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New results on copulas and related concepts

Tesi presentata per il conseguimento del titolo di Dottore di Ricerca

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Settore Disciplinare: Probabilità e Statistica Matematica (MAT/06)

Riassunto

Questa dissertazione è dedicata principalmente allo studio delle copule. Nello specifico, una copula è la restrizione all'ipercubo $[0,1]^n$ $(n \geq 2)$ di una funzione di ripartizione (= f.r.) n–dimensionale avente f.r. marginali uniformemente distribuite sull'intervallo $[0,1]$. Una copula è univocamente associata ad ogni vettore n–dimensionale di variabili aleatorie continue, di cui descrive le proprietà di dipendenza. Una delle principali ragioni dell'interesse degli statistici nelle copule risiede nel fatto che la costruzione di funzioni di ripartizioni multidimensionali (e quindi di modelli che descrivono fenomeni aleatori complessi) può essere divisa in due più semplici passi: la costruzione delle leggi marginali e la costruzione di un'opportuna famiglia di copule.

La maggior parte dei risultati presentati è dedicata alla costruzione di f.r. bidimensionali sia attraverso la costruzione esplicita di nuove famiglie di copule (dipendenti da uno o più parametri) sia attraverso l'introduzione di nuovi metodi costruttivi che permettono di associare a due f.r. (o copule) già note un'altra funzione nella stessa classe.

In particolare, si introducono tre famiglie di copule. La prima famiglia si adatta a sistemi bidimensionali con una dipendenza positiva. La seconda è collegata a due variabili aleatorie X e Y di cui sia noto il comportamento del loro massimo, $\max\{X, Y\}$. La terza, pur non avendo una diretta interpretazione probabilistica, generalizza la famiglia delle copule archimedeee, che sono ampiamente utilizzate nelle applicazioni grazie alla loro grande flessibilità.

Inoltre, per ogni coppia di f.r. A e B e per ogni operazione H su $[0,1]$, si caratterizzano tutte le f.r. F indotte puntualmente da A e B, cioè $F(x,y) = H(A(x,y), B(x,y))$. Tale caratterizzazione richiede la definizione di un nuova proprietà delle funzioni bidimensionali, denominata "P–increasing", che generalizza il concetto di supermodularità. Si considera, quindi, una forma leggermente modificata di tale operazione nella classe delle copule, dove si fornisce un metodo per aggiungere parametri ad una copula rendendola adatta a descrivere anche modelli multivariati in cui le variabili aleatorie in gioco non sono scambiabili.

Recentemente, le copule sono state utilizzate anche nella definizione dei concetti di affidabilità ed invecchiamento per sistemi scambiabili bidimensionali. In particolare, Bassan e Spizzicino (2005) hanno introdotto la cosiddetta "funzione bidimensionale di invecchiamento" che consente di definire nel caso bidimensionale alcuni concetti (unidimensionali) di affidabilità già noti, quali, ad esempio, IFR, DFR e NBU. Tale funzione è denominata "semicopula" in quanto verifica alcune, ma non tutte, le proprietà di una copula. La classe delle semicopule e le sue proprietà si studiano nei particolari, evidenziando il ruolo che tali funzioni svolgono anche nelle logiche a più valori e nella teoria delle misure "fuzzy" (anche note come capacità). Si considera, quindi, un metodo per trasformare una (semi–) copula in un'altra (semi–)copula, evidenziando l'utilizzo di tale trasformazione nella teoria dei valori estremi. Sempre nel contesto dell'affidabilità, si inserisce anche lo studio della Schur–concavità nella classe delle copule.

Classificazione AMS 2000: *Primaria:* 60E05, 62H10 *Secondaria:* 62H20, 60E15, 60B10, 03E72.

Parole chiave: Copule; Misure di associazione; Concetti di dipendenza; Supermodularità; Schur–convessità; Quasi–copule, Operatori di aggregazione; Semicopule.

Abstract

This dissertion is devoted, mainly, to the study of copulas. Specifically, a copula is the restriction on the n–cube $[0,1]^n$ ($n \geq 2$) of an n–dimensional distribution function (=d.f.) with marginal d.f.'s uniformly distributed on $[0,1]$. A copula is uniquely associated with an n–dimensional vector of continuous random variables and describes its dependence properties. One of the main reasons of the interest of statisticians to copulas consists in the fact that the construction of multivariate d.f.'s (and, hence, of models describing random phaenomena) can be divided into two easier steps: the construction of the marginal d.f.'s and the construction of a suitable family of copulas.

The major part of the presented results is devoted to the construction of bivariate d.f.'s by means both of the construction of new families of copulas (depending on one or more parameters) and of the introduction of new construction methods that allow to associate with two d.f.'s (or copulas) already known another function in the same class.

In particular, three families of copulas are introduced. The first family is suitable to describe bivariate systems with positive dependence. The second one is connected to two random variables X and Y such that the behaviour of their maximum, $\max\{X, Y\}$, is known. The third one, which does not have a probabilistic interpretation, generalizes the family of Archimedean copulas that are largely used in applications thanks to their great flexibility.

Moreover, for all d.f.'s A and B and for every binary operation H on $[0,1]$, we characterize the d.f.'s F pointwise induced by A and B, viz. $F(x,y) = H(A(x,y), B(x,y))$. Such characterization needs of the definition of a new property of bivariate functions, called "P–increasing", which generalizes the concept of supermodularity. A slight modified form of this operation is, hence, considered in the class of copulas, where we give a method for adding parameters to a copula in order to transform it into another copula suitable to describe also multivariate models whith non–exchangeable random variables.

Recently, the copulas have been also used in the definition of the concepts of reliability and aging for exchangeable bivariate system. In particular, Bassan and Spizzichino (2005) introduced the so–called "bivariate aging function", which allows to define in the bivariate case some (univariate) concepts of reliability already known, like, for instance, IFR, DFR and NBU. Such a function is called “semicopula” because it verifies some, but not all, properties of a copula. The class of semicopulas and its properties are studied in details by underlining the rôle that such functions have also in multivalued logic and in the theory of fuzzy measures (also called capacities). A method of transforming a (semi–) copula into another one is then considered and its use in extreme value theory is underlined. In connection with reliability theory, we study also the Schur–concavity in the class of copulas.

Mathematics Subject Classification 2000: *Primary:* 60E05, 62H10 *Secondary:* 62H20, 60E15, 60B10, 03E72.

Keywords: Copulas; Measures of Association; Dependence concepts; Supermodularity; Schur–convexity; Quasi–copulas, Aggregation operators; Semicopulas.

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Introduction

The official history of the concept of *copula* began with the following words, contained in the seminal paper by Abe Sklar ([149]):

Nous appelerons copule (à n dimensions) toute fonction C_n continue et non décroissante (au sense employé pour une fonction de répartition à n dimensions) définie sur le produit Cartésien de n intervalles fermés $[0,1]$ et satisfaisante aux conditions:

$$C_n(0,0,\dots,0)=0, \qquad C_n(1,\dots,1,\alpha,1,\dots,1)=\alpha.$$

Copulas have been introduced in order to answer a question posed by M. Fréchet on the determining of the classes of multidimensional probability distribution functions with given margins. This problem had occupied several researchers for some years (see, for example, [55, 53, 22]) and the proposed solution states in the following result, since then called *Sklar's Theorem.*

If G is an n-dimensional distribution function with margins $F_1, \dots, F_n$, then there exists a copula C_n such that

$$G(x_1,\dots,x_n)=C_n(F_1(x_1),\dots,F_n(x_n)),$$

and, if each F_i is continuous, then C is unique. Conversely, given the univariate distribution functions $F_1, \dots, F_n$, and a copula C_n, the function G, defined as above, is an n–dimensional distribution function.

Therefore, the Fréchet problem can be reduced to the study of the class of copulas.

At the beginning, many results on copulas were obtained in connection with problems arising in the theory of probabilistic metric spaces, a promising research field developed by B. Schweizer and A. Sklar following the original idea of K. Menger ([106, 141]). As explicitly said by B. Schweizer ([138]), in those years there were no "*ideas of possible statistical applications of copulas and the statistical community took little note of this new concept*".

The initial poor diffusion of this new concept is testified by the fact that, since 1959, copulas appeared implicitely, and under different names, in the works of several authors. In 1960, M. Sibuya considered a *dependence function* associated with a pair of random variables ([148]). In 1975, G. Kilmedorf and A.R. Sampson introduced the *uniform representation* and studied it as a tool to define various dependence notions ([77, 78]). Succesively, analogous concepts were introduced by P. Deheuvels, J. Galambos, D.S. Moore and M.C. Spruill (see [138] for more details). It is also important here to mention that a concept similar to that one of copula was introduced in a paper of W. Hoeffding published in 1940, but unknown largely for many years (see [138, 54]).

The situation changed after the paper [142], in which B. Schweizer and E.F. Wolff announced their first results on the use of copulas for defining a rank–based measure of dependence among random variables. This work, published after some years in the *Annals of Statistics* ([143]), gave the input to a large development of copulas in the study of dependence. In fact, as noted by B. Schweizer and E.F. Wolff,

"*it is precisely the copula which captures those properties of the joint distribution which are invariant under a.s. strictly increasing transformations. Hence the study of rank statistics – insofar as it is the study of properties invariant under such transformations – may be characterized as the study of copulas and copula–invariant properties*".

Some years later, only to make few examples, M. Scarsini showed the importance of copulas in the definition of a measure of concordance between random variables ([135]); C. Genest and J. Mac Kay studied the so–called *Archimedean* copulas, which can be easily constructed and simulated ([62, 63]); W.F. Darsow *et al.* used the copulas in the study of Markov processes (see [24, 125] and also [144]).

An important help to the diffusion of the copula concept has been given by the international conferences devoted to this idea: Rome (1990), Seattle (1993), Prague (1996), Barcelona (2000), Québec (2004); and by their published proceedings ([23, 133, 8, 19]). But, one should also mention the books by B. Schweizer and A. Sklar ([141]), by H. Joe ([74]) and by R.B. Nelsen ([114]), the most cited references in all the papers concerning this topic. A complete history of the development of this field is given in the papers by B. Schweizer ([138]) and by A. Sklar ([151]).

But, it is precisely in the last five years that the theory of copulas is growing into a central topic in the multivariate models and in the study of the dependence concepts. The explosion of the interest in copulas is testified by the fact that the number of papers reviewed by *Mathematical Reviews* since 2000 and mentioning anywhere the word "copula" is greater than the analogous number of papers in the first "40 years of the life" of the copula notion!

Such growing importance is due mainly to the fact that the copula function has been discovered by many researchers working in different areas of applied mathematics: for instance, in actuarial science ([58, 61]), finance ([51, 15]) and hydrology ([134]).

Nowadays, there are many results on copulas and many applications of them in the real problems. Paraphrasing the words of R.B. Nelsen in the introduction of his book, we could say that "*the study of copulas is a subject still in its youth*".

