Chapter 1

Introduction to Elasticity

This introductory chapter presents some basic concepts of continuum mechanics, symbols and notations for future reference.

1.1 Kinematics of finite deformations

We call $\mathcal{B}$ a *material body*, defined to be a three-dimensional differentiable manifold, the elements of which are called *particles* (or *material points*) P. This manifold is referred to a system of co-ordinates which establishes a one-to-one correspondence between particles and a region B (called a *configuration* of $\mathcal{B}$) in three-dimensional Euclidean space by its position vector $\boldsymbol{X}(P)$. As the body deforms, its configuration changes with time. Let $t \in I \subset \mathbb{R}$ denote time, and associate a unique B_t, the configuration at time t of $\mathcal{B}$; then the one-parameter family of all configurations $\{B_t : t \in I\}$ is called a *motion* of $\mathcal{B}$.

It is convenient to identify a *reference configuration*, B_r say, which is an arbitrarily chosen fixed configuration at some prescribed time r. Then we label by $\boldsymbol{X}$ any particle P of $\mathcal{B}$ in B_r and by $\boldsymbol{x}$ the position vector of P in the configuration B_t (called *current configuration*) at time t. Since B_r and B_t are configurations of $\mathcal{B}$, there exists a bijection mapping $\boldsymbol{\chi} : B_r \to B_t$ such that

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}) \quad \text{and} \quad \boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}). \tag{1.1}$$

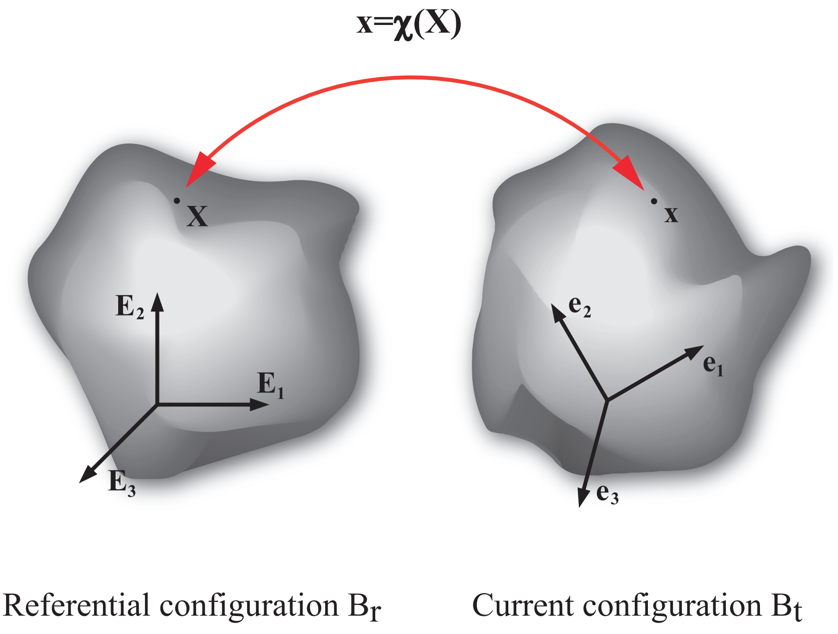
The mapping $\boldsymbol{\chi}$ is called the *deformation* of the body from B_r to B_t and since the latter depends on t, we write

$$\boldsymbol{x} = \boldsymbol{\chi}_t(\boldsymbol{X}) \quad \text{and} \quad \boldsymbol{X} = \boldsymbol{\chi}_t^{-1}(\boldsymbol{x}), \tag{1.2}$$

instead of (1.1), or equivalently,

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t) \quad \text{and} \quad \boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}, t), \tag{1.3}$$

for all $t \in I$. For each particle P (with label $\boldsymbol{X}$), $\boldsymbol{\chi}_t$ describes the motion of P with t as parameter, and hence the motion of $\mathcal{B}$. We assume that a sufficient number of derivatives of $\boldsymbol{\chi}_t$ (with respect to position and time) exists and that they are continuous.



The *velocity* $\boldsymbol{v}$ and the *acceleration* $\boldsymbol{a}$ of a particle P are defined as

$$\boldsymbol{v} \equiv \dot{\boldsymbol{x}} = \frac{\partial}{\partial t}\boldsymbol{\chi}(\boldsymbol{X}, t) \tag{1.4}$$

and

$$\boldsymbol{a} \equiv \dot{\boldsymbol{v}} \equiv \ddot{\boldsymbol{x}} = \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}(\boldsymbol{X}, t), \tag{1.5}$$

respectively, where the superposed dot indicates differentiation with respect to t at fixed $\boldsymbol{X}$, i.e. the material time derivative.

