property PCI can also be thought of as a very weak form of IIA, and is therefore satisfied by the Nash bargaining solution and its asymmetric variants. Furthermore, observe that PCI is the counterpart to *Disagreement Point Convexity* [DPC], one of the axioms used by Peters and van Damme (1991) to characterize the class of asymmetric Nash bargaining solutions. This property requires that the solution outcome $\varphi(S,d)$ remain unchanged if we replace d

by a point $\hat{d}$ on the line segment connecting the disagreement point d and the solution outcome $\varphi(S, d)$.

It is worth noting that the characterization of $\mathcal{D}^N$ makes no use of *Pareto Optimality* [PO]. In fact, in the Appendix, we use an argument similar as in Roth (1977) to show that PO is implied by our axioms. The following theorem states our characterization result.

Theorem 6.1 *Let $\varphi : I^N \to \mathbb{R}^N$ be a bargaining solution. Then $\varphi \in \mathcal{D}^N$ if and only if φ satisfies SIR, SI, IM, and PCI.*

7 Concluding Remarks

We have established that bargaining games with n possibly loss averse players, $n \geq 2$, have exactly one self-supporting outcome under the Kalai-Smorodinsky solution. Moreover, we defined an asymmetric n-player version of the KS solution which directly captures the asymmetry resulting from the bargainers' degrees of loss aversion. We may summarize these results in the following corollary.

Corollary 7.1 *Let $\lambda \in \mathbb{R}_+^N$ be the profile of loss aversion. For every $(S, d) \in I^N$, an outcome $z \in S$ is self-supporting under the KS solution if and only if $z = D^k(S, d)$, where $k_i := \frac{1+\lambda_n}{1+\lambda_i}$ for all $i \in \bar{N}$.*

We have characterized the class $\mathcal{D}^N$ of all bargaining solutions D^k by the axioms of *Strong Individual Rationality*, *Scale Invariance*, *Individual Monotonicity*, and *Proportional Concession Invariance*. While the first three properties are standard in the axiomatic bargining literature, the last one is newly defined in this article.

A   Collected Proofs

In this appendix we prove the claims made in the article.

A.1   The proof of Lemma 3.1

Let $(S,d) \in B^N$, $\lambda \in \mathbb{R}_+^N$, and $r \in \mathbb{R}^N$. Since the Shalev transformation is continuous, one-to-one, and preserves the ordering of payoffs, we have that

- $w(S,\lambda,r)$ is non-empty, closed, and comprehensive,

- $w(d,\lambda,r) \in w(S,\lambda,r)$,

- there are $z \in w(S,\lambda,r)$ with $z > w(d,\lambda,r)$, and

- $w(S_d,\lambda,r)$ is bounded.

Left to show is that $w(S,\lambda,r)$ is convex. Let $x,y \in S$ and $t \in [0,1]$. By convexity of S, we have $tx + (1-t)y \in S$. By the fact that w is concave in the first coordinate, we have

$$w(tx + (1-t)y, \lambda, r) \geq tw(x,\lambda,r) + (1-t)w(y,\lambda,r).$$

By comprehensiveness of $w(S,\lambda,r)$ this implies

$$tw(x,\lambda,r) + (1-t)w(y,\lambda,r) \in w(S,\lambda,r).$$

This implies convexity of $w(S,\lambda,r)$. ∎

A.2   The proof of Theorem 5.1

First, we show the following.

Lemma A.1 *Let $k \in \mathbb{R}_{++}^{\bar{N}}$. Then the correspondence $\vartheta^k(t)$, $t \in [1,n]$, is a monotonic curve.*