In this dissertation we present, mainly, several new results in the theory of copulas. However, a great attention is also given to some concepts that are a direct extension of the copula function (e.g., triangular norm, quasi–copula, semicopula, aggregation operator) and which have been introduced in other fields, such as probabilistic metric spaces, semigroup theory, reliability and fuzzy theory: an introduction to these notions is presented in chapter 1.

Taking into account the origin of the problems that spurred the investigations here presented, this dissertation can be divided into three parts, which overlap in several points and which are written in a mixed sequence.

The first part is devoted to the construction of new families of bivariate probability distribution functions. This problem has received great attention in the years ([73]) and, as written by N.I. Fisher in the *Encyclopedia of Statistical Sciences* ([54]), it is one of the main reasons of the interest to statisticians in copulas.

In chapter 4 we study a family of copulas that depend on a univariate function. Specifically, we give necessary and sufficient conditions on a function $f : [0,1] \to [0,1]$ that ensure that the mapping $C_f(x,y) := \min\{x,y\} f(\max\{x,y\})$ is a copula. This method provides several examples and, among others, it is shown that the Cuadras–Augé copulas belong to this class. Such a C_f is suitable to describe the positive dependence between random variables (namely, it is positively quadrant dependent) and, moreover, it has also an interesting probabilistic interpretation.

In chapter 5 we characterize the copulas that can be constructed beginning from their diagonal sections. Note that, if C is the copula associated with two random variables X and Y, then the diagonal section of C, namely $\delta_C(t) := C(t,t)$, expresses the behaviour of the maximum between X and Y. Constructions of this type have been already considered in [56, 57]; in particular, our class is a distinguished subset of the *Bertino class of copulas*, formed by those copulas satisfying a functional equation studied, in the class of triangular norms, by G. Mayor and J. Torrens ([105]).

The study of a generalization of the Archimedean class of copulas is, instead, the topic of chapter 6. This class is larger than the two other ones presented in chapters 4 and 5 and might include both singular and absolutely continuous copulas. Although, as in the Archimedean case, no probabilistic interpretation is given, their simple form and flexibility makes this class suitable to be used in the statistical modelling.

Finally, in chapter 7 we characterize a binary operation on the class of bivariate distribution functions. Such an operation was considered, in the univariate case, by C. Alsina *et al.* ([4]), but their extension to the bivariate case is a bit intricate and stimulate us to introduce the new concept of P–increasing function. Some considerations about bivariate distribution functions with fixed marginal d.f.'s and the convergence of distribution functions are then investigated.

The second part of this dissertation is directly inspired by the work of B. Bassan and F. Spizzichino ([7]). In their investigations on multivariate aging through the analysis of the Schur–concavity of the survival distribution functions, they introduced the concept of *semicopula*, which generalizes the copula function, and studied some of its properties. Following these ideas, we investigate the class of semicopulas (chapter 2) and study a transformation method for copulas, also used in other contexts (chapter 9). Moreover, we notice that semicopulas have an interest of their own in fuzzy logic, where it can considered as a generalization of the boolean conjunction from the set $\{0, 1\}$ to the interval $[0, 1]$, and in fuzzy measures. Chapter 10 is, instead, devoted to the study of Schur–concavity of copulas, which allows us to make some considerations about the properties of associative copulas.

The third part of this dissertation is connected with the theory of aggregation operators. Aggregation or fusion of several inputs into a single output is a basic problem in many practical applications and various categories and several approaches have been proposed and investigated. In particular, this field is especially useful for researchers interested in artificial intelligence and multicriteria decision making, where the aggregation of several inputs is the most difficult and controversial problem. In particular, the aggregation of a finite number of real inputs involves functions already known in a mathematical context as triangular norms, quasi–copulas, copulas and, by now, semicopulas. Through all the dissertation, we often present the results in this most general form, and this point of view is especially underlined in chapter 3, where the class of binary aggregation operator sharing the 2–increasing property is analized in details, and in chapter 8, where another kind of composition is introduced for special subclasses of aggregation operators (semicopulas, quasi–copulas, etc). In particular, this last method is applied to copulas, where it gives a valuable method to construct non–symmetric families.

Chapter 1

Preliminaries

In this chapter, we recall several definitions and properties that will be used in the sequel. We begin with some notations on sets and functions (section 1.1) and, in particular, we present the construction of the pseudo–inverse of a monotone function. Section 1.2 is devoted to the presentation of the main concepts and results about the majorization ordering. Binary operations and, in particular, *triangular norms* are presented in sections 1.3 and 1.4.

After recalling some facts about distribution functions (section 1.5), we present the concept of *copula* and its applications to dependence concepts (sections 1.6–1.9). Two generalizations of the copula function are presented in the sections 1.10 and 1.11.

1.1 Sets and functions

We denote by $\mathbb{R}$ the ordinary set of real numbers $(-\infty, +\infty)$ and by $\overline{\mathbb{R}}$ its extension $[-\infty, +\infty]$. For every positive integer $n \geq 2$, $\mathbb{R}^n$ and $\overline{\mathbb{R}}^n$ denote, respectively, the cartesian product of n copies of $\mathbb{R}$ and $\overline{\mathbb{R}}$. We use vector notations for the points in $\overline{\mathbb{R}}^n$, e.g. $\mathbf{x} = (x_1, \dots, x_n)$, and we write $\mathbf{x} \leq \mathbf{y}$ when $x_i \leq y_i$ for all $i \in \{1, 2, \dots, n\}$.

A *n–box* B is a subset of $\overline{\mathbb{R}}^n$ given by the cartesian product of n closed intervals, $B = [x_1, y_1] \times \cdots \times [x_n, y_n]$, and we write it also in the form $[\mathbf{x}, \mathbf{y}]$, where we suppose $x_i < y_i$ for at least an index $i \in \{1, 2, \dots, n\}$. In particular, $[0, 1]^n$ indicates the cartesian product of n copies of the unit interval, i.e. the unit n–cube. The *vertices* of the n–box $B = [\mathbf{x}, \mathbf{y}]$ are the points $\mathbf{c} = (c_1, \dots, c_n) \in B$ such that $c_i \in \{x_i, y_i\}$ for all $i \in \{1, 2, \dots, n\}$. In every vertex $\mathbf{c}$, we can define the following function

$$sgn(\mathbf{c}) := \begin{cases} 1, & \text{if } card\{i \in \{1, 2, \dots, n\} \mid c_i = x_i\} \text{ is an even number;} \\ -1, & \text{if } card\{i \in \{1, 2, \dots, n\} \mid c_i = x_i\} \text{ is an odd number.} \end{cases}$$

An *n–place real function* H is a function whose domain, $DomH$, is a subset of $\overline{\mathbb{R}}^n$ and whose range, $RanH$, is a subset of $\overline{\mathbb{R}}$. As a convention, a 1–place real function is called simply *real function*. The *partial derivative* of H with respect to the i-th variable x_i is denoted by $\partial_{x_i} H$ or $\partial_i H$. If S is a subset of $\overline{\mathbb{R}}^n$, 1_S denote the *indicator function* of S defined by

$$1_S(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in S; \\ 0, & \text{if } \mathbf{x} \notin S. \end{cases}$$

A statement about the points of a set $S \subseteq \mathbb{R}^n$ is said to hold *almost everywhere* (briefly, a.e.) if the set of points of S where the statement fails to hold has Lebesgue measure zero.

Given a real function f and an accumulation point x_0 of $Dom f$, we denote the *left–hand limit* of f at x_0 (if it exists) by $f(x_0^-)$, and the *right–hand limit* of f at x_0 (if it exists) by $f(x_0^+)$. Analogously, $f'(x_0^-)$ and $f'(x_0^+)$ denote, resp., the *left–hand derivative* and *right–hand derivative* of f at x_0. Moreover, if $S \subseteq \mathbb{R}$, we will denote by id_S the *identity function* of S, i.e. $\mathrm{id}_S(x) := x$ for every $x \in S$.

A real function f is *increasing* (resp., *strictly increasing*) if, for every $x < y$, $f(x) \le f(y)$ (resp., $f(x) < f(y)$). Similarly, f is *decreasing* (resp., *strictly decreasing*) if, for every $x < y$, $f(x) \ge f(y)$ (resp., $f(x) > f(y)$). A function f is (*strictly*) *monotone* if f is either (strictly) increasing or (strictly) decreasing.

Let $f : I \to \mathbb{R}$ be a real function whose domain I is an interval of $\mathbb{R}$. The function f is said to be *convex* on I if, for every $x, y \in I$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y).$$

The function f is called *Jensen–convex* on I (or *mid–convex*) if, for every $x, y \in I$,

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2}.$$

A function f is said to be (*Jensen–*)*concave* on I if the function $-f$ is (Jensen–)convex.

Proposition 1.1.1 ([69]). *Let f be a continuous real function defined on an interval I of $\mathbb{R}$. Then f is convex if, and only if, f is Jensen–convex.*

In the same manner, we could define the convexity for an n–place real function whose domain is a convex subset of $\mathbb{R}^n$.

Some notations from lattice theory will be also necessary (see [25]). Let $(X, \le)$ be a partially ordered set, $X \ne \emptyset$. For all $x, y \in X$, let $U(x, y) := \{z \in X : x \le z, y \le z\}$. If $U(x, y)$ has a unique smallest element $\widetilde{z}$ such that $\widetilde{z} \le z$ for all $z \in U(x, y)$, then $\widetilde{z}$ is called the *supremum* of x and y, denoted by $x \vee y$ or $\sup\{x, y\}$. Similarly, if there is a unique greatest element z' smaller than x and y, then this is called the *infimum*, denoted by $x \wedge y$ or $\inf\{x, y\}$. If, for all $x, y \in X$, $x \wedge y$ and $x \vee y$ exist in X, then $(X, \le)$ is called *lattice*. Moreover, for every $S \subseteq X$, we denote by $\bigvee S$ the supremum of the elements of S and by $\bigwedge S$ the infimum of the elements of S. If, for every $S \subseteq X$, $\bigvee S$ and $\bigwedge S$ exist in X, then $(X, \le)$ is called *complete lattice*.

1.1.1 The pseudo–inverse of a real function

Definition 1.1.1. Let $[a, b]$ and $[c, d]$ be intervals of $\overline{\mathbb{R}}$ and let $f : [a, b] \to [c, d]$ be a monotone function. The *pseudo–inverse* of f is the function $f^{[-1]} : [c, d] \to [a, b]$ defined by

$$f^{[-1]}(y) = \begin{cases} \sup\{x \in [a, b] \mid f(x) < y\}, & \text{if } f(a) < f(b); \\ \sup\{x \in [a, b] \mid f(x) > y\}, & \text{if } f(a) > f(b); \\ a, & \text{if } f(a) = f(b). \end{cases}$$

Notice that, if f is a bijection, then the pseudo–inverse coincides with the inverse.