We assume that the body is a contiguous collection of particles; we call this body a *continuum* and we define the *deformation gradient tensor* $\boldsymbol{F}$ as a second-order tensor,

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \text{Grad } \boldsymbol{x} \equiv \text{Grad } \boldsymbol{\chi}(\boldsymbol{X}, t). \tag{1.6}$$

Here and henceforth, we use the notation Grad, Div, Curl (respectively grad, div, curl) to denote the gradient, divergence and curl operators in the reference (respectively, current) configuration, i.e with respect to $\boldsymbol{X}$ (respectively, $\boldsymbol{x}$).

We introduce the quantity

$$J = \det \boldsymbol{F} \tag{1.7}$$

and assume that $J \neq 0$, in order to have $\boldsymbol{F}$ invertible, with inverse

$$\boldsymbol{F}^{-1} = \text{grad } \boldsymbol{X}. \tag{1.8}$$

In general the deformation gradient $\boldsymbol{F}$ depends on X, i.e. varies from point to point and such deformation is said to be *inhomogeneous*. If, on the other hand, $\boldsymbol{F}$ is independent of $\boldsymbol{X}$ for the body in question then the deformation is said to be *homogeneous*. If the deformation is such that there is no change in volume, then the deformation is said to be *isochoric*, and

$$J \equiv 1. \tag{1.9}$$

A material for which (1.9) holds for all deformations is called an *incompressible material*.

The polar decomposition theorem of linear algebra applied to the nonsingular tensor $\boldsymbol{F}$ gives two unique multiplicative decompositions:

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} \quad \text{and} \quad \boldsymbol{F} = \boldsymbol{V}\boldsymbol{R}, \tag{1.10}$$

where $\boldsymbol{R}$ is the *rotation tensor* (and characterizes the local rigid body rotation of a material element), $\boldsymbol{U}$ is the *right stretch tensor*, and $\boldsymbol{V}$ is the *left stretch tensor* of the deformation ($\boldsymbol{U}$ and $\boldsymbol{V}$ describe the local deformation of the element). Using this decomposition for $\boldsymbol{F}$, we define two tensor measures of deformation called the *left* and *right Cauchy-Green strain tensors*, respectively, by

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T = \boldsymbol{V}^2, \qquad \boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{U}^2. \tag{1.11}$$

The couples ($\boldsymbol{U}$,$\boldsymbol{V}$) and ($\boldsymbol{B}$,$\boldsymbol{C}$) are *similar* tensors, that is, they are such that

$$\boldsymbol{V} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T, \quad \boldsymbol{B} = \boldsymbol{R}\boldsymbol{C}\boldsymbol{R}^T, \tag{1.12}$$

and therefore $\boldsymbol{U}$ and $\boldsymbol{V}$ have the same principal values $\lambda_1, \lambda_2, \lambda_3$, say, and $\boldsymbol{B}$ and $\boldsymbol{C}$ have the same principal values $\lambda_1^2, \lambda_2^2, \lambda_3^2$. Their respective principal directions $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are related by the rotation $\boldsymbol{R}$,

$$\boldsymbol{\nu} = \boldsymbol{R}\boldsymbol{\mu}. \tag{1.13}$$

The λ's are the stretches of the three principal material lines; they are called *principal stretches*.

1.2 Balance laws, stress and equations of motion

Let A_r, in the reference configuration, be a set of points occupied by a subset $\mathcal{A}$ of a body $\mathcal{B}$. We define a function m called a *mass function* in the following way

$$m(A_r) = \int_{A_r} \rho_r \, \mathrm{d}V, \tag{1.14}$$

where ρ_r is the density of mass per unit volume V. In the current configuration, the mass of A_t is calculated as

$$m(A_t) = \int_{A_t} \rho \, \mathrm{d}v, \tag{1.15}$$

where in this case ρ is the density of mass per unit volume v. The *local mass conservation law* is expressed by

$$\rho = J^{-1}\rho_r, \tag{1.16}$$

or equivalentely in the form

$$\dot{\rho} + \rho \mathrm{div}\boldsymbol{v} = 0. \tag{1.17}$$

This last form of mass conservation equation is also known as the *continuity equation*.

