Chapter 2

Strain energy functions

The aims of *constitutive theories* are to develop mathematical models for representing the real behavior of matter, to determine the material response and in general, to distinguish one material from another. As described in the preceding chapter, constitutive equations for hyperelastic materials postulate the existence of a strain energy function W. There are several theoretical frameworks for the analysis and derivation of constitutive equations, for example the *Rivlin-Signorini method* where the governing idea is to expand the strain energy function in a power series of the invariants, or the *Valanis-Landel approach* expressing the strain energy directly in terms of the principal stretches [115].

In this chapter, we make no attempt at presenting these methods but instead, we present some classical explicit forms of strain-energy functions used in the literature for some isotropic hyperelastic materials. Many other models have been proposed (for example, a collection of constitutive models for rubber can be found in [32]).

2.1 Strain energy functions for incompressible materials

2.1.1 Neo-Hookean model

The *neo-Hookean model* is one of the simplest strain energy functions. It involves a single parameter and provides a mathematically simple and reliable constitutive model for the non-linear deformation behavior of isotropic rubber-like materials. Its strain energy function is

$$W = \frac{\mu}{2}(I_1 - 3), \tag{2.1}$$

where $\mu > 0$ is the shear modulus for infinitesimal deformations. The neo-Hookean model comes out of the molecular theory, in which vulcanized rubber is regarded as a three-dimensional network of long-chain molecules that are connected at a few points. The elementary molecular theory of networks is based on the postulate that the elastic free energy of a network is equal to the sum of the elastic free energies of the individual chains. In order to derive (2.1), a Gaussian distribution for the

probability of the end-to-end vector of the single chain is also assumed. While in a phenomenological theory the constitutive parameters are dictated only by the functional form considered, in a molecular theory the parameters are introduced on the basis of the modeled phenomena and consequently, are related *ex ante* to physical quantities. In this framework the constitutive parameter μ is determined by micromechanics parameters, as

$$\mu = nkT, \tag{2.2}$$

where n is the chain density, k is the Boltzmann constant and T is the absolute temperature. Although it poorly captures the basic features of rubber behaviour, the neo-Hookean model is much used in finite elasticity theory because of its "good" mathematical properties (for example a huge number of exact solutions to boundary value problems may be found using this model).

2.1.2 Mooney-Rivlin model

To improve the fitting to data, Rivlin introduced a dependence of the strain energy function on both the first and second invariants. A slightly more general model than neo-Hookean is therefore a simple, or two-term, *Mooney-Rivlin model*, for which the strain energy function is assumed to be linear in the first and second invariant of the Cauchy-Green strain tensor. This model is of purely phenomenological origin, and was originally derived by Mooney [84]. The strain energy may be written as

$$W = \tfrac{1}{2}\left(\tfrac{1}{2}+\gamma\right)\mu(I_1-3) + \tfrac{1}{2}\left(\tfrac{1}{2}-\gamma\right)\mu(I_2-3), \tag{2.3}$$

where γ is a dimensionless constant in the range $-1/2 \leq \gamma \leq 1/2$ and $\mu > 0$ is the shear modulus for infinitesimal deformations. When $\gamma = 1/2$, we recover the neo-Hookean model (2.1). Mooney [84] showed that the form (2.3) is the most general one which is valid for large deformations of an incompressible hyperelastic material, isotropic in its undeformed state, for the relation between the shearing force and amount of simple shear to be linear. Hence the constant μ is also the shear modulus for large shears.

By considering the expansion of the strain energy function in power series of $(I_1 - 3)$ and $(I_2 - 3)$ terms, it can be shown that for small deformations, the quantities $(I_1 - 3)$ and $(I_2 - 3)$ are, in general, small quantities of the same order, so that (2.3) represents an approximation valid for sufficiently small ranges of deformations, extending slightly the range of deformations described by the neo-Hookean model. This is pointed out in the figures (2.1 - 2.4) where the classical experimental data of Treloar [126] for simple tension and of Jones and Treloar [69] for equibiaxial tension are plotted (their numerical values having been obtained from the original experimental tables), and compared with the predictions of neo-Hookean and Mooney-Rivlin models.