The graph of the pseudo–inverse of a non–constant monotone function f can be easily constructed by the following procedure:

(i) draw the graph of f and complete it, if it is necessary, by adding vertical line segments connecting the points $(x_0, f(x_0^-))$ and $(x_0, f(x_0^+))$ at each discontinuity point x_0 of f;

(ii) reflect the graph so obtained with respect to the graph of $\mathrm{id}_{\mathbb{R}}$, namely with respect to the bisector of the first quadrant;

(iii) remove all but the smallest point from any vertical line contained in the reflected graph.

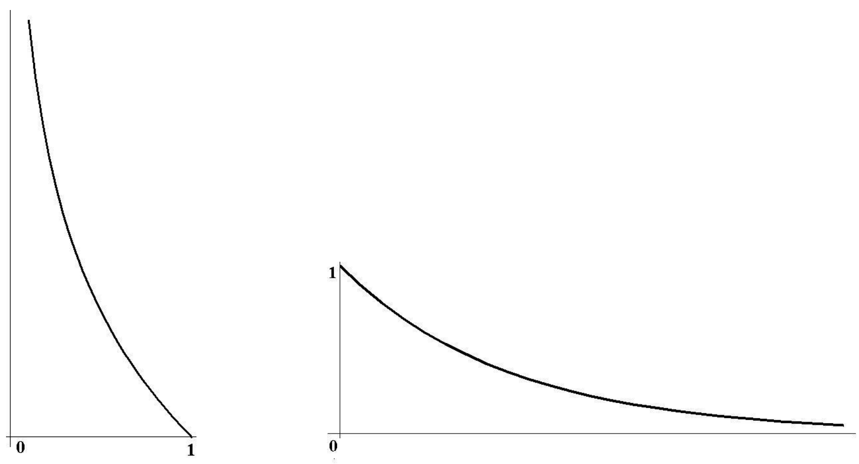

Figure 1.1: A function f and its inverse f^{-1}

Now, we consider a pseudo–inverse construction in two special cases.

Example 1.1.1. Let us consider a function $\varphi : [0,1] \to [0,+\infty]$ that is continuous and strictly decreasing with $\varphi(1) = 0$. The pseudo–inverse of φ is given by

$$\varphi^{[-1]}(t) := \begin{cases} \varphi^{-1}(t), & \text{if } t \in [0, \varphi(0)]\,; \\ 0, & \text{if } t \in [\varphi(0), +\infty]\,. \end{cases}$$

Note that $\varphi^{[-1]}$ is continuous and decreasing on $[0,+\infty]$ and strictly decreasing on $[0, \varphi(0)]$. Furthermore, for all $t \in [0,1]$,

$$\varphi^{[-1]}\left(\varphi(t)\right) = t \tag{1.1}$$

and, for all $t \in [0,+\infty]$,

$$\varphi\left(\varphi^{[-1]}(t)\right) = \min\{t, \varphi(0)\}. \tag{1.2}$$

Example 1.1.2. Given a function $h : [0,1] \to [0,1]$ that is continuous and strictly increasing with $h(1) = 1$, its pseudo–inverse $h^{[-1]} : [0,1] \to [0,1]$ is defined by

$$h^{[-1]}(t) := \begin{cases} h^{-1}(t), & \text{if } t \in [h(0), 1]; \\ 0, & \text{if } t \in [0, h(0)]. \end{cases}$$

Notice that $h^{[-1]}$ is continuous and increasing on $[0,1]$ and strictly increasing on $[h(0), 1]$ and, for all $t \in [0,1]$

$$h^{[-1]}\left(h(t)\right) = t \qquad \text{and} \qquad h\left(h^{[-1]}(t)\right) = \max\{t, h(0)\}.$$

1.2 Majorization ordering

In this section we recall the concepts of majorization ordering on $\mathbb{R}^n$ and Schur–convexity, which can be found in the book by A.W. Marshall and I. Olkin (see [103]).

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be two points in $\mathbb{R}^n$ and denote by

$$x_{[1]}, x_{[2]}, \ldots, x_{[n]} \qquad \text{and} \qquad y_{[1]}, y_{[2]}, \ldots, y_{[n]}$$

the components of $\mathbf{x}$ and $\mathbf{y}$ rearranged in decreasing order, and by

$$x_{(1)}, x_{(2)}, \ldots, x_{(n)} \qquad \text{and} \qquad y_{(1)}, y_{(2)} \cdots, y_{(n)}$$

the components of $\mathbf{x}$ and $\mathbf{y}$ rearranged in increasing order.

Definition 1.2.1. The point $\mathbf{x}$ is *majorized* by $\mathbf{y}$ (and we write $\mathbf{x} \prec \mathbf{y}$) if

(i) $\displaystyle\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}$ for every $k \in \{1, 2, \ldots, n-1\}$;

(ii) $\displaystyle\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$.

Definition 1.2.2. The point $\mathbf{x}$ is said to be *weakly submajorized* by $\mathbf{y}$ (and we write $\mathbf{x} \prec_w \mathbf{y}$) if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \quad \text{for every } k \in \{1, 2, \ldots, n\}.$$

The point $\mathbf{x}$ is said to be *weakly supermajorized* by $\mathbf{y}$ (and we write $\mathbf{x} \prec^w \mathbf{y}$) if

$$\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)} \quad \text{for every } k \in \{1, 2, \ldots, n\}.$$

In the case $n = 2$, the previous definitions take these forms.

$$(x_1, x_2) \prec (y_1, y_2) \iff \begin{cases} \max\{x_1, x_2\} \leq \max\{y_1, y_2\} \\ x_1 + x_2 = y_1 + y_2 \end{cases}$$

$$(x_1, x_2) \prec_w (y_1, y_2) \iff \begin{cases} \max\{x_1, x_2\} \leq \max\{y_1, y_2\} \\ x_1 + x_2 \leq y_1 + y_2 \end{cases}$$

$$(x_1, x_2) \prec^w (y_1, y_2) \iff \begin{cases} \min\{x_1, x_2\} \geq \min\{y_1, y_2\} \\ x_1 + x_2 \geq y_1 + y_2. \end{cases}$$

The following theorems characterize the majorization orderings ([68, 69, 103]).

Theorem 1.2.1 (Hardy, Littlewood and Pólya). *Given two points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, the following statements are equivalent:*

(i) $\mathbf{x} \prec \mathbf{y}$;

(ii) *a doubly stochastic matrix P exists such that $\mathbf{x} = P\mathbf{y}$.*

Corollary 1.2.1. *Given two points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$, the following statements are equivalent:*

(i) $\mathbf{x} \prec \mathbf{y}$;

(ii) *there exists $\alpha \in [0, 1]$ such that*

$$x_1 = \alpha y_1 + (1 - \alpha) y_2 \quad \text{and} \quad x_2 = (1 - \alpha) y_1 + \alpha y_2.$$

Theorem 1.2.2 (Hardy, Littlewood and Pólya). *Given two points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, the following statements are equivalent:*

(a) $\mathbf{x} \prec \mathbf{y}$;

(b) *for every continuous convex function $g : \mathbb{R} \to \mathbb{R}$*

$$\sum_{i=1}^{n} g(x_i) \leq \sum_{i=1}^{n} g(y_i).$$

The following result, which extends Theorem 1.2.2 to the weak majorization ordering and which will be necessary in the sequel, can be found in [103] since it was published in a journal of difficult access ([155]).

Theorem 1.2.3 (**Tomić**). *Given two points* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{R}^n$*, the following statements are equivalent:*

(a) $\mathbf{x} \prec_w \mathbf{y}$*;*

(b) *for every continuous, increasing and convex function* $g : \mathbb{R} \to \mathbb{R}$

$$\sum_{i=1}^{n} g(x_i) \le \sum_{i=1}^{n} g(y_i).$$

Similarly, the following statements are equivalent

(a) $\mathbf{x} \prec^w \mathbf{y}$*;*

(b) *for every continuous, decreasing and convex function* $g : \mathbb{R} \to \mathbb{R}$

$$\sum_{i=1}^{n} g(x_i) \le \sum_{i=1}^{n} g(y_i).$$

Definition 1.2.3. A function $\varphi : A \subset \mathbb{R}^n \to \mathbb{R}$ is said to be *Schur–convex* on A if it is increasing with respect to the majorization order $\prec$, namely if, for all $\mathbf{x}, \mathbf{y} \in A$, $\mathbf{x} \prec \mathbf{y}$ implies $\varphi(\mathbf{x}) \le \varphi(\mathbf{y})$. If, in addition, $\varphi(\mathbf{x}) < \varphi(\mathbf{y})$ whenever $\mathbf{x} \prec \mathbf{y}$ but $\mathbf{x}$ is not a permutation of $\mathbf{y}$, then φ is said to be *strictly Schur–convex* on A.
Similarly, φ is said to be *Schur–concave* on A if, for all $\mathbf{x}, \mathbf{y} \in A$, $\mathbf{x} \prec \mathbf{y}$ implies $\varphi(\mathbf{x}) \ge \varphi(\mathbf{y})$. Moreover, φ is said to be *Schur–constant* if it is, at same time, Schur–convex and Schur–concave.

The next result characterizes continuously differentiable Schur–concave functions ([137, 126]).

Theorem 1.2.4 (**Schur, Ostrowski**). *Let* I *be an open interval in* $\mathbb{R}$ *and let* $\varphi : I^n \to \mathbb{R}$ *be a continuously differentiable function. Then* φ *is Schur–concave on* I^n *if, and only if,*

(i) φ *is symmetric, viz.* $\varphi(\mathbf{x}) = \varphi(\mathbf{x}\Pi)$ *for every permutation* Π*;*

(ii) *for all* $\mathbf{z} = (z_1, z_2, \dots, z_n) \in I^n$ *and* $i \neq j$

$$(z_i - z_j)\,(\partial_i \varphi(\mathbf{z}) - \partial_j \varphi(\mathbf{z})) \le 0.$$

1.3 Binary operations

Definition 1.3.1. A 2–place real function H is *binary operation* on a nonempty set $S \subseteq \mathbb{R}$ if $Dom\ H = S \times S$ and $Ran\ H \subseteq S$.

Let H be a binary operation on $[0,1]$.

Definition 1.3.2. The *horizontal section of* H *at* $b \in [0,1]$ is the function $h_b : [0,1] \to [0,1]$ defined by $h_b(x) := H(x,b)$; the *vertical section of* H *at* $a \in [0,1]$ is the function $v_a : [0,1] \to [0,1]$ defined by $v_a(y) := H(a,y)$. The sections h_0, h_1, v_0 and v_1 are also called *margins* of H.

The *diagonal section of* H is the function $\delta_H : [0,1] \to [0,1]$ defined by $\delta_H(t) := H(t,t)$; the *opposite diagonal section of* H is the function $\delta_H^* : [0,1] \to [0,1]$ defined by $\delta_H^*(t) := H(t, 1-t)$.