Proof.   Let $\bar{s} \in [1,n]$. We show that there is a unique point $z^* \in \mathbb{R}_+^N$ such that $\vartheta^k(\bar{s}) = z^*$. That is, we show that the system of equations

$$(1 - z_n)z_i = k_i(1 - z_i)z_n \text{ for all } i \in \bar{N} \tag{4}$$

$$\sum_{j=1}^n z_j = \bar{s}, \tag{5}$$

has exactly one solution z^* in $\mathbb{R}_+^N$. Suppose the system has a solution $z \in \mathbb{R}_+^N$, and suppose $z_n > 1$. Then for each $i \in \bar{N}$ we either have $z_i > 1$ or $z_i < 0$, which by $z \in \mathbb{R}_+^N$ implies $z_i > 1$ for all $i \in N$. Since this is a violation of (5), we must have $z_n \leq 1$. Since $(1 - k_i)^{-1} \notin [0,1]$ for all $i \in \bar{N}$, we can write

$$z_i = \frac{k_i z_n}{1 - z_n(1 - k_i)} \text{ for all } i \in \bar{N}.$$

Then for all $i \in \bar{N}$, z_i is strictly increasing in z_n on the domain $[0, 1]$. The observations that

$$\sum_{j=1}^{n} z_j = 0 \leq \bar{s} \qquad \text{for } z_n = 0, \text{ and}$$

$$\sum_{j=1}^{n} z_j = n \geq \bar{s} \qquad \text{for } z_n = 1,$$

together with continuity of $\sum_{j=1}^{n} z_j$ in z_n then imply that there is exactly one $z^* \in \mathbb{R}_+^N$ that solves the system of equations (4) and (5). It follows that the correspondence ϑ^k is a map. Moreover, for all $1 \leq s \leq t \leq n$ we have $0 < \vartheta^k(s) \leq \vartheta^k(t) \leq e^N$ and $\sum_{i \in N} \vartheta_i^k(s) = s$. It follows that ϑ^k is a monotonic curve. ∎

To prove Theorem 5.1, we show that each map $D^k \in \mathcal{D}^N$ is the bargaining solution associated with the monotonic curve ϑ^k as defined in (3).

Proof of Theorem 5.1. Consider a normalized bargaining problem $T \in I_0^N$, i.e. $u(T) = e^N$. Let $k \in \mathbb{R}_{++}^{\bar{N}}$ and observe that by (2) we have

$$D^k(T) = P(T) \cap G^k.$$

By convexity of T we have $P(T) \subset \{z \in \mathbb{R}_+^N \mid \sum_{j \in N} z_j \geq 1\}$, implying

$$D^k(T) = P(T) \cap \{\vartheta^k(t) \mid t \in [1, n]\},$$

where ϑ^k is defined by (3). It follows that $D^k(T) = \left\{\rho^{\vartheta^k}(T)\right\}$. From this it is easily established that

$$D^k(S, d) = \left\{\rho^{\vartheta^k}(S, d)\right\}$$

for all $(S, d) \in I^N$. ∎

A.3 The Proof of Theorem 5.2

We start with the following lemma.

Lemma A.2 *Let $(S, d) \in I^N$. Then $D^k(S, d) = KS(S, d)$ if and only if $k = e^{\bar{N}}$.*

Proof. It is easy to show that for any $t \in [1, n)$, we have $\vartheta^k(t) = \hat{\vartheta}(t)$ if and only if $k = e^{\bar{N}}$. The result then follows from Theorem 5.1 and the definition of KS. ∎

Proof of Theorem 5.2 Let $(S, d) \in I^N$ and $\lambda \in \mathbb{R}_+^N$, and let $u \equiv u(S, d)$. By Lemma A.2 and the fact that $KS(S, d) \geq d$, we have

$KS(w((S, d), \lambda, z)) = \{x \in P(w(S, \lambda, z)) \mid \text{for all } i \in \bar{N} \text{ we have}$

$$(u_n - x_n)(x_i - (1 + \lambda_i)d_i + \lambda_i z_i) = (u_i - x_i)(x_n - (1 + \lambda_n)d_n + \lambda_n z_n)\}.$$

Observe that $z \in Self^{KS}((S,d), \lambda)$ if and only if $z = KS(w((S,d), \lambda, z))$. That is, $z \in Self^{KS}((S,d), \lambda)$ iff

(i) $z \in P(w(S, \lambda, z))$, and

(ii) $(u_n - z_n)(z_i - (1 + \lambda_i)d_i + \lambda_i z_i) = (u_i - z_i)(z_n - (1 + \lambda_n)d_n + \lambda_n z_n)$ for all $i \in \bar{N}$.