In the first case, *simple tension*, the principal stresses are

$$t_1 = t, \qquad t_2 = t_3 = 0, \tag{2.4}$$

and requiring for the principal stretches

$$\lambda_1 = \lambda, \qquad \lambda_2 = \lambda_3 = \lambda^{-1/2}, \tag{2.5}$$

we obtain from the relation (1.40)

$$t = 2\left(\lambda^2 - \frac{1}{\lambda}\right)\left(W_1 + \frac{1}{\lambda}W_2\right). \tag{2.6}$$

In Figure (2.1) we report the classical data of Treloar, by plotting the Biot stress $f = t/\lambda$ defined per unit reference cross-sectional area against the stretch λ. In Figure (2.2), we used the so-called *Mooney plot* (widely used in the experiment literature to compare the different models) because it is sensitive to relative errors. It represents the Biot stress $f = t/\lambda$ divided by the universal geometrical factor $2\,(\lambda - 1/\lambda^2)$, plotted against $1/\lambda$:

$$\frac{f}{2\,(\lambda - 1/\lambda^2)} = W_1 + \frac{1}{\lambda}W_2. \tag{2.7}$$

The Mooney-Rivlin model, fitting to data, improves the neo-Hookean model for small and moderate stretches. In fact, in the case of simple extension, the curves in (2.1) and (2.2) for the models under examination are obtained considering only the early part of the data. For large extensions, the Mooney-Rivlin curve gives a bad fitting. This fact may be emphasized by the Mooney plot (2.2), where the Mooney-Rivlin curve is a straight line, and is seen to fit only a reduced range of data.

For the *equibiaxial tension* test we let

$$t_1 = t_2 = t, \quad t_3 = 0, \tag{2.8}$$

and require the principal stretches to be

$$\lambda_1 = \lambda_2 = \lambda, \quad \lambda_3 = \lambda^{-2}, \tag{2.9}$$

so that we obtain by (1.40) the following relation for the principal stress,

$$t = 2\left(\lambda^2 - \frac{1}{\lambda^4}\right)\left(W_1 + \lambda^2 W_2\right). \tag{2.10}$$

In Figure (2.3), we report the classical data of Jones and Treloar [69] by plotting the Biot stress $f = t/\lambda$ against the stretch λ. In Figure (2.4) we represent the Mooney plot for the Biot stress divided by $2\,(\lambda - 1/\lambda^5)$, plotted against λ^2:

$$\frac{f}{2\,(\lambda - 1/\lambda^5)} = W_1 + \lambda^2 W_2. \tag{2.11}$$

The Mooney plot (2.4) reveals how the Mooney-Rivlin model extends slightly the range of data approximation compared to the neo-Hookean model, but cannot fit all of them.

The Mooney-Rivilin model has been studied extensively even though no rubber-like material seems to be described by it to within errors of experiment. It is used as the first illustration for every general result for isotropic incompressible materials for which several analytical solutions have been found.