Definition 1.3.3. An element 0_H of $[0,1]$ is said to be *annihilator* of H (or *zero, null element* of H) if $H(0_H, s) = 0_H = H(s, 0_H)$ for every s in $[0,1]$.

An element 1_H of $[0,1]$ is said to be *neutral element* of H if $H(1_H, s) = s = H(s, 1_H)$ for every s in $[0,1]$.

Definition 1.3.4. An element a of $[0,1]$ is said to be *idempotent* under H if $H(a,a) = \delta_H(a) = a$, namely if a is a fixed point for δ_H.

Definition 1.3.5. The *transpose* of H is the function H^T given by

$$H^T(x,y) = H(y,x) \qquad \text{for every } x, y \in [0,1].$$

A binary operation H is said to be *commutative* (or *symmetric*) if

$$H(x,y) = H(y,x) \qquad \text{for every } x, y \in [0,1], \tag{1.3}$$

viz. $H = H^T$.

Definition 1.3.6. A binary operation H is said to be *associative* if

$$H(H(x,y),z) = H(x, H(y,z)) \qquad \text{for every } x, y, z \in [0,1]. \tag{1.4}$$

Definition 1.3.7. Let H be a binary operation on $[0,1]$ and let x be an element of $[0,1]$. The *H–powers* of x are the elements of $[0,1]$ given recursively by

$$x_H^1 = x \qquad \text{and} \qquad x_H^{n+1} = H(x_H^n, x)$$

for all positive integers n.

1.4 Triangular norms

A *triangular norm* (briefly, t–norm) is a distinguished type of binary operation on the unit interval $[0,1]$ that has been introduced (in a simplified form) by K. Menger

([106]) in order to extend the triangle inequality from the setting of metric spaces to probabilistic metric spaces. Since then, triangular norms were largely studied in this context and B. Schweizer and A. Sklar provided the axioms of t–norms as they are commonly used today (see the book [141] for an extended bibliography); but they also are widely used in statistics ([62, 65]) and in fuzzy logic, as a generalization of the classical logic connectives (see [160, 83]). For a complete discussion also on the recent developments of the theory of triangular norm, we refer to [139, 82, 3].

Definition 1.4.1. A binary operation T on $[0,1]$ is a *triangular norm* (briefly, *t–norm*) if it satisfies the following properties:

(T1) T is associative;

(T2) T is commutative;

(T3) T is increasing in each place;

(T4) T has neutral element 1.

The following functions are examples of t–norms:

$$M(x,y) := \min\{x,y\}; \qquad W(x,y) := \max\{x+y-1,0\};$$
$$\Pi(x,y) := xy; \qquad Z(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1[^2\,; \\ \min\{x,y\}, & \text{otherwise.} \end{cases}$$

They are called, resp., *minimum, Łukasiewicz, product and drastic t–norm* and are also denoted by T_M, T_L, T_P and T_D.

These four basic t–norms are remarkable for several reasons. For every t–norm T, we have

$$Z(x,y) \leq T(x,y) \leq M(x,y) \qquad \text{for all } (x,y) \in [0,1]^2.$$

The t–norms Π and W are prototypical examples of two important subclasses of t–norms called, respectively, *strict* and *nilpotent* t–norms ([83]). Moreover, M, Π and W play an important role in the theory of copulas, as we shall underline in the sequel.

An example of parametrized family of t–norm is the *Yager family* $\{T_\alpha\}_{\alpha\geq 0}$ (see [157]), given by

$$T_\alpha(x,y) = \begin{cases} Z(x,y), & \text{if } \alpha = 0; \\ M(x,y), & \text{if } \alpha = +\infty; \\ \max\{1 - [(1-x)^\alpha + (1-y)^\alpha]^{1/\alpha}\}, & \text{otherwise.} \end{cases}$$

Now, we present a simple way of constructing a new t–norm beginning from already known ones. This method goes back to some investigations by A.H. Clifford ([17]) on the theory of semigroups (see [141, 83] for more details).

Let $\{T_i : i \in \mathfrak{I}\}$ be a (possibly countable) collection of binary operations on $[0,1]$ that are increasing and bounded from above by M, namely $T_i(x,y) \leq M(x,y)$ for every $i \in \mathfrak{I}$ and all $(x,y) \in [0,1]^2$. Let $\{J_i := [a_i, b_i]\}_{i\in\mathfrak{I}}$ be a family of closed, non overlapping (except at the end points), non degenerate subintervals of $[0,1]$. Then the function T, given by

$$T(x,y) := \begin{cases} a_i + (b_i - a_i)\, T_i\left(\dfrac{x-a_i}{b_i-a_i}, \dfrac{y-a_i}{b_i-a_i}\right), & \text{if } (x,y) \in [a_i,b_i]^2; \\ \min\{x,y\}, & \text{otherwise;} \end{cases}$$

is a binary operation on $[0,1]$, called the *ordinal sum* of the *summands* $\langle a_i, b_i, T_i\rangle$, $i \in \mathfrak{I}$, and we shall write $T = (\langle a_i, b_i, T_i\rangle)_{i\in\mathfrak{I}}$.

Theorem 1.4.1 (Theorem 5.3.8, [141])**.** *An ordinal sum of t–norms is a t–norm.*

Clearly, every t–norm T can be viewed as a trivial ordinal sum with only one summand $\langle 0,1,T\rangle$ only, viz. $T = (\langle 0,1,T\rangle)$. Moreover, the t–norm M can be viewed as an empty ordinal sum of t–norms, when the index set $\mathfrak{I}$ is the empty set. Notice that, for an ordinal sum of the above type, the points a_i and b_i ($i \in \mathfrak{I}$) are the idempotent elements of T.

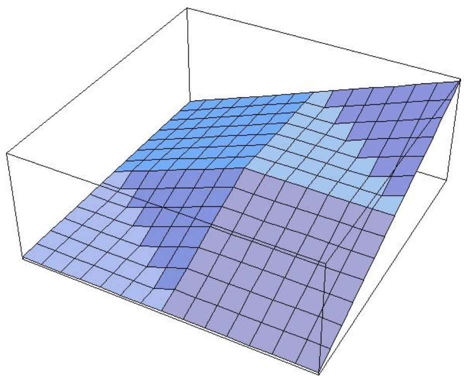

Figure 1.2: The ordinal sum $T = (\langle 0, 1/2, W\rangle, \langle 1/2, 1, W\rangle)$

Using ordinal sums, parametric families of t–norms can be easily constructed.

Example 1.4.1 (Mayor–Torrens family). Given $\alpha \in [0,1]$, consider the following family

$$T_\alpha(x,y) := \begin{cases} \max\{0, x+y-\alpha\}, & \text{if } (x,y) \in [0,\alpha]^2; \\ \min\{x,y\}, & \text{otherwise.} \end{cases} \tag{1.5}$$

This family is known as the *Mayor–Torrens* family of t–norms and every T_α is an ordinal sum, $T = (\langle 0, \alpha, W\rangle)$.

An important property that a t–norm can have is the Archimedean one.

Definition 1.4.2. A t–norm T is called *Archimedean* if, for each $(x, y) \in]0,1[^2$ there is an $n \in \mathbb{N}$ such that $x_T^n < y$.

For continuous Archimedean t–norms, we have the following representation (see [97, 3]).

Theorem 1.4.2. *For a binary operation* $T : [0,1]^2 \to [0,1]$ *the following statements are equivalent:*

(i) T *is a continuous Archimedean t–norm;*

(ii) *there exists a mapping* $\varphi : [0,1] \to [0,+\infty]$ *continuous and strictly decreasing with* $\varphi(1) = 0$ *such that, for every* $(x, y) \in [0,1]$,

$$T(x,y) = \varphi^{[-1]}\left(\varphi(x) + \varphi(y)\right). \tag{1.6}$$

The function φ is said to be *additive generator* of T. A continuous and Archimedean t–norm T is said to be *strict* if it has an additive generator φ such that $\varphi(0) = +\infty$.

Theorem 1.4.3 (Representation of continuous t–norms). *Let* T *be a binary operation on* $[0,1]$ *such that:*

(i) T *has annihilator element* 0*;*

(ii) $T(1,1) = 1$*;*

(iii) T *is associative;*

(iv) T *is jointly continuous.*

Then T *admits one of the following representations:*

(a) $T = M$*;*

(b) $T(x,y) = \varphi^{[-1]}\left(\varphi(x) + \varphi(y)\right)$*, where* $\varphi : [0,1] \to [0,+\infty]$ *is a continuous and strictly decreasing function with* $\varphi(1) = 0$*;*

(c) T *is an ordinal sum of t–norms* T_i*, each of them representable in the form* (b).

1.5 Distribution Functions

Let n be a natural number, $n \in \mathbb{N}$.

Definition 1.5.1. Let H be an n–place real function and let $B = [\mathbf{x}, \mathbf{y}]$ be an n–box whose vertices belong to $DomH$. The *H–volume* of B is given by

$$V_H(B) = \sum sgn(\mathbf{c}) H(\mathbf{c}),$$

where the sum is taken over all the vertices $\mathbf{c}$ of B.

Definition 1.5.2. Let $S_1, \dots, S_n$ be nonempty subsets of $\overline{\mathbb{R}}$ and let H be an n–place real function such that $DomH = S_1 \times \cdots \times S_n$. The function H is said to be *n–increasing* if $V_H(B) \geq 0$ for every n–box B whose vertices lie in $DomH$.

In particular:

- ▷ a 1–increasing function is an increasing function in the classical sense;
- ▷ a 2–increasing function H satisfies the following condition

$$H(x_1, y_1) + H(x_2, y_2) \geq H(x_1, y_2) + H(x_2, y_1), \tag{1.7}$$

for every $x_1 \leq x_2$ and $y_1 \leq y_2$.

Definition 1.5.3. A function $H : \overline{\mathbb{R}}^n \to [0, 1]$ is an *n–dimensional distribution function* (briefly n–d.f.) if

(i) H is n–increasing;

(ii) H is left–continuous in each place;

(iii) $H(+\infty, \dots, +\infty) = 1$;

(iv) $H(\mathbf{x}) = 0$, whenever $x \in \overline{\mathbb{R}}^n$ and $\min\{x_1, x_2, \dots, x_n\} = -\infty$.

The class of all n–dimensional d.f.'s will be denoted by Δ^n.

Specifically:

- ▷ $F : \overline{\mathbb{R}} \to [0, 1]$ is a (unidimensional) d.f. it it is increasing and left–continuous with $F(-\infty) = 0$ and $F(+\infty) = 1$;
- ▷ $H : \overline{\mathbb{R}}^2 \to [0, 1]$ is a bivariate d.f. if it is 2–increasing and left–continuous in each place, with $H(+\infty, +\infty) = 1$ and $H(x, -\infty) = 0 = H(-\infty, y)$ for all $x, y \in \overline{\mathbb{R}}$.