From $z = w(z, \lambda, z)$, $P(w(S, \lambda, z)) = w(P(S), \lambda, z)$, and the fact that the Shalev transformation is one-to-one, it follows that $z \in P(w(S, \lambda, z))$ is equivalent to $z \in P(S)$.

Define $k := (k_1, \ldots, k_{n-1})$ where $k_i := \frac{1 + \lambda_n}{1 + \lambda_i}$ for all $i \in \bar{N}$. Observe that the statement in (ii) is equivalent to

$$(u_n - z_n)(z_i - d_i) = k_i(u_i - z_i)(z_n - d_n) \text{ for all } i \in \bar{N}.$$

It follows that $z \in Self^{KS}((S,d), \lambda)$ is equivalent to

$$z \in \left\{ x \in P(S) \mid (u_n - x_n)(x_i - d_i) = k_i(u_i - x_i)(x_n - d_n) \text{ for all } i \in \bar{N} \right\}.$$

Hence, $Self^{KS}((S,d), \lambda) = D^k(S,d)$. ∎

A.4 The proof of Theorem 6.1

The standard axiom of Pareto Optimality is useful for the proof.

(PO) $\varphi : B^N \to \mathbb{R}^N$ satisfies PO if $\varphi(S,d) \in P(S)$ for all $(S,d) \in B^N$.

From Peters and Tijs (1984) we obtain the following.

Lemma A.3 (Peters and Tijs, 1984) *Let $\varphi : I^N \to \mathbb{R}^N$ be a bargaining solution. Then φ satisfies PO, SI, and IM, if and only if $\varphi = \rho^\vartheta$ for some $\vartheta \in \Theta^N$.*

This we use to establish the following result.

Proposition A.4 *Let $\varphi : I^N \to \mathbb{R}^N$ be a bargaining solution in $\mathcal{D}^N$. Then φ satisfies SIR, SI, IM, and PCI.*

Proof. Since $\varphi \in \mathcal{D}^N$ we have $\varphi = D^k$ for some $k \in \mathbb{R}^{\bar{N}}_{++}$. By Theorem 5.1, we have $\varphi = \rho^\vartheta$ where $\vartheta \in \Theta^N$, which by Lemma A.3 implies that φ satisfies SI and IM.

Consider a bargaining problem $(S,d) \in I^N$, and the solution $\varphi(S,d) =: z$. To see that φ satisfies SIR, suppose there is an $i \in N$ such that $z_i = d_i$. Observe that $z \in P(S)$ and

$$(u_n(S,d) - z_n)(z_j - d_j) = k_j(u_j(S,d) - z_j)(z_n - d_n) \text{ for all } j \in \bar{N}.$$

If $i = n$, then $z_j = d_j$ for all $j \in \bar{N}$, implying $z = d$. Let $i \in \bar{N}$, and observe that $z_i = d_i$ implies $z_n = d_n$, and thus $z = d$. Since $d \notin P(S)$, we arrive at a contradiction. It follows that $z > d$.

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To see that φ satisfies PCI, consider the problem $(\hat{S}, d) \in I^N$ where

$$\hat{S} := \{x \in S \mid x \leq \hat{u}\},$$

with $\hat{u} = \alpha z + (1-\alpha) u(S, d)$ for some $\alpha \in (0, 1)$. Then $\hat{u} - z = (1-\alpha)(u(S, d) - z)$, implying that

$$(\hat{u}_n - z_n)(z_i - d_i) = k_i(\hat{u}_i - z_i)(z_n - d_n) \text{ for all } i \in \bar{N}. \tag{6}$$

Since $z \in \hat{S}$ and $z \in P(S)$, we have $z \in P(\hat{S})$. This and (6) together imply $D^k(\hat{S}, d) = z$. Hence, φ satisfies PCI. ∎

For the reverse implication we require two additional lemmas.