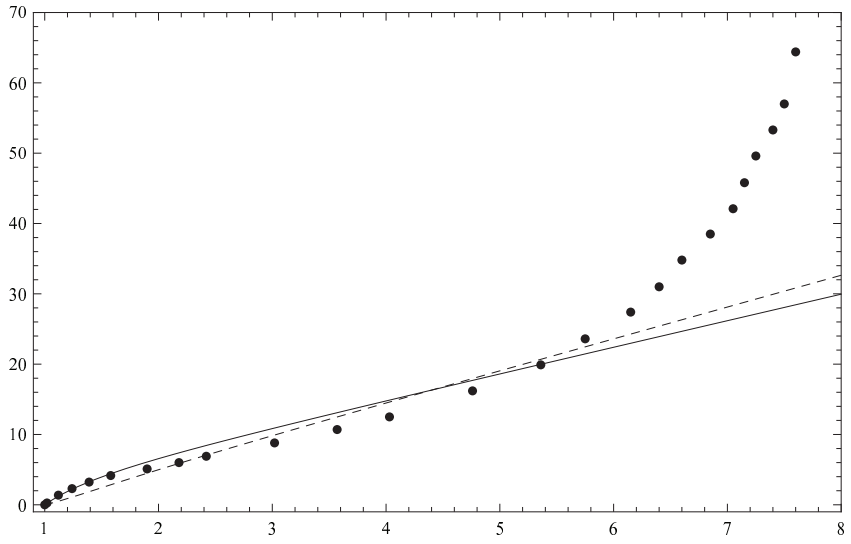

Figure 2.1: Plot of the simple tension data (circles) of Treloar [126] against the stretch λ, compared with the predictions of the Mooney-Rivlin model (dashed curve) and the neo-Hookean model (continuous curve). (In the figure, both models were optimized to fit the first 16 points, i.e. data for which $\lambda \in (1, 6.15)$).

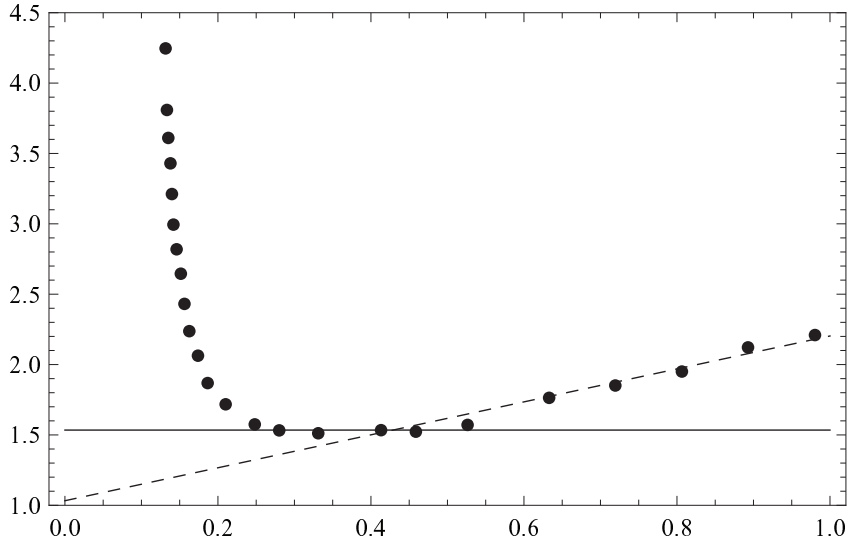

Figure 2.2: Plot of the simple tension data (circles) of Treloar [126] normalized by $2(\lambda - 1/\lambda^2)$ (λ is the stretch), against $1/\lambda$, compared with the predictions of the Mooney-Rivlin model (dashed curve) and the neo-Hookean model (continuous curve). (In the figure, the Mooney-Rivlin model has been optimized to fit the nine points for which $1/\lambda \in (0.33, 0.99)$ and the neo-Hookean model has been optimized to the five points for which $1/\lambda \in (0.28, 0.53)$).