Definition 1.5.4. Let $n \in \mathbb{N}$, $n \geq 2$. Let $\{i_1, i_2, \dots, i_k\}$ be a nonempty set of k indices in $\{1, 2, \dots, n\}$ $(1 \leq k < n)$ and let H be an n–distribution function. The *k–margins* of H $(1 \leq k < n)$ are the $\binom{n}{k}$ functions $H_{i_1,\dots,i_k} : \overline{\mathbb{R}}^k \to [0, 1]$ defined, for every $\mathbf{y} \in \overline{\mathbb{R}}^k$ by

$$H_{i_1,\dots,i_k}(\mathbf{y}) = H(\mathbf{x}),$$

where $\mathbf{x}$ is a point in $\overline{\mathbb{R}}^n$ defined by

$$x_j = \begin{cases} y_j, & \text{if } j \in \{i_1, \dots, i_k\}; \\ +\infty, & \text{if } j \notin \{i_1, \dots, i_k\}. \end{cases}$$

Proposition 1.5.1. *Given an n–dimensional d.f. H, every k–margin of H $(1 \leq k < n)$ is a k–dimensional distribution function.*

In particular, we shall generally denote the 1–margins of an n–d.f. H by $F_1, \dots, F_n$ instead of $H_1, \dots, H_n$ and we shall refer to them briefly as *margins* or *marginal d.f.'s*.

Remark 1.5.1. Given a probability space $(\Omega, \mathcal{F}, P)$ and a random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$, namely $\mathbf{X} : \Omega \to \overline{\mathbb{R}}^n$ is a measurable mapping with respect to the σ–algebra $\mathcal{F}$ and the Borel σ–algebra over $\overline{\mathbb{R}}^n$, the function

$$H(\mathbf{x}) := P\left(\bigcap_{i=1}^{n} \{\omega \in \Omega : X_i(\omega) < x_i\}\right) \tag{1.8}$$

is a n–d.f.. Conversely, in view of the classical *Kolmogorov's compatibility Theorem* (see [94]), given an n–dimensional d.f. H it is possible to construct a probability space $(\Omega, \mathcal{F}, P)$ and a random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$, such that equation (1.8) holds for every $\mathbf{x}$ in $\mathbb{R}^n$.

Remark 1.5.2. In many applications on reliability theory, the random variables of interest represent lifetimes of individuals or objects and then it is very important to study the *survival d.f.* instead of the d.f.. For a r.v. X, its survival d.f. is defined by $\overline{F}(t) := P(X \geq t) = 1 - F_X(t)$. In general, the *joint survival d.f.* of the vector $(X_1, X_2, \dots, X_n)$ is defined by

$$\overline{H}(x_1, x_2, \dots, x_n) := P(X_1 \geq x_1, X_2 \geq x_2, \dots, X_n \geq x_n).$$

For a random pair (X, Y) with joint d.f. H and margins F_1 and F_2, the survival d.f. is given by

$$\overline{H}(x, y) = 1 - F_1(x) - F_2(y) + H(1 - x, 1 - y).$$

Finally, we recall the concept of Fréchet class, introduced in [55].

Definition 1.5.5. The *Fréchet class* determined by the univariate d.f.'s $F_1, F_2, \dots, F_n$ is the set $\Gamma[F_1, F_2, \dots, F_n]$ of all n–d.f.'s whose margins are $F_1, F_2, \dots, F_n$.

Notice that, for every choice of a set of n univariate d.f.'s, the corrisponding Fréchet class is not empty, because it contains the independence d.f. given by the product of the margins.

1.6 Copulas

In this section, we introduce the concept of copula. For simplicity's sake, first, we limit ourselves to consider two–dimensional copulas; the multivariate case ($n \geq 3$) will be, instead, considered briefly in section 1.9. For a deeper discussion of this topic, we refer to the book by R.B. Nelsen ([114]) and to chap. 6 of the book by B. Schweizer and A. Sklar ([141]) (see also the recent papers [128, 116]).

Definition 1.6.1. A function $C\colon [0,1]^2 \to [0,1]$ is a (bivariate) *copula* if it satisfies:

(C1) the *boundary conditions*,

$$\forall x \in [0,1] \quad C(x,0) = C(0,x) = 0 \quad \text{and} \quad C(x,1) = C(1,x) = x;$$

(C2) the 2–*increasing property*, i.e. for all x, x', y, y' in $[0,1]$, with $x \le x'$ and $y \le y'$,

$$V_C\left([x,x'] \times [y,y']\right) := C(x',y') - C(x,y') - C(x',y) + C(x,y) \ge 0.$$

In particular, every copula is *increasing in each place*, viz.

$$C(x,y) \le C(x',y) \qquad \text{and} \qquad C(x,y) \le C(x,y') \quad \text{for } x \le x',\, y \le y', \tag{1.9}$$

and satisfies the 1–*Lipschitz condition*, i.e. for all $x, x', y, y' \in [0,1]$

$$|C(x,y) - C(x',y')| \le |x - x'| + |y - y'|. \tag{1.10}$$

Moreover, if $C\colon [0,1]^2 \to [0,1]$ is twice continuously differentiable, condition (C2) is equivalent to

$$\frac{\partial^2 C(x,y)}{\partial_x \partial_y} \ge 0 \qquad \text{for all } (x,y) \in [0,1]^2. \tag{1.11}$$

In order to prove that a function $F : [0,1]^2 \to [0,1]$ satisfies the so–called *rectangular inequality* (C2), the following technical Proposition will be useful. But, first, we denote by Δ_+ and Δ_- the subsets of the unit square given by:

$$\Delta_+ := \{(x,y) \in [0,1]^2 : x \ge y\}, \qquad \Delta_- := \{(x,y) \in [0,1]^2 : x \le y\}, \tag{1.12}$$

and we prove

Lemma 1.6.1. *For every $F : [0,1]^2 \to [0,1]$, the F–volume $V_F(R)$ of any rectangle $R \subseteq [0,1]^2$ can be expressed as the sum $\sum_i V_F(R_i)$ of at most three terms, where the rectangles R_i may have a side in common and belong to one of the following types:*

(a) $R_i \subseteq \Delta_+$;

(b) $R_i \subseteq \Delta_-$;

(c) $R_i = [s,t] \times [s,t]$.

Proof. Let a rectangle $R \subseteq [0,1]^2$ be given; if it belongs to one of the three types (a), (b) or (c) there is nothing to prove. Then, consider the other possible cases: R may have one, two or three vertices in Δ_-.

If $R = [x_1,x_2] \times [y_1,y_2]$ has one vertex in Δ_+ and three vertices in Δ_-, then, since $y_2 > x_2 > y_1 > x_1$, we can write

$$R = ([x_1,y_1] \times [y_1,y_2]) \cup ([y_1,x_2] \times [y_1,x_2]) \cup ([y_1,x_2] \times [x_2,y_2])\,;$$

of these rectangles, the first and the third one are of type (b), while the second one is of type (c). Now

$$\begin{aligned} V_F([x_1, y_1] \times [y_1, y_2]) &= F(y_1, y_2) - F(y_1, y_1) - F(x_1, y_2) + F(x_1, y_1), \\ V_F([y_1, x_2] \times [y_1, x_2]) &= F(x_2, x_2) - F(x_2, y_1) - F(y_1, x_2) + F(y_1, y_1), \\ V_F([y_1, x_2] \times [x_2, y_2]) &= F(x_2, y_2) - F(x_2, x_2) - F(y_1, y_2) + F(y_1, x_2). \end{aligned}$$

Therefore, summing these equalities we have

$$\begin{aligned} &V_F([x_1, y_1] \times [y_1, y_2]) + V_F([y_1, x_2] \times [y_1, x_2]) + V_F([y_1, x_2] \times [x_2, y_2]) \\ &\quad = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1) = V_F([x_1, x_2] \times [y_1, y_2]), \end{aligned}$$

which proves the assertion in this case. The other cases can be proved in a similar manner. □

Proposition 1.6.1. *A binary operation $F : [0,1]^2 \to [0,1]$ is 2–increasing if, and only if, the three following conditions hold:*

(a) $V_F(R) \geq 0$ *for every rectangle* $R \subseteq \Delta_+$*;*

(b) $V_F(R) \geq 0$ *for every rectangle* $R \subseteq \Delta_-$*;*

(c) $V_F(R) \geq 0$ *for every rectangle* $R = [s,t] \times [s,t] \subseteq [0,1]^2$*.*

Proof. If F is 2–increasing, (a), (b) and (c) follow easily. Conversely, let R be a rectangle of $[0,1]^2$. Then, because of the previous Lemma, R can be decomposed into the union of at most three sub–rectangles R_i of type (a), (b) and (c); and for each of them $V_F(R_i) \geq 0$ holds. Therefore $V_F(R) = \sum V_F(R_i) \geq 0$. □

For every $(x,y) \in [0,1]^2$ and for every copula C

$$W(x,y) \leq C(x,y) \leq M(x,y); \tag{1.13}$$

this inequality is known as the *Fréchet–Hoeffding bounds inequality* ([109]), and W and M are copulas, called also *Fréchet–Hoeffding bounds*, in honour of the pioneering works of Hoeffding ([71]) and Fréchet ([55]). Hence the graph of a copula is a surface within the unit cube $[0,1]^3$ that lies between the graphs of the copulas W and M.

A third important copula is the *product copula* Π.

Notice that a copula is the restriction to $[0,1]^2$ of the bivariate d.f. H_C, given by

$$H_C(x,y) := \begin{cases} 0, & \text{if } \min\{x,y\} < 0; \\ C(x,y), & \text{if } (x,y) \in [0,1]^2; \\ x, & \text{if } x \in [0,1] \text{ and } y > 1; \\ y, & \text{if } x > 1 \text{ and } y \in [0,1]; \\ 1, & \text{if } x > 1 \text{ and } y > 1; \end{cases}$$

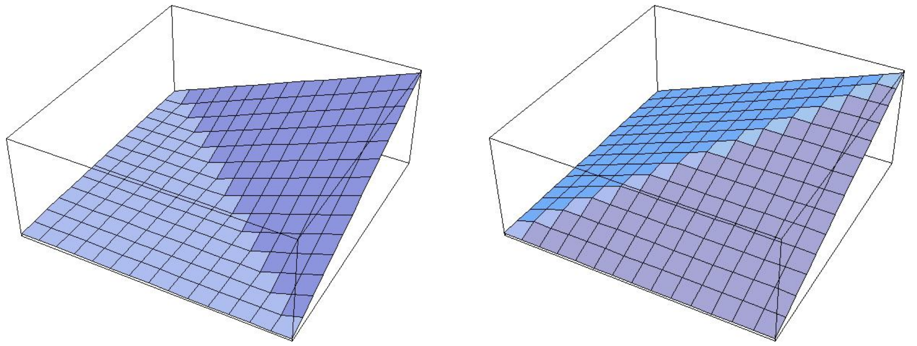

Figure 1.3: The copulas W and M

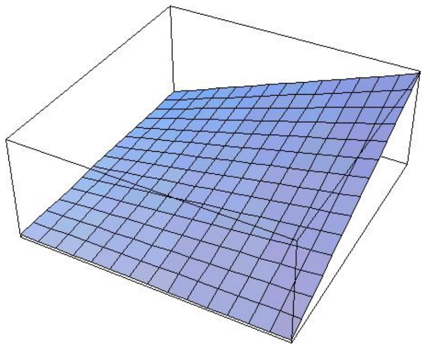

Figure 1.4: The copula Π

whose margins are uniformly distributed on $[0,1]$.