Lemma A.5 *Let $(S, d) \in I^N$, and $z \in S \setminus P(S)$. Then for the function $f :$ $[0, 1] \to \mathbb{R}^N$ defined as*

$$f(\alpha) := (1 - \alpha)z + \alpha u(S, d),$$

there is exactly one $\alpha^* \in [0, 1]$ such that $f(\alpha^*) \in P(S)$.

Proof. By compactness of S we have that

$$\alpha^* := \max\{\alpha \mid f(\alpha) \in S\}$$

is well-defined. We now show that $f(\alpha^*) \in P(S)$. Suppose $f(\alpha^*) \notin P(S)$. By condition (1), it follows that for each $i \in N$ with $z_i < u_i(S, d)$, there is an $\varepsilon_i > 0$ such that $f(\alpha^*) + \varepsilon_i e^i \in S$. Then by convexity of S there is an $\varepsilon > 0$, such that

$$f(\alpha^*) + \varepsilon(u(S, d) - z) \in S.$$

But then there is a $\beta > \alpha^*$ with $f(\beta) \in S$. This is a contradiction.
To show uniqueness, let $\alpha_1, \alpha_2 \in [0, 1]$ with $\alpha_1 \neq \alpha_2$, and suppose $f(\alpha_1), f(\alpha_2) \in P(S)$. Without loss of generality, assume $\alpha_2 > \alpha_1$. Then since $u(S, d) \geq z$ and $u(S, d) \neq z$, we have $f(\alpha_2) \geq f(\alpha_1)$ and $f(\alpha_2) \neq f(\alpha_1)$. Since $f(\alpha_1) \in P(S)$, this implies $f(\alpha_2) \notin S$, a contradiction. ∎

Lemma A.6 *Let $\varphi : I^N \to \mathbb{R}^N$ be a solution satisfying SIR, SI, and PCI. Then φ satisfies PO.*

Proof. Let $\varphi : I^N \to \mathbb{R}^N$ be a bargaining solution satisfying SIR, SI, and PCI. By SI it is sufficient to restrict attention to bargaining problems in I_0^N. Let $S \in I_0^N$. By SIR we have $\varphi(S) > 0$. Now assume $\varphi(S) \notin P(S)$. By Lemma A.5, there is a single $z^* \in P(S)$, such that

$$z^* = (1 - \alpha)\varphi(S) + \alpha u(S) \text{ for some } \alpha \in (0, 1].$$

Define $\hat{S} := \{x \in S \mid x \leq z^*\}$, and observe that by PCI we have

$$\varphi(\hat{S}) = \varphi(S).$$

Similarly, for the set $T := \{x \in S \mid x \leq \varphi(S)\}$ we have $\varphi(T) = \varphi(S)$. Now observe that $\hat{S} = [z^*(\varphi(S))^{-1}]T$. Then by SI we have

$$\varphi(\hat{S}) = z^*(\varphi(S))^{-1}\varphi(T) = z^*(\varphi(S))^{-1}\varphi(S) = z^*.$$

This contradicts $\varphi(\hat{S}) = \varphi(S)$. ∎

Proposition A.7 *Let $\varphi : I^N \to \mathbb{R}^N$ be a bargaining solution satisfying SIR, SI, IM, and PCI. Then $\varphi \in \mathcal{D}^N$.*

Proof.   Since φ satisfies SIR, SI, and PCI, it follows from Lemma A.6 that φ satisfies PO. Then by Lemma A.3 it follows that $\varphi \equiv \rho^{\vartheta^*}$ for some monotonic curve $\vartheta^* \in \Theta^N$.

Consider the problem

$$H := Conv\left(\{e^i \mid i \in N\} \cup \{0\}\right),$$

and observe that $H \in I_0^N$. If there is an $i \in N$ with $\varphi_i(H) = 1$, then $\varphi_j(H) = 0$ for all $j \neq i$, which is a violation SIR. Hence, $0 < \varphi(H) < e^N$. Hence, $k := (k_1, \ldots, k_{n-1})$, where

$$k_i := \frac{1 - \varphi_n(H)}{1 - \varphi_i(H)} \cdot \frac{\varphi_i(H)}{\varphi_n(H)}, \tag{7}$$

is well-defined and $k \in \mathbb{R}_{++}^{\bar{N}}$.