The forces that act on any part $A_t \subset B_t$ of a continuum $\mathcal{B}$ are of two kinds: a distribution of contact forces, which we denote $\boldsymbol{t_n}$ per unit area of the boundary ∂A_t of A_t, and a distribution of body forces, denoted $\boldsymbol{b}$ per unit volume of A_t. Applying the *Cauchy theorem*, we know that there exists a second-order tensor called the *Cauchy stress tensor*, which we denote $\boldsymbol{T}$, such that

(i) for each unit vector $\boldsymbol{n}$,

$$\boldsymbol{t_n} = \boldsymbol{T}\boldsymbol{n}, \tag{1.18}$$

where $\boldsymbol{T}$ is independent of $\boldsymbol{n}$,

(ii)

$$\boldsymbol{T}^T = \boldsymbol{T}, \tag{1.19}$$

and

(iii) $\boldsymbol{T}$ satisfies the *equation of motion*,

$$\text{div}\boldsymbol{T} + \rho\boldsymbol{b} = \rho\boldsymbol{a}. \tag{1.20}$$

Often, the Cauchy stress tensor is inconvenient in solid mechanics because the deformed configuration generally is not known *a priori*. Conversely, it is convenient to use the material description. To this end, we introduce the *engineering stress tensor* $\boldsymbol{T_R}$, also known as the *first Piola-Kirchhoff stress tensor*, in order to define the contact force distribution $\boldsymbol{t_N} \equiv \boldsymbol{T_R}\boldsymbol{N}$ in the reference configuration

$$\boldsymbol{T_R} = J\boldsymbol{T}\boldsymbol{F}^{-T}. \tag{1.21}$$

It is then possible to rewrite the balance laws corresponding to (1.18), (1.19) and (1.20), in the following form

$$\boldsymbol{t_N} = \boldsymbol{T_R}\boldsymbol{N}, \tag{1.22}$$

$$\boldsymbol{T_R}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{T_R}^T, \tag{1.23}$$

$$\text{Div}\boldsymbol{T_R} + \rho_r\boldsymbol{b}_r = \rho_r\ddot{\boldsymbol{x}}, \tag{1.24}$$

where $\boldsymbol{b}_r$ denotes the body force per unit volume in the reference configuration.

1.3 Isotropy and hyperelasticity: constitutive laws

We call *nominal stress tensor* the transpose of $\boldsymbol{T_R}$ that we denote by

$$\boldsymbol{S} = \boldsymbol{T_R}^T \tag{1.25}$$

and we call *hyperelastic* a solid whose elastic potential energy is given by the strain energy function $W(\boldsymbol{F})$ and such that

$$\boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{F}}(\boldsymbol{F}), \tag{1.26}$$

holds, relating the nominal stress and the deformation, or equivalently, such that

$$\boldsymbol{T} = J^{-1} \frac{\partial W}{\partial \boldsymbol{F}}^T \boldsymbol{F}^T, \tag{1.27}$$

relating the Cauchy stress and the deformation. In component form (1.26) and (1.27) read, respectively,

$$S_{ji} = \left(\frac{\partial W}{\partial F_{ij}} \right), \qquad T_{ij} = J^{-1} \frac{\partial W}{\partial F_{i\alpha}} F_{j\alpha}. \tag{1.28}$$

A material having the property that at a point $\boldsymbol{X}$ of undistorted state, every direction is an axis of material symmetry, is called *isotropic at* $\boldsymbol{X}$. A hyperelastic material which is isotropic at every material point in a global undistorted material is called an *isotropic hyperelastic material*; in this case, the strain energy density function can be expressed uniquely as a symmetric function of the principal stretches or in terms of the principal invariants I_1, I_2, I_3 of $\boldsymbol{B}$ (or equivalently, the principal invariants of $\boldsymbol{C}$, because in the isotropic case they coincide for every deformation $\boldsymbol{F}$), or in terms of the principal invariants i_1, i_2, i_3 of $\boldsymbol{V}$. Thus,

$$W = \hat{W}(\lambda_1, \lambda_2, \lambda_3) = \bar{W}(I_1, I_2, I_3) = \tilde{W}(i_1, i_2, i_3), \tag{1.29}$$

say, where

$$I_1 = \text{tr}\boldsymbol{B}, \quad I_2 = \tfrac{1}{2}[(\text{tr}\boldsymbol{B})^2 - \text{tr}\boldsymbol{B}^2], \quad I_3 = \det \boldsymbol{B}. \tag{1.30}$$

The principal invariants I_1, I_2, I_3 of $\boldsymbol{B}$ are given in terms of the principal stretches by

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \\ I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2. \end{aligned} \tag{1.31}$$

The principal invariants of $\boldsymbol{V}$ (and hence of $\boldsymbol{U}$), i_1, i_2, i_3, are given by:

$$\begin{aligned} i_1 &= \text{tr}\boldsymbol{V} = \lambda_1 + \lambda_2 + \lambda_3, \\ i_2 &= \tfrac{1}{2}[i_1^2 - \text{tr}\boldsymbol{V}] = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \\ i_3 &= \det\boldsymbol{V} = \lambda_1\lambda_2\lambda_3. \end{aligned} \tag{1.32}$$