Every copula C induces a probability measure P_C on $[0,1]^2$ given, for every rectangle R, by $P_C(R) := V_C(R)$. In particular, such a probability measure P_C is *doubly stochastic*, namely $P(J \times [0,1]) = P([0,1] \times J) = \lambda(J)$, where J is a Borel set in $[0,1]$ and λ is the the Lebesgue measure on $[0,1]$. The *support of a copula* C is the complement of the union of all open subsets of $[0,1]^2$ with P_C measure equal to zero. If a Borel set $R \subseteq [0,1]^2$ has P_C–measure equal to $m \in]0,1]$, we said that the *probability mass* of C on R is m (or C *spreads* a mass m on R). For every copula C, we have the decomposition

$$C(x,y) = C_A(x,y) + C_S(x,y),$$

where

$$C_A(x,y) := \int_0^x \int_0^y \frac{\partial^2}{\partial s \partial t} C(s,t)\, dsdt, \quad C_S(x,y) = C(x,y) - C_A(x,y).$$

The function C_A is the *absolutely continuous component* of C and C_S is the *singular component* of C. If $C = C_A$, then it is called *absolutely continuous* (e.g. Π) and the mixed second derivative of C, $\partial_{12}C$ is called *density* of C. If $C = C_S$, then it is called *singular* (e.g. M and W). If one of the first derivatives of C has a jump discontinuity, then C has a singular component (see [74, page 15]).

When a copula C is singular, then its support has Lebesgue measure zero, and conversely. For example, the support of M is the main diagonal of $[0,1]^2$, $\{(x,y) \in [0,1]^2 \mid x = y\}$, namely M is singular. Also W is singular and its support is the opposite diagonal of $[0,1]^2$, $\{(x,y) \in [0,1]^2 \mid x + y = 1\}$.

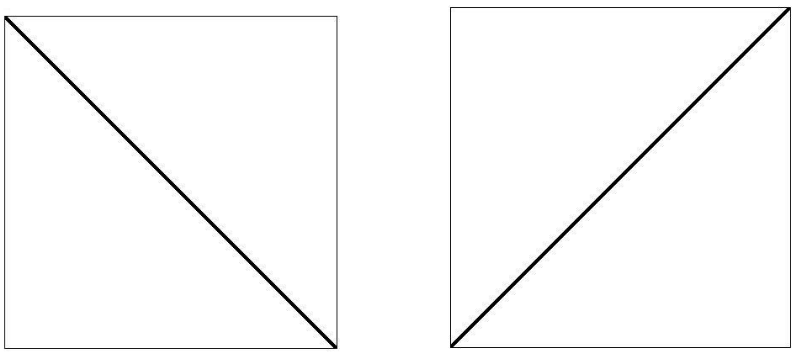

Figure 1.5: Supports of the copulas W and M

We shall denote by $\mathcal{C}$ (or $\mathcal{C}_2$) the class of all the (bivariate) copulas. The set $\mathcal{C}$ is convex and compact under the topology induced by the norm $\| \, \|_\infty$, given for every A in $\mathcal{C}$ by

$$\|A\|_\infty := \max\left\{|A(x,y)| : (x,y) \in [0,1]^2\right\}.$$

Moreover, pointwise convergence in $\mathcal{C}$ is equivalent to uniform convergence, in the sense that, if a sequence $\{C_n : n \in \mathbb{N}\}$ of copulas converges pointwise to a copula C, then it converges also uniformly.

Notice that, since the set $\mathcal{C}$ of copulas is a convex and compact subset of the class of real–valued continuous functions defined on $[0,1]^2$, equipped with the $\| \, \|_\infty$ norm, from the classical Krein–Milman's Theorem (see, e.g., [32]) it follows that $\mathcal{C}$ is the convex hull of its extremal points.

Given two copulas C and D, D is said to be *more concordant* (or *more PQD*) than C ($C \leq D$, for short) if $C(x,y) \leq D(x,y)$ for every x, y in $[0,1]$ (see [74]). The concordance order is only a partial ordering; however, some parametric families of copulas are totally ordered. In particular, we say that a family $\{C_\theta : \theta \in I \subseteq R\}$ is *positively ordered* (resp., *negatively*) if $C_\alpha \leq C_\beta$ whenever $\alpha \leq \beta$ (resp., $\alpha \geq \beta$).

1.6.1 Copulas and random variables

Sklar's Theorem (see [149, 150, 151]) is surely the most important result in the theory of copulas and it is the foundation of many of the applications of copulas to statistics. From that, it is clear in which sense we say that "*a copula is a function which joins or couples a bivariate distribution function to its one–dimensional margins*".

Theorem 1.6.1 (**Sklar, 1959**). *If X and Y are random variables with unidimensional d.f.'s F and G, respectively, and joint d.f. H, then there exists a copula C (uniquely determined on $\mathrm{Ran}\,F \times \mathrm{Ran}\,G$, and hence unique when X and Y are continuous) such that*

$$\forall (x,y) \in \overline{\mathbb{R}}^2 \qquad H(x,y) = C(F(x), G(y)). \tag{1.14}$$

Conversely, given a copula C and two univariate d.f.'s F and G, the function H defined in (1.14) *is a bivariate d.f. with margins F and G.*

Given a joint d.f. H with continuous margins F and G, it is easy to construct the corresponding copula is given by:

$$C(x,y) = H(F^{[-1]}(x), G^{[-1]}(y)),$$

where $F^{[-1]}(t) = \sup\{x : F(x) \leq t\}$ is the pseudo–inverse of F (and similarly for $G^{[-1]}$). Conversely, given a copula C and two univariate d.f.'s F and G, the equality (1.14) allows us to construct a bivariate d.f. H.

Note as well that, if X and Y are continuous r.v.'s with d.f.'s F and G, C is the joint d.f. of the r.v.'s $U = F(X)$ and $V = G(Y)$.

The following result gives an interesting probabilistic interpretation of the three basic copulas M, Π and W.

Theorem 1.6.2. *For continuous r.v.'s X and Y with copula C the following statements hold:*

- ▷ *X and Y are independent if, and only if, $C = \Pi$;*
- ▷ *Y is almost surely an increasing function of X if, and only if, $C = M$;*
- ▷ *Y is almost surely a decreasing function of X if, and only if, $C = W$.*

In general, Sklar's Theorem allows us to study the dependence properties of a random vector by examination of the copula alone, *if the r.v.'s are continuous*. This last assumption is essential because, for discontinuous r.v.'s, the copula is not unique and many problems arise, as discussed, e.g., in [100, 146, 124].

Example 1.6.1. Let X and Y be r.v.'s with d.f.'s $F_X = 1_{]a,+\infty]}$ and $F_Y = 1_{]b,+\infty]}$, with $a < b$. Then the joint d.f. of X and Y is

$$H(x,y) = \begin{cases} 1, & \text{if } (x,y) \geq (a,b); \\ 0, & \text{otherwise.} \end{cases}$$

Notice that, in view of Sklar's Theorem, there exists a (not uniquely determined) copula C such that (1.14) holds. In this case, C has to satisfy only the assumptions

$$C(1,1) = 1, \qquad C(0,1) = C(1,0) = C(0,0) = 0.$$

Therefore, every copula can be associated with the random pair (X,Y).

In the sequel, when we speak about "the copula of the random pair (X,Y)", we assume that X and Y are continuous and, therefore, the copula is unique and it will also be denoted by C_{XY}.

Remark 1.6.1. The first–order derivatives of a copula have a nice interpretation. If C is the copula of the random pair (U,V) of two r.v.'s uniformly distributed on $[0,1]$, then

$$\frac{\partial C(u,v)}{\partial u} = P(V \leq v \mid U = u) \quad \text{and} \quad \frac{\partial C(u,v)}{\partial v} = P(U \leq u \mid V = v).$$

Now, we express the copula of a random vector obtained from another one by strictly monotone transformations.

Theorem 1.6.3. *Let X and Y be continuous r.v.'s with copula C. Let α and β be two functions strictly monotone on $\mathrm{Ran}X$ and $\mathrm{Ran}Y$, respectively.*

(i) *If α and β are both strictly increasing, then*

$$C_{\alpha(X)\beta(Y)} = C_{XY}.$$

(ii) *If α is strictly increasing and β is strictly decreasing, then*

$$C_{\alpha(X)\beta(Y)}(x,y) = x - C_{XY}(x, 1-y).$$

(iii) *If α is strictly decreasing and β is strictly increasing, then*

$$C_{\alpha(X)\beta(Y)}(x,y) = y - C_{XY}(1-x, y).$$

(iii) *If α and β are both strictly decreasing, then*

$$C_{\alpha(X)\beta(Y)}(x,y) = x + y - 1 + C_{XY}(1-x, 1-y).$$

From the above result we have that, given a copula C, the following function are copulas (see [84]):

$$C_{0,1}(x,y) := x - C(x, 1-y), \tag{1.15}$$

$$C_{1,0}(x,y) := y - C(1-x, y), \tag{1.16}$$

$$C_{1,1}(x,y) := x + y - 1 + C(1-x, 1-y). \tag{1.17}$$

In particular, $C_{1,1}$ is called *survival copula* and it is denoted more frequently by $\hat{C}$. It has a large use in reliability theory, where Sklar's Theorem can be reformulated under the following form:

Theorem 1.6.4. *Let X and Y be two continuous r.v.'s with copula C. Let $\overline{H}$ be the joint survival d.f. of (X,Y) and let $\overline{F}$ and $\overline{G}$ be the univariate survival d.f.'s. Then*

$$\overline{H}(x,y) = \hat{C}\left(\overline{F}(x), \overline{G}(y)\right),$$

where $\hat{C}$ is the survival copula of C.

Remark 1.6.2. Notice that the survival copula $\hat{C}$ is not the joint survival d.f. $\overline{C}$ of two r.v.'s unifomly distributed on $[0,1]$ whose joint d.f. is the copula C. In such a case, in fact, we have $\overline{C}(x,y) := 1 - x - y + C(x,y)$.

The symmetry properties of a random pair can also be expressed in terms of the associated copula (see [114, 84] for more details).

Definition 1.6.2. Two r.v.'s X and Y are *exchangeable* if, and only if, (X,Y) and (Y,X) are identically distributed.

Proposition 1.6.2. *Let X and Y be continuous r.v.'s with margins d.f.'s F and G, respectively, and copula C. Then X and Y are exchangeable if, and only if, $F = G$ and C is symmetric.*

Definition 1.6.3. Let X and Y be r.v.'s and let (a,b) be a point in $\mathbb{R}^2$.