In what follows, we show that

$$\vartheta^*(t) = \vartheta^k(t) \tag{8}$$

for all $1 \leq t \leq n$. We do this in three steps. Let $\bar{t} \in [1, n]$. Then

1. we construct a specific problem $S \in I_0^N$,

2. we show that $\varphi(S) = \vartheta^k(\bar{t})$, and

3. we show that $\varphi(S) = \vartheta^*(\bar{t})$.

By (8) we have $\rho^{\vartheta^k} = \rho^{\vartheta^*}$, which together with $\varphi = \rho^{\vartheta^*}$ and $D^k = \rho^{\vartheta^k}$ establishes $\varphi = D^k$.

Step 1: Define the function $g : [0, 1] \to [1, n]$ as

$$g(\alpha) := \sum_{i \in N} \frac{\varphi_i(H)}{\alpha\varphi_i(H) + (1 - \alpha)}. \tag{9}$$

From the fact that g is strictly increasing[2] and continuous, and the fact that $g(0) = 1$ and $g(1) = n$, it follows that for each $t \in [1, n]$ there is a unique $\alpha \in [0, 1]$ such that $g(\alpha) = t$.

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[2]This follows from $\varphi(H) \neq 0$ and $\varphi(H) \neq e^N$.

Let $\beta := \bar{\alpha}\varphi(H) + (1 - \bar{\alpha})e^N$, where $\bar{\alpha}$ is such that $g(\bar{\alpha}) = \bar{t}$. Then define the problem $S \in I_0^N$ as

$$S := \{\beta^{-1}z \mid z \in H \text{ and } z \leq \beta\}.$$

Since φ satisfies SI and PCI, we have

$$\varphi(S) = \beta^{-1}\varphi(H). \tag{10}$$

It follows from (10) and (9), and the fact that $g(\bar{\alpha}) = \bar{t}$ that $\sum_{i \in N} \varphi_i(S) = \bar{t}$.

Step 2: Rewriting (7) yields

$$(1 - \varphi_n(H))\varphi_i(H) = k_i(1 - \varphi_i(H))\varphi_n(H) \text{ for all } i \in \bar{N}.$$

From the definition of β we have $\beta - \varphi(H) = (1 - \bar{\alpha})(e^N - \varphi(H))$. Thus,

$$(\beta_n - \varphi_n(H))\varphi_i(H) = k_i(\beta_i - \varphi_i(H))\varphi_n(H) \text{ for all } i \in \bar{N}.$$

For each $i \in \bar{N}$, we can multiply both sides of the equation by $\frac{1}{\beta_n\beta_i}$. By (10) this yields

$$(1 - \varphi_n(S))\varphi_i(S) = k_i(1 - \varphi_i(S))\varphi_n(S) \text{ for all } i \in \bar{N}.$$

It follows that $\varphi(S) \in G^k$. Since $\sum_{i \in N} \varphi_i(S) = \bar{t}$, we have

$$\varphi(S) = \vartheta^k(\bar{t}). \tag{11}$$

Step 3: Since $\varphi \equiv \rho^{\vartheta^*}$, and $S \in I_0^N$ with $u(S) = e^N$, we have

$$\{\varphi(S)\} = P(S) \cap \{\vartheta^*(t) \mid t \in [1, n]\},$$

which implies $\varphi(S) = \vartheta^*(t^*)$ for some $t^* \in [1, n]$. From the definition of monotonic curves we obtain

$$t^* = \sum_{i \in N} \vartheta_i^*(t^*) = \sum_{i \in N} \varphi_i(S) = \bar{t}.$$

Hence,

$$\varphi(S) = \vartheta^*(\bar{t}). \tag{12}$$

Combining (11) and (12) yields the desired result. ∎

Theorem 6.1 is then established by Propositions A.4 and A.7.

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