The principal invariants of $\boldsymbol{B}$, given in (1.31), are connected with the principal invariants of $\boldsymbol{V}$ given in (1.32) by the relations

$$I_1 = i_1^2 - 2i_2, \qquad I_2 = i_2^2 - 2i_1 i_3, \qquad I_3 = i_3^2. \tag{1.33}$$

It is usual to require (for convenience) that the strain-energy function W should vanish in the reference configuration, where $\boldsymbol{F} = \boldsymbol{I}$, $I_1 = I_2 = 3$, $I_3 = 1$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Thus,

$$\bar{W}(3, 3, 1) = 0, \qquad \hat{W}(1, 1, 1) = 0. \tag{1.34}$$

After some algebraic manipulations, follow two useful forms for the general constitutive equation, which we write as

$$\boldsymbol{T} = \alpha_0 \boldsymbol{I} + \alpha_1 \boldsymbol{B} + \alpha_2 \boldsymbol{B}^2, \tag{1.35}$$

or, using the Cayley-Hamilton theorem, as

$$\boldsymbol{T} = \beta_0 \boldsymbol{I} + \beta_1 \boldsymbol{B} + \beta_{-1} \boldsymbol{B}^{-1}, \tag{1.36}$$

where

$$\alpha_i = \alpha_i(I_1, I_2, I_3), \qquad \beta_j = \beta_j(I_1, I_2, I_3), \tag{1.37}$$

$i = 0, 1, 2$; $j = 0, 1, -1$, are called *the material* or *elastic response functions*. In terms of the strain energy function they are given by

$$\begin{aligned}
\beta_0(I_1, I_2, I_3) &= \alpha_0 - I_2\alpha_2 = \frac{2}{\sqrt{I_3}}\left[I_2\frac{\partial W}{\partial I_2} + I_3\frac{\partial W}{\partial I_3}\right],\\
\beta_1(I_1, I_2, I_3) &= \alpha_1 + I_1\alpha_2 = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_1},\\
\beta_{-1}(I_1, I_2, I_3) &= I_3\alpha_2 = -2\sqrt{I_3}\frac{\partial W}{\partial I_2}.
\end{aligned} \tag{1.38}$$

When the hyperelastic isotropic material is also incompressible, it is possible to rewrite (1.35) and (1.36) as

$$\boldsymbol{T} = -p\boldsymbol{I} + \alpha_1 \boldsymbol{B} + \alpha_2 \boldsymbol{B}^2, \tag{1.39}$$

and

$$\boldsymbol{T} = -p\boldsymbol{I} + \beta_1 \boldsymbol{B} + \beta_{-1} \boldsymbol{B}^{-1}, \tag{1.40}$$

respectively, where p is an undetermined scalar function of $\boldsymbol{x}$ and t (p is a Lagrange multiplier). The undetermined parameter p differs in (1.39) and (1.40) by a $2I_2(\partial W/\partial I_2)$ term. Then the material response coefficients $\alpha_i = \alpha_i(I_1, I_2)$ and $\beta_j = \beta_j(I_1, I_2)$ with $i = 1, 2$ and $j = 1, -1$ are defined respectively by

$$\beta_1 = \alpha_1 + I_1\alpha_2 = 2\frac{\partial W}{\partial I_1}, \qquad \beta_{-1} = \alpha_2 = -2\frac{\partial W}{\partial I_2}. \tag{1.41}$$

We say that a body $\mathcal{B}$ is *homogeneous* if it is possible to choose a single reference configuration B_r of the whole body so that the response functions are the same for all particle.

The formulae (1.35), (1.36), (1.39) and (1.40), may be replaced by any other set of three independent symmetric invariants, for example by i_1, i_2, i_3, the principal invariants of $\boldsymbol{V}$. When the strain energy function W depends by the principal stretches, the principal Cauchy stress components (that we denote by T_i, $i = 1, 2, 3$) are given by

$$T_i = \frac{\lambda_i}{J}\frac{\partial W}{\partial \lambda_i} \tag{1.42}$$

for compressible materials, and by

$$T_i = \lambda_i\frac{\partial W}{\partial \lambda_i} - p, \tag{1.43}$$

for incompressible materials.