- ▷ (X,Y) is *radially symmetric* about (a,b) if the joint d.f. of $(X-a)$ and $(Y-b)$ is the same as the joint d.f. of $(a-X)$ and $(b-Y)$.
- ▷ (X,Y) is *jointly symmetric* about (a,b) if the following four pairs of r.v.'s have a common joint d.f.: $(X-a, Y-b)$, $(X-a, b-Y)$, $(a-X, Y-b)$ and $(a-X, b-Y)$.

Note that the joint symmetry implies the radial symmetry.

Proposition 1.6.3. *Let X and Y be continuous r.v.'s with marginal d.f.'s F and G, respectively, and copula C. Given a point $(a,b) \in \mathbb{R}^2$, assume that $(X-a)$ has the same d.f. as $(a-X)$, and $(Y-b)$ has the same d.f. as $(b-Y)$. Then:*

- ▷ *(X,Y) is radially symmetric about (a,b) if, and only if, $C = \widehat{C}$;*
- ▷ *(X,Y) is jointly symmetric about (a,b) if, and only if, $C = C_{0,1}$ and $C = C_{1,0}$ (and then also $C = \widehat{C}$).*

1.6.2 Families of copulas

For many years, statisticians have been fascinated by the following problem: given two univariate d.f.'s F and G, find a bivariate d.f. H having F and G as its margins, and having useful properties such as a simple analytic expression, a simple stochastic representation, some desirable dependence properties, and a suitable number of parameters. Many methods and procedures for constructing such joint distributions have been introduced and studied in the literature (see, for example [75, 73]). As noted in subsection 1.6.1, thanks to Sklar's Theorem, we can decompose this problem into two easier steps: the construction of a copula and the construction of two univariate margins.

Having several families of bivariate distributions at disposal is of great importance in statistical applications. In fact, for many years, multivariate models have been often constructed either under the assumption of the independence of their components or by assuming the components are connected by a multivariate normal distribution (see, e.g., [58]). Copulas, instead, allow to study models with a more flexible and wide range of dependence.

In [74, 77], some criteria are given in order to ensure that a family of copulas is a "good" family, in the sense that it can be useful in certain statistical applications. Here we list some desirable properties for a parametric class of copulas C_α, where α belongs to an interval of the real line:

- ▷ *interpretability*, which means having a probabilistic interpretation;
- ▷ *flexible and wide range of dependence*, which implies that the copula Π and at least one of the Fréchet–Hoeffding bounds W and M belong to the class;
- ▷ *closed form*, in the sense that every copula of the class is absolutely continuous or has a simple representation;
- ▷ *ordering*, with respect, for example, to concordance.

Now, we present some families of copulas (see [114] for more details).

Example 1.6.2 (Fréchet family). For all $x, y \in [0,1]$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$, the family

$$C_{\alpha,\beta}(x,y) = \alpha\, M(x,y) + (1-\alpha-\beta)\,\Pi(x,y) + \beta\, W(x,y)$$

is a family of copulas, known as the *Fréchet family.* A slight modification of this family is the so–called *linear Spearman copula* (see [72] and [74, family B11]), given, for every $\alpha \in [-1,1]$, by

$$C_\alpha(x,y) = (1-|\alpha|)\cdot\Pi(x,y) + |\alpha|\cdot C_{sgn(\alpha)}(x,y),$$

where $C_{sgn(\alpha)} = M$, if $\alpha \geq 0$, and $C_{sgn(\alpha)} = W$, otherwise.

Example 1.6.3 (**FGM family**)**.** For all $x, y \in [0,1]$ and $\alpha \in [-1,1]$

$$C_\alpha(x,y) = xy + \alpha xy\,(1-x)\,(1-y)$$

is a family of copulas, known as the *Farlie-Gumbel-Morgenstern family* (often abbreviated FGM) and contains as its members copulas with sections that are quadratic in both x and y.

Example 1.6.4 (**Cuadras–Augé family**)**.** For every $\alpha \in [0,1]$, the following function

$$C_\alpha(x,y) := \begin{cases} xy^{1-\alpha}, & \text{if } x \le y; \\ x^{1-\alpha}y, & \text{if } x \ge y; \end{cases}$$

is a copula, belonging to the family introduced by Cuadras and Augé ([18]). Notice that C_α is the weighted geometric mean of M and Π; in particular, $C_0 = \Pi$ and $C_1 = M$.

Example 1.6.5 (**Marshall–Olkin family**)**.** For every α and β in $[0,1]$, the following function

$$C_{\alpha,\beta}(x,y) := \begin{cases} x^{1-\alpha}y, & \text{if } x^\alpha \ge y^\beta; \\ xy^{1-\beta}, & \text{if } x^\alpha \le y^\beta; \end{cases}$$

is a copula, belonging to the family introduced by Marshall and Olkin ([101, 102]), which contains the family given in Example 1.6.4 for $\alpha = \beta$.

Example 1.6.6 (**BEV Copula**)**.** Let $A : [0,1] \to [1/2,1]$ be a convex function such that $\max\{t, 1-t\} \le A(t) \le 1$ for every $t \in [0,1]$. The following function

$$C_A(x,y) := \exp\left[(\ln x + \ln y) A\left(\frac{\ln x}{\ln x + \ln y}\right)\right]$$

is a copula, known as *bivariate extreme value copula* (briefly, BEV) (see [74, chap. 6]). This copula satisfies the equality $C^n(x,y) = C(x^n, y^n)$ for every $n \in \mathbb{N}$. The name of this class arises from the theory of extreme statistics. In fact, let $(X_1, Y_1), \dots, (X_n, Y_n)$ be a random sample from bivariate distribution H, define $M_n := \max\{X_1, \dots, X_n\}$ and $N_n := \max\{Y_1, \dots, Y_n\}$ and suppose that there exist constants a_{1n}, a_{2n}, b_{1n} and b_{2n}, with $a_{1n} > 0$ and $a_{2n} > 0$, for which the pair

$$\left(\frac{M_n - b_{1n}}{a_{1n}}, \frac{N_n - b_{2n}}{a_{2n}}\right)$$

has a non-degenerate joint limiting distribution H^*. Then the copula associated with H^* is a BEV copula (see [59, 129]).

Example 1.6.7 (**Normal copula**)**.** Let $N_\rho(x,y)$ denote the standard bivariate normal joint d.f. with correlation coefficient ρ. Then the corresponding copula is

$$C_\rho(x,y) = N_\rho(\Phi^{-1}(x), \Phi^{-1}(y)),$$

where Φ denotes the standard normal d.f.. Because Φ^{-1} does not have a closed form, there is no closed form for C_ρ, which can be only evaluated approximately.

Example 1.6.8 (Shuffle of Min). The copulas known as *shuffles of* M were introduced in [110] and do not have a simple explicit expression. However, the procedure to obtain their mass distribution can be easily described:

1. spread uniformly the mass on the main diagonal of $[0,1]^2$,
2. cut $[0,1]^2$ vertically into a finite number of strips,
3. shuffling the strips with perhaps some of them flipped around their vertical axes of symmetry,
4. reassembling them to form the square again.

The resulting mass distribution corresponds to a copula called *shuffle of* M. Formally, a shuffle of M is determined by a partition $\{J_i\}_{i=1,2,\dots,n}$, a permutation of $(1,2,\dots,n)$ and an orientation n–ple $(i_1,i_2,\dots,i_n)$ such that $i_k=-1$ or 1 according to whether or not the strip $J_i \times [0,1]$ is flipped.

For instance, the shuffle given by $\{[0,1/2],[1/2,1]\}$, permutation $(2,1)$ and orientation $(-1,-1)$ is W. Moreover, the shuffle of M with partition $\{[0,a],[a,1-a],[1-a,1]\}$, $(a\in[0,1/2])$, permutation $(3,2,1)$ and orientation $(-1,+1,-1)$ is the copula $C_\alpha(x,y)=\max\{W(x,y),M(x,y)-\alpha\}$.

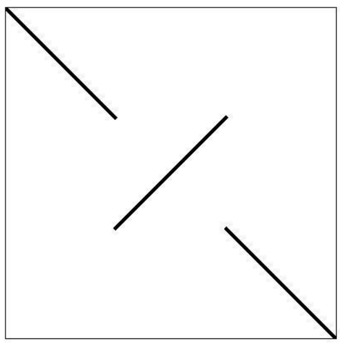

Figure 1.6: Support of the copula C_α for $\alpha=1/3$

A way of constructing new copulas is given by the *ordinal sum* construction, a method already presented in section 1.4, and reproduced here.

Theorem 1.6.5. *Let $C=(\langle a_i,b_i,C_i\rangle)_{i\in\mathfrak{I}}$ be an ordinal sum such that C_i is a copula for every $i\in\mathfrak{I}$. Then C is a copula.*

1.6.3 Diagonal sections of copulas

Given a copula C, it is easily proved that its diagonal δ satisfies the following properties:

(D1) $\delta(1) = 1$;

(D2) $\delta(t) \leq t$ for all $t \in [0, 1]$;

(D3) δ is increasing;

(D4) $|\delta(t) - \delta(s)| \leq 2|t - s|$ for all $t, s \in [0, 1]$.

The set of functions $\delta : [0, 1] \to [0, 1]$ satisfying (D1)–(D3) will be denoted by $\mathcal{D}$, instead $\mathcal{D}_2$ will denote the subset of $\mathcal{D}$ of the functions satisfying also (D4).

For each function $\delta \in \mathcal{D}_2$, there is always a copula whose diagonal section coincides with δ. Consider, for example, the *diagonal copula*

$$K_\delta(x, y) := \min\left\{x, y, \frac{\delta(x) + \delta(y)}{2}\right\}, \tag{1.18}$$

introduced in [117, 56]. Another example is given by the *Bertino copula* ([9, 57])

$$B_\delta(x, y) := \min\{x, y\} - \min\{t - \delta(t) : t \in [x \wedge y, x \vee y]\}. \tag{1.19}$$

In particular, a Bertino copula is called *simple* if it can be expressed in the form

$$B_\delta(x, y) := \min\{x, y\} - \min\{x - \delta(x), y - \delta(y)\}. \tag{1.20}$$

From a probabilistic point of view, investigations on diagonal sections of copulas are of interest because, if X and Y are random variables with the same distribution function F and copula C, then the distribution function of $\max\{X, Y\}$ is $\delta_C(F(t))$. Moreover, copulas with given diagonal section have important consequences in finding the bounds on arbitrary subsets of joint d.f.'s (see [121]). An absolutely continuous copula with given diagonal section is also given in the recent paper [52].