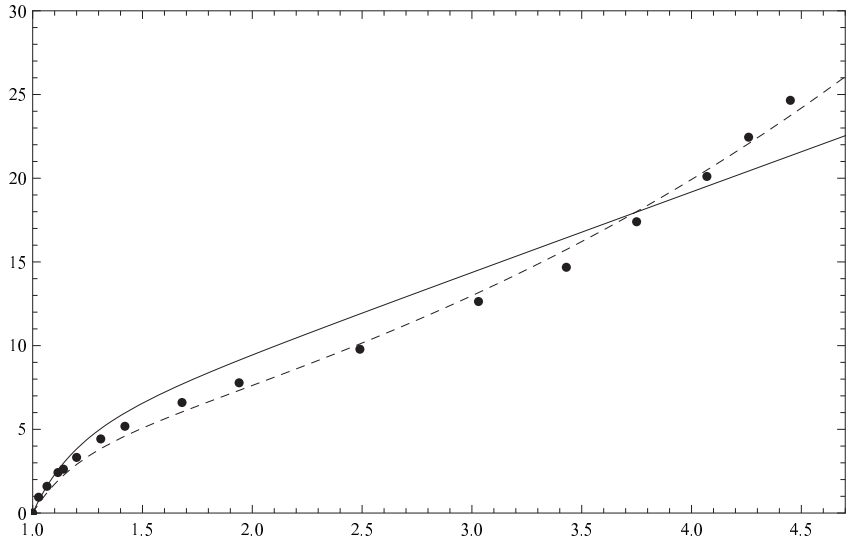

Figure 2.3: Plot of the equibiaxial tension data (circles) of Jones and Treloar [69], against the stretch λ, compared with the predictions of the Mooney-Rivlin model (dashed curve) and the neo-Hookean model (continuous curve). (In the figure, both models have been optimized to fit all seventeen points.)

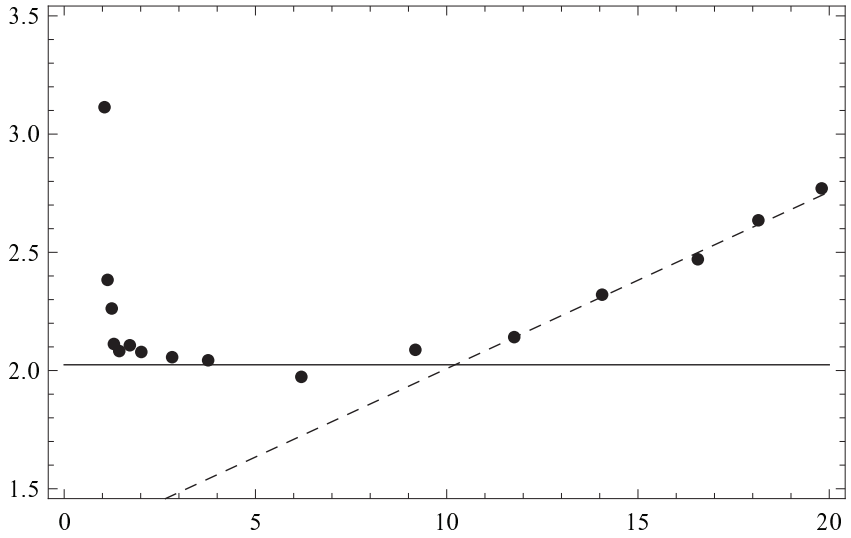

Figure 2.4: Plot of the equibiaxial tension data (circles) of Jones and Treloar [69] normalized by $2(\lambda - 1/\lambda^5)$ (λ is the stretch), against λ^2, compared with the predictions of the Mooney-Rivlin model (dashed curve) and the neo-Hookean model (continuous curve). (In the figure, the Mooney-Rivlin model has been optimized to fit the five data for which $\lambda^2 \in (11.76, 19.81)$ and the neo-Hookean model has been optimized to fit the three points for which $\lambda^2 \in (2.8, 6.2)$).