1.4 Restrictions and empirical inequalities

The response functions β_j are not completely arbitrary but must meet some requirements. First of all, if we ask our compressible (incompressible) model to be stress free in the reference configuration, then they must satisfy

$$\bar{\beta}_0 + \bar{\beta}_1 + \bar{\beta}_{-1} = 0, \quad (-\bar{p} + \bar{\beta}_1 + \bar{\beta}_{-1} = 0), \tag{1.44}$$

where $\bar{\beta}_j = \beta(3, 3, 1)$ (and $\bar{p} = p(3, 3, 1)$) are the values of the material functions (1.37) in the reference configuration. In general (to have hydrostatic stress $\boldsymbol{T}_0$) they must satisfy

$$\boldsymbol{T}_0 = (\bar{\beta}_0 + \bar{\beta}_1 + \bar{\beta}_{-1})\boldsymbol{I}, \quad (\boldsymbol{T}_0 = (-\bar{p}_0 + \bar{\beta}_1 + \bar{\beta}_{-1})\boldsymbol{I}). \tag{1.45}$$

The question of what other restrictions should be imposed in general on the strain energy functions of hyperelasticity theory, in order to capture the actual physical behavior of isotropic materials in finite deformation is of no less importance, and forms the substance of *Truesdell's problem*. To model *real* material behavior, we assume that the response functions β_j are compatible with fairly general empirical descriptions of mechanical response, derived from carefully controlled large deformation tests of isotropic materials. To this end we assume that the empirical inequalities imposed by Truesdell and Noll hold (see [127]). They are, in the compressible case,

$$\beta_0 \leq 0, \quad \beta_1 > 0, \quad \beta_{-1} \leq 0, \tag{1.46}$$

and in the incompressible case,

$$\beta_1 > 0, \quad \beta_{-1} \leq 0. \tag{1.47}$$

1.5 Linear elasticity and other specializations

In the special case of *linear* (linearized) elasticity, some constitutive restrictions must be considered also in order to reflect the real behavior of the material, and these restrictions lead to some important assumptions on the physical constants. Hence, let $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ be the mechanical displacement. In the case of small strains, the linear theory of elasticity is based on the following equations

$$\boldsymbol{T} = \boldsymbol{\mathcal{C}}[\boldsymbol{\epsilon}], \tag{1.48}$$

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right), \tag{1.49}$$

$$\text{Div}\boldsymbol{T} + \boldsymbol{b}_r = \rho \ddot{\boldsymbol{u}}, \tag{1.50}$$

where $\boldsymbol{\epsilon}$ denotes the *infinitesimal strain tensor* and $\boldsymbol{\mathcal{C}}$ the fourth-order tensor of elastic stiffness. These three equations represent the *stress-strain law*, *strain-displacement relation*, and the *equation of motion*, respectively. When the body is homogeneus and isotropic, the constitutive equation (1.48) reduces to

$$\boldsymbol{T} = 2\mu\boldsymbol{\epsilon} + \lambda(\text{tr}\boldsymbol{\epsilon})\boldsymbol{I}, \tag{1.51}$$

where μ and λ are the so-called *Lamé constants* or, in the inverted form,

$$\boldsymbol{\epsilon} = \frac{1}{E}\left[(1+\nu)\boldsymbol{T} - \nu(\mathrm{tr}\boldsymbol{T})\boldsymbol{I}\right], \tag{1.52}$$

where

$$E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}, \quad \nu = \frac{\lambda}{2(\mu+\lambda)}. \tag{1.53}$$

The second Lamé constant μ determines the response of the body in shear, at least within the linear theory, and for this reason is called the *shear modulus*. The constant E is known as *Young's modulus*, the constant ν as Poisson's ratio, and the quantity $\kappa = (2/3)\mu + \lambda$ as the *modulus of compression* or *bulk modulus*.

A linearly elastic solid should increase its length when pulled, should decrease its volume when acted on by a pure pressure, and should respond to a positive shearing strain by a positive shearing stress. These restrictions are equivalent to either sets of inequalities

$$\mu > 0, \quad \kappa > 0; \tag{1.54}$$

$$E > 0, \quad -1 < \nu \leq 1/2. \tag{1.55}$$

In the incompressible case, the constitutive equation (1.51) is replaced by

$$\boldsymbol{T} = 2\mu\boldsymbol{\epsilon} - p\boldsymbol{I}, \tag{1.56}$$

in which p is an arbitrary scalar function of $\boldsymbol{x}$ and t, independent of the strain $\boldsymbol{\epsilon}$. In the limit of incompressibility ($\mathrm{tr}\boldsymbol{\epsilon} \to 0$)

$$\kappa \to \infty, \quad \lambda \to \infty, \quad \mu = \frac{E}{3}, \quad \nu \to \frac{1}{2} \tag{1.57}$$

so that the strain-stress relation (1.52) becomes

$$\boldsymbol{\epsilon} = \frac{1}{2E}\left[3\boldsymbol{T} - (\mathrm{tr}\boldsymbol{T})\boldsymbol{I}\right]. \tag{1.58}$$