2.1.3 Generalized neo-Hookean model

Despite the idea of Rivlin to introduce the dependence of W of the second invariant I_2, there are several models of strain energy functions depending on the first invariant I_1 only. In the molecular theory, I_1 is connected to the mean squared end-to-end distance of the chains, but in general the chains cannot assume a completly arbitrary form and length. To overcome this constraint, the second invariant I_2, which is connected instead with the surface extension of material, is needed. Often the introduction of this invariant renders the calculations cumbersome, and from there follows the wide use of strain energies functions depending in a nonlinear manner on the first invariant only. A function of this form is called *generalized neo-Hookean model*,

$$W = W(I_1). \tag{2.12}$$

To account for the finite extensibility of the polymeric chains composing the elastomer network (since Gaussian statistics give rise to a probability density function without *compact support*), some models of the form (2.12) introduce a distribution function for the end-to-end distance of the polymeric chain which is not Gaussian. These models are usually called *non-Gaussian models*. From the phenomenological point of view these models can be divided into two classes: models with *limiting chain extensibility*, and *power-law* models. An example of the first class is due to Gent [45], who proposed the following strain energy density

$$W = -\frac{\mu}{2b}\ln\left[1 - b(\,I_1 - 3\,)\right], \tag{2.13}$$

where $b > 0$ is a limiting parameter value constant for I_1, accounting for limiting polymeric chain extensibility and $\mu > 0$ is the shear modulus for infinitesimal deformations. An example of the second class, widely used in biomechanics, was proposed by Fung [44] as follows

$$W = \frac{\mu}{2b}\exp\left[b\,(I_1 - 3) - 1\right], \tag{2.14}$$

where the dimensionless constant $b > 0$ is a stiffining parameter, and $\mu > 0$ is the shear modulus for infinitesimal deformations. Both classes behave as neo-Hookean solids in the small b/small-deformation limit, since they both obey

$$W(I_1, b) = \frac{\mu}{2}\,(I_1 - 3) + \frac{\mu b}{4}\,(I_1 - 3)^2 + O\left(b^2\,(I_1 - 3)^3\right) \tag{2.15}$$

as $b\,(I_1 - 3) \to 0$. Another power-law constitutive model was proposed by Knowles [73]. It can be written as

$$W = \begin{cases} \dfrac{\mu}{2\alpha\beta}\left[(1 + \beta\,(I_1 - 3))^{\alpha} - 1\right], & \text{if } \alpha \neq 0 \text{ and } \beta \neq 0, \\ \dfrac{\mu}{2\beta}\log\left(1 + \beta\,(I_1 - 3)\right), & \text{if } \alpha = 0 \text{ and } \beta \neq 0, \\ \dfrac{\mu}{2}\,(I_1 - 3), & \text{if } \beta = 0,\ (\forall\alpha), \end{cases} \tag{2.16}$$

where α and β are constants; when $\alpha = 1$ the neo-Hookean model (2.1) is recovered. Knowles introduced this model to describe both strain-stiffening and strain-softening effects in elastomeric materials and biological soft tissues. For a careful study of the analytical properties of the Knowles potential, see [15].

Even though some classical experimental data suggest that constitutive equations of the form (2.12) may have limited applicability, they nevertheless often lead to closed-form analytical solution for many interesting problems. Such solutions are useful for a better understanding of the mechanical properties of the matter and also as benchmarks for more complex numerical computations.

2.1.4 Other models

Rivlin and Saunders [112] showed that both neo-Hookean and Money-Rivlin models are not adequate to describe accurately the experimental properties of rubber. Their conclusion was that $\partial W/\partial I_1$ is independent of both I_1 and I_2, and that $\partial W/\partial I_2$ is independent of I_1 and decreases with increasing I_2. They thus deduced the strain energy function in the form

$$W = C(I_1 - 3) + f(I_2 - 3), \tag{2.17}$$

where C is a constant and f is a function whose slope diminishes continuously with increasing I_2. In the more recent work of Obata [92], it is found that neither $\partial W/\partial I_1$ nor $\partial W/\partial I_2$ can be regarded as constant, and that each should depend on both I_1 and I_2.

Valanis and Landel [128] proposed that the strain energy function W may be expressible as the sum of three functions of the principal stretches,

$$W = w(\lambda_1) + w(\lambda_2) + w(\lambda_3), \tag{2.18}$$

in which the function $w(\lambda)$ is, by symmetry, the same for each of the extension ratios. Equivalent to (2.18) is the expansion due to Ogden [93],

$$W = \sum_{m=1}^{\infty} \mu_m(\lambda_1^{\alpha_m} + \lambda_2^{\alpha_m} + \lambda_3^{\alpha_m} - 3)/\alpha_m \tag{2.19}$$

in terms of powers of the principal stretches, where each μ_m and α_m are material constants, not necessarily integers [93]. Jones and Treloar [69] and Ogden [115] show how the biaxial strain experiments are consistent with the Valanis-Landel model (2.18) and the Ogden expansion (2.19).

2.2 Strain energy functions for compressible materials

In the compressible case, as well as (1.34), a further assumption is required for W: it should approach infinity as I_3 tends to infinity or zero$^+$. In other words, an infinite amount of energy is required in order to expand the body to infinite volume or to compress it to a point with vanishing volume, so that

$$\lim_{I_3 \to +\infty} W = +\infty, \quad \lim_{I_3 \to 0^+} W = +\infty. \tag{2.20}$$

2.2.1 Hadamard model

Hadamard [51] introduced a class of elastic materials characterised by the property that infinitesimal longitudinal waves may propagate in every direction, when they are maintained in an arbitrary state of finite static homogeneous deformation. This constitutive model, called *Hadamard model* by John [68], describes also the only compressible isotropic homogeneous elastic material for which three linearly-polarized *finite* amplitude plane waves, one longitudinal and two transverse, may propagate in every direction when it is homogeneously deformed [24, 68]. The strain energy function is defined by

$$W = c_1(I_1 - 3) + c_2(I_2 - 3) + H(I_3), \tag{2.21}$$

where $c_1,\ c_2$ are material constants such that $c_1 > 0,\ c_2 \geq 0$, or $c_1 \geq 0,\ c_2 > 0$ and $H(I_3)$ is an arbitrary function to be specified on the basis of constitutive arguments. The connection with the Lamé constants of the linear theory is made through the relations

$$c_1 = \mu + H'(1), \quad c_2 = -\frac{\mu}{2} - H'(1), \quad 4H''(1) = \lambda + 2\mu. \tag{2.22}$$

An example for the function $H(I_3)$, accounting for the effects of compressibility, is given by Levinson and Burgess[1] [79]. They propose the following explicit form for the material function $H(I_3)$,

$$H(I_3) = (\lambda + \mu)\,(I_3 - 1) - (\lambda + 2\mu)\,(\sqrt{I_3} - 1). \tag{2.23}$$

2.2.2 Blatz-Ko model

The *Blatz-Ko model* is one of much used models describing the behavior of rubber in the compressible case. Replacing the principal invariants I_k by another set of independent invariants of $\boldsymbol{B}$, J_k defined by

$$J_1 \equiv I_1 = \mathrm{tr}\boldsymbol{B}, \quad J_2 \equiv I_2/I_3 = \mathrm{tr}\boldsymbol{B}^{-1}, \quad J_3 \equiv I_3^{1/2} = \det \boldsymbol{F}, \tag{2.24}$$

the strain energy function may be written as $W(J_1, J_2, J_3)$. Introducing (2.24) into (1.38), we find that

$$\beta_0 = \frac{\partial W}{\partial J_3}, \quad \beta_1 = \frac{2}{J_3}\frac{\partial W}{\partial J_1}, \quad \beta_{-1} = -\frac{2}{J_3}\frac{\partial W}{\partial J_2}. \tag{2.25}$$