The components of the strain tensor (1.49) must satisfy the *compatibility conditions* of Saint Venant, which can be written in terms of the strain components as

$$\epsilon_{ij,hk} + \epsilon_{hk,ij} - \epsilon_{ik,jh} + \epsilon_{jh,ik} = 0, \tag{1.59}$$

where $i, j, h, k = 1, 2, 3$ and $\epsilon_{ij,hk} = \partial^2\epsilon_{ij}/(\partial x_h \partial x_k)$. Writing (1.59) in full, the 81 possible equations reduce to six essential equations, which are

$$2\epsilon_{12,12} = \epsilon_{11,22} + \epsilon_{22,11}, \tag{1.60}$$

and a further two by cyclic exchanges of indices, and

$$\epsilon_{11,23} = \left(\epsilon_{12,3} + \epsilon_{31,2} - \epsilon_{23,1}\right)_1, \tag{1.61}$$

and a further two by cyclic exchanges of indices. Introducing the equations (1.52) and (1.50) into the compatibility conditions (1.59) in the isotropic and homogeneous case, we obtain *Michell's equations*

$$T_{ij,kk} + \frac{1}{1+\nu}T_{kk,ij} = -\frac{\nu}{1-\nu}\delta_{i,j}b_{k,k} - (b_{i,j} + b_{j,i}), \tag{1.62}$$

or *Beltrami's* simpler *equations*, in the case of no or constant body forces,

$$T_{ij,kk} + \frac{1}{1+\nu} T_{kk,ij} = 0. \tag{1.63}$$

Let us consider the shear modulus $\mu > 0$ and the bulk modulus $\kappa > 0$ and go back to the hyperelastic case. For consistency with the linearized isotropic elasticity theory, the strain-energy function must satisfy

$$\begin{aligned} &\bar{W}_1 + 2\bar{W}_2 + \bar{W}_3 = 0, \\ &\bar{W}_{11} + 4\bar{W}_{12} + 4\bar{W}_{22} + 2\bar{W}_{13} + 4\bar{W}_{23} + \bar{W}_{33} = \frac{\kappa}{4} + \frac{\mu}{3}, \end{aligned} \tag{1.64}$$

where $\bar{W}_i = \partial\bar{W}/\partial I_i$, $\bar{W}_{ij} = \partial^2\bar{W}/(\partial I_i \partial I_j)$ $(i,j = 1,2,3)$ and the derivatives are evaluated for $I_1 = I_2 = 3$, and $I_3 = 1$. We can observe that $(1.64)_1$ is equivalent to (1.44). The analogues of (1.64) for $\tilde{W}(i_1, i_2, i_3)$ are

$$\begin{aligned} &\tilde{W}_1 + 2\tilde{W}_2 + \tilde{W}_3 = 0, \\ &\tilde{W}_{11} + 4\tilde{W}_{12} + 4\tilde{W}_{22} + 2\tilde{W}_{13} + 4\tilde{W}_{23} + \tilde{W}_{33} = \kappa + \frac{4}{3}\mu, \end{aligned} \tag{1.65}$$

where $\tilde{W}_i = \partial\tilde{W}/\partial i_i$, $\tilde{W}_{ij} = \partial^2\tilde{W}/(\partial i_i \partial i_j)$ $(i,j = 1,2,3)$ and the derivatives are evaluated for $i_1 = i_2 = 3$, and $i_3 = 1$. If instead of (1.64) and (1.65), the form $\hat{W}(\lambda_1, \lambda_2, \lambda_3)$ of the strain energy function is considered, then it must satisfy

$$\begin{aligned} &\hat{W}_i(1,1,1) = 0 \\ &\hat{W}_{ij}(1,1,1) = \kappa - \frac{2}{3}\mu \ (i \neq j), \quad \hat{W}_{ii} = \kappa + \frac{4}{3}\mu, \end{aligned} \tag{1.66}$$

where, in the latter, no summation is implied by the repetition of the index i, the notation $\hat{W}_i = \partial\hat{W}/\partial\lambda_i$, $\hat{W}_{ij} = \partial^2\hat{W}/\partial(\lambda_i \partial\lambda_j)$ $(i,j = 1,2,3)$ is adopted, and the derivatives are evaluated for $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

1.6 Incremental elastic deformations

Let us consider the deformation of a body $\mathcal{B}$ relative to a given reference configuration $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X})$ and then suppose that the deformation is changed to $\boldsymbol{x}' = \boldsymbol{\chi}'(\boldsymbol{X})$. The displacement of a material particle due to this change is $\dot{\boldsymbol{x}}$ say, defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{x}' - \boldsymbol{x} = \boldsymbol{\chi}'(\boldsymbol{X}) - \boldsymbol{\chi}(\boldsymbol{X}) \equiv \dot{\boldsymbol{\chi}}(\boldsymbol{X}), \tag{1.67}$$

and its gradient is

$$\mathrm{Grad}\dot{\boldsymbol{\chi}} = \mathrm{Grad}\boldsymbol{\chi}' - \mathrm{Grad}\boldsymbol{\chi} \equiv \dot{\boldsymbol{F}}. \tag{1.68}$$