Let us now consider a special class of materials whose response functions in (2.25) depend on J_3 alone. This is possible if and only if

$$\beta_0 = W_3(J_3), \quad \beta_1 = \frac{\alpha}{J_3}, \quad \beta_{-1} = -\frac{\beta}{J_3}, \tag{2.26}$$

where $W_3 \equiv \partial W/\partial J_3$ and α and β are constants. It can be shown that

$$\beta_1(1) - \beta_{-1}(1) = \alpha + \beta = \mu, \tag{2.27}$$

[1]We observe that Levinson and Burgess give an explicit form of $H(I_3)$ that does not verify $(2.20)_2$.

and introducing another constant f such that

$$\alpha = \mu f, \quad \beta = \mu(1-f), \tag{2.28}$$

the equation for the Cauchy stress for this special class of material is derived from (1.40) in the form

$$\boldsymbol{T} = W_3(J_3) + \frac{\mu f}{J_3}\boldsymbol{B} - \frac{\mu(1-f)}{J_3}\boldsymbol{B}^{-1}. \tag{2.29}$$

Considering a simple tensile loading

$$T_1 = t, \qquad T_2 = 0, \qquad T_3 = 0, \tag{2.30}$$

with principal stretches $(\lambda, \lambda_2, \lambda_3)$, Blatz and Ko [18] assumed (since in their experiment with $f = 0$ they found $J_3 = \lambda^{1/2}$) the following general constitutive assumption of volume control

$$J_3 = \lambda^n. \tag{2.31}$$

It follows from Batra's theorem [7] that

$$\lambda_2 = \lambda_3, \tag{2.32}$$

and from (2.31), that

$$\lambda_2(\lambda) = \lambda^{(n-1)/2}. \tag{2.33}$$

From (1.49) the infinitesimal strains are of the form $\epsilon_k = \lambda_k - 1$. Following [12] we define the *Poisson function* $\nu(\lambda)$ as

$$\nu(\lambda) = -\frac{\epsilon_3}{\epsilon_1} = \frac{1-\lambda_2(\lambda)}{\lambda - 1}, \tag{2.34}$$

from which the infinitesimal Poisson ratio is deduced in the limit

$$\nu = \lim_{\lambda \to 1} \nu(\lambda) = -\frac{(n-1)}{2}. \tag{2.35}$$

Therefore a Blatz-Ko material must verify

$$\lambda_2(\lambda) = \lambda^{-\nu}, \tag{2.36}$$

and consequently

$$\lambda = J_3^{1/(1-2\nu)}. \tag{2.37}$$

Blatz and Ko integrated the expression W_3 by making use of condition (2.37) and the condition $W(3,3,1) = 0$ in the natural state. They thus obtained the following general expression for the strain energy

$$W(J_1, J_2, J_3) = \frac{\mu f}{2}[(J_1 - 3) - \frac{2}{q}(J_3^q - 1)] + \frac{\mu(1-f)}{2}[(J_2 - 3) - \frac{2}{q}(J_3^{-q} - 1)], \tag{2.38}$$

where

$$q = \frac{n-1}{n} = \frac{-2\nu}{1-2\nu}. \tag{2.39}$$

Two special models of this expression (2.38), $f = 0$ and $f = 1$, are often used in applications. The former characterizes the class of *foamed, polyurethane elastomers* and the latter describes the class of *solid, polyurethane rubbers* studied in the Blatz-Ko experiments. We note that in the limit $I_3 \to 1$ it is possible to obtain the Mooney-Rivlin strain energy density for incompressible materials from (2.38). Thus (2.38) may be viewed as a generalization of the Mooney-Rivlin model to compressible materials. In the literature, a special compressible material of the first case ($f = 0$) is often used at $q = -1$, for which the strain energy, rewritten in terms of invariants I_k, is given by

$$W(I_1, I_2, I_3) = \frac{\mu}{2}\left(\frac{I_2}{I_3} + 2I_3^{1/2} - 5\right). \tag{2.40}$$

2.3 Weakly non-linear elasticity

To study small-but-finite elastic effects, the *weakly non-linear elasticity* theory [76], considers an expansion for the strain energy function in the following form

$$W = \frac{1}{2!}C_{ijkl}E_{ij}E_{kl} + \frac{1}{3!}C_{ijklmn}E_{ij}E_{kl}E_{mn} + \ldots, \tag{2.41}$$

where $C_{ijk\ldots}$ are constant moduli and $\boldsymbol{E} = \boldsymbol{E}^T$ is the Lagrange, or Green, strain tensor, defined as $\boldsymbol{E} = (\boldsymbol{C} - \boldsymbol{I})/2$. In the isotropic case, the strain energy (2.41) has the following expansion to the second order (*second-order elasticity*) as

$$W = \frac{\lambda}{2}(\mathrm{tr}\boldsymbol{E})^2 + \mu\ \mathrm{tr}(\boldsymbol{E}^2), \tag{2.42}$$

where λ and μ are the Lamé constants. At the third order (*third-order elasticity*), the expansion is (see [101] for example)

$$W = \frac{\lambda}{2}(\mathrm{tr}\boldsymbol{E})^2 + \mu\ \mathrm{tr}(\boldsymbol{E}^2) + \frac{\mathcal{A}}{3}\mathrm{tr}(\boldsymbol{E}^3) + \mathcal{B}(\mathrm{tr}\boldsymbol{E})\,\mathrm{tr}(\boldsymbol{E}^2) + \frac{\mathcal{C}}{3}(\mathrm{tr}\boldsymbol{E})^3, \tag{2.43}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are the *Landau third-order elastic constants*.

For incompressible solids the second-order expansion involves only one material constant: μ, and the third-order expansion involves only two material constants: μ and $\mathcal{A}$. They are written respectively as

$$W = \mu\ \mathrm{tr}(\boldsymbol{E}^2), \tag{2.44}$$

and

$$W = \mu\ \mathrm{tr}(\boldsymbol{E}^2) + \frac{\mathcal{A}}{3}\mathrm{tr}(\boldsymbol{E}^3). \tag{2.45}$$

Rivlin and Saunders [112] showed that the Mooney-Rivlin strain-energy function (2.3) of exact non-linear incompressible elasticity coincides, at the same order of

approximation, with the general weakly nonlinear third-order elasticity expansion (2.45). Introducing the following constants

$$C_1 = \frac{1}{2}\left(\frac{1}{2} + \gamma\right)\mu, \quad C_2 = \frac{1}{2}\left(\frac{1}{2} - \gamma\right)\mu, \tag{2.46}$$

in (2.3), the connections between the material constants are

$$\mu = 2(C_1 + C_2), \quad \mathcal{A} = -8(C_1 + 2C_2). \tag{2.47}$$

Notes

This presentation of theoretical framework for the constitutive equations includes many but not all models proposed in literature. One of the main problems encountered in the applications of mechanics of continua is the complete and accurate determination of the constitutive relations necessary for the mathematical description of the behavior of real materials. Indeed people working with rubber know very well that the mechanical behavior of this material is very complex and outside of the forecast possibilities of nonlinear elasticity (see Saccomandi [115]).

One of the omissions, in this chapter is the so-called Rivlin-Signorini method. First Murnaghan [85] and then Rivlin [110] and Signorini [118] approximated the material response functions by polynomials in the appropriate invariants. In this way, a particular material is then characterized by the constant coefficients of the polynomial rather than by functions. Applications of the Rivlin-Signorini method can be found in [81, 120]. Although from a theoretical point of view, any complete set of invariants is equivalent to another, it has been observed by several authors that the approach used by Rivlin considering the principal invariants it is not very practical in fitting experimental data, because of the possible propagation of experimental errors (see for example [128]). Therefore it may be interesting to consider the possibility of expressing the strain energy directly in terms of the principal stretches and to overcome some difficulties related to the symmetry. That is why Valanis and Landel [128] postulated that the strain energy function be a sum of functions each depending on a single stretch (see (2.18)).