When $\dot{\boldsymbol{x}}$ is expressed as a function of $\boldsymbol{x}$ we call it the *incremental mechanical displacement*, $\boldsymbol{u} = \dot{\boldsymbol{x}}(\boldsymbol{x})$. For a compressible hyperelastic material (1.26), the associated nominal stress difference is

$$\dot{\boldsymbol{S}} = \boldsymbol{S}' - \boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{F}}(\boldsymbol{F}') - \frac{\partial W}{\partial \boldsymbol{F}}(\boldsymbol{F}), \tag{1.69}$$

which has the linear approximation

$$\dot{\boldsymbol{S}} = \boldsymbol{\mathcal{A}}\dot{\boldsymbol{F}}, \tag{1.70}$$

where $\boldsymbol{\mathcal{A}}$ is the fourth-order tensor of elastic moduli, with components

$$\mathcal{A}_{\alpha i\beta j} = \mathcal{A}_{\beta j\alpha i} = \frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}. \tag{1.71}$$

The component form of (1.70) is

$$\dot{S}_{\alpha i} = \mathcal{A}_{\alpha i\beta j}\dot{F}_{j\beta}, \tag{1.72}$$

which provides the convention for defining the product appearing in (1.70). The corresponding form of (1.70) for incompressible materials is

$$\dot{\boldsymbol{S}} = \boldsymbol{\mathcal{A}}\dot{\boldsymbol{F}} - \dot{p}\boldsymbol{F}^{-1} + p\boldsymbol{F}^{-1}\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}, \tag{1.73}$$

where $\dot{p}$ is the increment of p and $\boldsymbol{\mathcal{A}}$ has the same form as in (1.71). Equation (1.73) is coupled with the incremental form of the incompressibility constraint (1.9),

$$\mathrm{tr}(\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}) = 0. \tag{1.74}$$

From the equilibrium equation (1.24) and its counterpart for $\boldsymbol{\chi}'$, we obtain by subtraction the equations of static equilibrium in absence of body forces,

$$\mathrm{Div}\dot{\boldsymbol{S}}^T = \boldsymbol{0}, \tag{1.75}$$

which does not involve approximation. In its linear approximation, $\dot{\boldsymbol{S}}$ is replaced by (1.70) or (1.73) with (1.74). When the displacement boundary conditions on ∂B_r are prescribed, the incremental version is written as

$$\dot{\boldsymbol{x}} = \dot{\boldsymbol{\xi}} \quad \text{on } \partial B_r \tag{1.76}$$

or in the case of tractions boundary conditions (1.22), as

$$\dot{\boldsymbol{S}}^T\boldsymbol{N} = \dot{\boldsymbol{\tau}} \quad \text{on } \partial B_r, \tag{1.77}$$

where $\dot{\boldsymbol{\xi}}$ and $\dot{\boldsymbol{\tau}}$ are the prescribed data for the incremental deformation $\dot{\boldsymbol{\chi}}$. It is often convenient to use the deformed configuration B_t as the reference configuration instead of the initial configuration B_r and one needs therefore to treat all incremental quantities as functions of $\boldsymbol{x}$ instead of $\boldsymbol{X}$. Making use of the following definitions

$$\boldsymbol{u}(\boldsymbol{x}) = \dot{\boldsymbol{\chi}}(\boldsymbol{\chi}^{-1}(\boldsymbol{x})), \quad \boldsymbol{\Gamma} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}, \quad \boldsymbol{\Sigma} = J^{-1}\boldsymbol{F}\dot{\boldsymbol{S}}, \tag{1.78}$$

and of the fourth-order (Eulerian) tensor $\boldsymbol{\mathcal{A}}_0$ of *instantaneous elastic moduli*, whose components are given in terms of those of $\boldsymbol{\mathcal{A}}$ by

$$\mathcal{A}_{0piqi} = J^{-1}F_{p\alpha}F_{q\beta}\mathcal{A}_{\alpha i\beta j}, \tag{1.79}$$

it follows that $\boldsymbol{\Gamma} = \text{grad}\boldsymbol{u}$ and the equilibrium equations (1.75) become

$$\text{div}\boldsymbol{\Sigma}^T = \mathbf{0}, \tag{1.80}$$

where for compressible materials

$$\boldsymbol{\Sigma} = \boldsymbol{\mathcal{A}}_0\boldsymbol{\Gamma}, \tag{1.81}$$

and for incompressible materials

$$\boldsymbol{\Sigma} = \boldsymbol{\mathcal{A}}_0\boldsymbol{\Gamma} + p\boldsymbol{\Gamma} - \dot{p}\boldsymbol{I}, \tag{1.82}$$

where now $J = 1$ in (1.79). The incompressibility constraint (1.74) takes the form

$$\text{tr}\boldsymbol{\Gamma} \equiv \text{div}\boldsymbol{u} = 0. \tag{1.83}$$

When the strain energy function W is given as a symmetrical function of the principal strains $W = \hat{W}(\lambda_1, \lambda_2, \lambda_3)$, the non-zero components, in a coordinate system aligned with the principal axes of strain, are given in general by [95]

$$\begin{aligned}
J\mathcal{A}_{0iijj} &= \lambda_i\lambda_j\hat{W}_{ij}, & & \\
J\mathcal{A}_{0ijij} &= (\lambda_i\hat{W}_i - \lambda_j\hat{W}_j)\lambda_i^2/(\lambda_i^2 - \lambda_j^2), & & i \neq j,\ \lambda_i \neq \lambda_j, \\
J\mathcal{A}_{0ijji} &= (\lambda_j\hat{W}_i - \lambda_i\hat{W}_j)\lambda_i\lambda_j/(\lambda_i^2 - \lambda_j^2), & & i \neq j,\ \lambda_i \neq \lambda_j, \\
J\mathcal{A}_{0ijij} &= (\mathcal{A}_{0iiii} - \mathcal{A}_{0iijj} + \lambda_i\hat{W}_i)/2, & & i \neq j,\ \lambda_i = \lambda_j, \\
J\mathcal{A}_{0ijji} &= \mathcal{A}_{0jiij} = \mathcal{A}_{0ijij} - \lambda_i\hat{W}_i, & & i \neq j,\ \lambda_i = \lambda_j,
\end{aligned} \tag{1.84}$$

(no sums), where $\hat{W}_{ij} \equiv \partial^2\hat{W}/(\partial\lambda_i\partial\lambda_j)$.

Notes

In this chapter we have only introduced some basic concepts, definitions, symbols and basic relationships of continuum mechanics in the field of elasticity. Although there is an extensive literature on the thermomechanics of elastomers, our setting here is purely isothermal and no reference is made to thermodynamics.

For literature on this introductory part, we refer mainly to: Atkin and Fox [4], Beatty [9], Gurtin [50], Holzapfel [57], Landau and Lifshitz [76], Leipholz [77], Ogden [95], Spencer [121] and Truesdell and Noll [127]. These books are an excellent survey of some selected topics in elasticity with an updated list of references.

In Truesdell and Noll [127] (see Section 43), a material is called *elastic* if it is *simple*[1] and if the stress at time t depends only on the local configuration at time t, and not on the entire past history of the motion. This means that the constitutive equation must be expressed as

$$\boldsymbol{T} = \mathcal{G}(\boldsymbol{F}), \tag{1.85}$$

where $\boldsymbol{T}$ is the Cauchy stress tensor, $\boldsymbol{F}$ is the deformation gradient at the present time, taken with respect to a fixed but arbitrary, local reference configuration and

[1]A material is simple if and only if its response to any deformation history is known as soon as its response to all homogeneous irrotational histories is specified (see Section 29 in [127]).

$\mathcal{G}$ is the response function of the elastic material. It is important to point out that in recent years, Rajagopal [103, 104] asserted that this interpretation is much too restrictive and he illustrated his thesis by introducing implicit constitutive theories that can describe the *non-dissipative* response of solids. Hence, Rajagopal gives the constitutive equation for the mathematical model of an elastic material in the form

$$\mathcal{F}(\boldsymbol{F}, \boldsymbol{T}) = \mathbf{0}, \tag{1.86}$$

and in [104] gives some interesting conceptual and theoretical reasons to adopt implicit constitutive equations. In [103], Rajagopal and Srinivasa show that the class of solids that are incapable of dissipation is far richer than the class of bodies that is usually understood as being elastic.

In the last section of this chapter, we introduced the linearized equations for incremental deformations. They constitute the first-order terms associated with a formal perturbation expansion in the incremental deformation. The higher-order (nonlinear) terms are for example required for weakly nonlinear analysis of the stability of finitely deformed configurations, see Chapter 10 in [43]. For a discussion of the mathematical structure of the incremental equations, see [54]. Applications of the linearized incremental equations for interface waves in pre-stressed solids can be found in Chapter 3 of [30].