1.6.4 Archimedean copulas

From a general point of view, copulas are special type of binary operations on $[0, 1]$, and many important copulas are also t–norms. In particular, the class of Archimedean copulas (i.e. associative copulas with the Archimedean property as defined in section 1.4), is a very useful subclass of copulas, both in the statistical context (see [62, 63, 113, 112]) and in applications, especially in finance, actuarial science ([58, 70]) and hydrology ([134]), due to their simple form and nice properties. Archimedean copulas are characterized here.

Theorem 1.6.6. *A function C is an Archimedean copula if, and only if, it admits the representation*

$$C(x,y) := \varphi^{[-1]}\left(\varphi(x)+\varphi(y)\right), \qquad \text{for all } x,y\in[0,1], \tag{1.21}$$

where $\varphi : [0,1] \to [0,+\infty]$ is continuous, strictly decreasing and convex with $\varphi(1) = 0$.

The function φ is said to be an *additive generator* of C and, therefore, C is also denoted as C_φ. Notice that, by setting $h(t) := \exp(-\varphi(t))$ for every $t \in [0,1]$, C_φ may be represented in the form

$$C_\varphi(x,y) = h^{[-1]}\left(h(x)\cdot h(y)\right) \qquad \text{for all } x,y\in[0,1]. \tag{1.22}$$

This function h is a *multiplicative generator* of C_φ and Theorem 1.6.6 may be rephrased in the following (multiplicative) form.

Theorem 1.6.7. *A function C is an Archimedean copula if, and only if, it admits the representation*

$$C(x,y) := h^{[-1]}\left(h(x)\cdot h(y)\right), \qquad \text{for all } x,y\in[0,1], \tag{1.23}$$

where $h : [0,1] \to [0,1]$ is continuous, strictly increasing and log–concave, viz. for every α, s and t in $[0,1]$, it satisfies the inequality

$$h^\alpha(s)\,h^{1-\alpha}(t) \le h\left(\alpha s + (1-\alpha)t\right).$$

Notice that, neither the additive nor the multiplicative generator of an Archimedean copula are unique. In fact, if φ is an additive generator of C, then every additive generator of C has the form $\varphi_1 := k\,\varphi$, for $k > 0$. Analogously, if h is a multiplicative generator of a copula D, then $h_1(t) := h(t^\alpha)$ $(\alpha > 0)$ is also a multiplicative generator for D. The next result yields a technique for finding generators of Archimedean copulas ([62]).

Theorem 1.6.8. *Let C be an Archimedean copula with generator φ. Then*

$$\varphi'(x)\cdot\partial_y C(x,y) = \varphi'(y)\cdot\partial_x C(x,y) \qquad \textit{a.e. on } [0,1]^2.$$

In Table 1.1 we list some known families of Archimedean copulas and their additive generators.

In the spirit of the representation of continuous t–norms (see 1.4.3), Archimedean copulas allow us to give a full characterization of associative copulas.

Theorem 1.6.9 (Representation of associative copulas). *Let C be an associative copula with diagonal section δ_C. Then:*

- ▷ *$C = M$ if, and only if, $\delta_C = id_{[0,1]}$;*

Family	Copula $C_\theta(x,y)$	$\theta \in$
Frank	$-\frac{1}{\theta}\ln\left(1+\frac{(e^{-\theta x}-1)(e^{-\theta y}-1)}{e^{-\theta}-1}\right)$	$[-\infty,+\infty]$
Clayton	$\max\left\{(x^{-\theta}+y^{-\theta}-1)^{-1/\theta},0\right\}$	$[-1,+\infty]$
Gumbel–Hougaard	$\exp\left(-\left((-\ln x)^\theta+(-\ln y)^\theta\right)^{1/\theta}\right)$	$[1,+\infty]$
Ali–Mikhail–Haq	$\frac{xy}{1-\theta(1-x)(1-y)}$	$[-1,1]$

Table 1.1: Families of Archimedean copulas

- ▷ *C is Archimedean if, and only if, $\delta_C(t) < t$ on $]0,1[$;*
- ▷ *C is an ordinal sum of Archimedean copulas if, and only if, $\delta_C(t) < t$ for some, but not all, t in $]0,1[$.*

In [14], the following generalization of an Archimedean copula is studied.

Example 1.6.9. [**Archimax copula**] Let $A : [0,1] \to [1/2,1]$ be a convex function such that $\max\{t,1-t\} \le A(t) \le 1$ for every $t \in [0,1]$. Let φ be an additive generator of an Archimedean copula. The following function

$$C_{\varphi,A}(x,y) := \varphi^{[-1]}\left[(\varphi(x)+\varphi(y))A\left(\frac{\varphi(x)}{\varphi(x)+\varphi(y)}\right)\right]$$

is a copula, known as *Archimax*. The family of Archimax copulas includes both Archimedean copulas and BEV copulas. The functions A and φ, which uniquely determine $C_{\varphi,A}$, are called, respectively, *dependence function* and *Archimedean generator.*

1.7 Dependence Properties

Here we recall some dependence properties between random variables that will be expressed in terms of copulas. For more details on this topic, see [114, chap. 5] and [74].

Definition 1.7.1. Let X and Y be random variables.

- ▷ X and Y are *positively quadrant dependent* (briefly, *PQD*) if, for every (x,y) in $\mathbb{R}^2$, $P(X \le x, Y \le y) \ge P(X \le x)P(Y \le y)$.
- ▷ X and Y are *negatively quadrant dependent* (briefly, *NQD*) if, for every (x,y) in $\mathbb{R}^2$, $P(X \le x, Y \le y) \le P(X \le x)P(Y \le y)$.

Proposition 1.7.1. *Let X and Y be continuous r.v.'s with copula C. X and Y are PQD (resp. NQD) if, and only if, $C \ge \Pi$ (resp. $C \le \Pi$).*

Definition 1.7.2. Let X and Y be random variables.

- ▷ Y is *left tail decreasing* in X (briefly, $LTD(Y|X)$) if, and only if, the mapping $x \mapsto P(Y \le y \mid X \le x)$ is a decreasing function for all y.
- ▷ X is *left tail decreasing* in Y (briefly, $LTD(X|Y)$) if, and only if, the mapping $y \mapsto P(X \le x \mid Y \le y)$ is a decreasing function for all x.
- ▷ Y is *right tail increasing* in X (briefly, $RTI(Y|X)$) if, and only if, the mapping $x \mapsto P(Y > y \mid X > x)$ is an increasing function for all y.
- ▷ X is *right tail increasing* in Y (briefly, $RTI(X|Y)$) if, and only if, the mapping $y \mapsto P(X > x \mid Y > y)$ is an increasing function for all x.

Proposition 1.7.2. *Let X and Y be continuous r.v.'s with copula C.*

- ▷ *$LTD(Y|X)$ if, and only if, for every $y \in [0,1]$,*

$$x \mapsto C(x,y)/x \qquad \textit{is decreasing.}$$

- ▷ *$LTD(X|Y)$ if, and only if, for every $x \in [0,1]$,*

$$y \mapsto C(x,y)/y \qquad \textit{is decreasing.}$$

- ▷ *$RTI(Y|X)$ if, and only if, for every $y \in [0,1]$,*

$$x \mapsto [y - C(x,y)]/(1-x) \qquad \textit{is decreasing.}$$

- ▷ *$RTI(Y|X)$ if, and only if, for every $x \in [0,1]$,*

$$y \mapsto [x - C(x,y)]/(1-y) \qquad \textit{is decreasing.}$$

Definition 1.7.3. Let X and Y be random variables.

- ▷ Y is *stochastically increasing* in X (briefly, $SI(Y|X)$) if, and only if, the mapping $x \mapsto P(Y > y \mid X = x)$ is an increasing function for all y.
- ▷ X is *stochastically increasing* in Y (briefly, $SI(X|Y)$) if, and only if, the mapping $y \mapsto P(X > x \mid Y = y)$ is an increasing function for all x.
- ▷ Y is *stochastically decreasing* in X (briefly, $SD(Y|X)$) if, and only if, the mapping $x \mapsto P(Y > y \mid X = x)$ is a decreasing function for all y.
- ▷ X is *stochastically decreasing* in Y (briefly, $SD(X|Y)$) if, and only if, the mapping $y \mapsto P(X > x \mid Y = y)$ is a decreasing function for all x.

Proposition 1.7.3. *Let X and Y be continuous r.v.'s with copula C.*

▷ *$SI(Y|X)$ if, and only if, $x \mapsto C(x,y)$ is concave for every $y \in [0,1]$.*

▷ *$SI(X|Y)$ if, and only if, $y \mapsto C(x,y)$ is concave for every $x \in [0,1]$.*

▷ *$SD(Y|X)$ if, and only if, $x \mapsto C(x,y)$ is convex for every $y \in [0,1]$.*

▷ *$SD(X|Y)$ if, and only if, $y \mapsto C(x,y)$ is convex for every $x \in [0,1]$.*

Definition 1.7.4. Let X and Y be random variables

▷ X and Y are *left corner set decreasing* (briefly , $LCSD(X,Y)$) if, and only if, $P(X \leq x, Y \leq y \mid X \leq x', Y \leq y')$ is decreasing in x' and in y' for all x and y.

▷ X and Y are *left corner set increasing* (briefly , $LCSI(X,Y)$) if, and only if, $P(X \leq x, Y \leq y \mid X \leq x', Y \leq y')$ is increasing in x' and in y' for all x and y.

▷ X and Y are *right corner set increasing* (briefly , $RCSI(X,Y)$) if, and only if, $P(X > x, Y > y \mid X > x', Y > y')$ is increasing in x' and in y' for all x and y.

▷ X and Y are *right corner set decreasing* (briefly , $RCSD(X,Y)$) if, and only if, $P(X > x, Y > y \mid X > x', Y > y')$ is increasing in x' and in y' for all x and y.

Proposition 1.7.4. *Let X and Y be r.v.'s uniformly distributed on $[0,1]$ with associated copula C.*

▷ *$LCSD(X,Y)$ if, and only if,*

$$C(x,y)C(x',y') \geq C(x,y')C(x',y)$$

for every x, x', y, y' in $[0,1]$, $x \leq x'$, $y \leq y'$.

▷ *$RCSI(X,Y)$) if, and only if,*

$$\widehat{C}(x,y)\widehat{C}(x',y') \geq \widehat{C}(x,y')\widehat{C}(x',y)$$

for every x, x', y, y' in $[0,1]$, $x \leq x'$, $y \leq y'$.

The scheme of implications among the various dependence concepts is presented in Table 1.2.

For the study of dependence between extreme values, the concept of *tail dependence* is useful and can be also expressed in terms of copula (see [74, 113]).

Definition 1.7.5. Let X and Y be continuous r.v.'s with d.f.'s, resp., F and G. If the following limits exist in $[0,1]$, then the *upper tail dependence parameter* λ_U of (X,Y) is defined by

$$\lambda_U = \lim_{t \to 1^-} P\left(Y > G^{[-1]}(t) \mid X > F^{[-1]}(t)\right);$$

$$\begin{array}{ccccc}
\mathrm{SI}(Y|X) & \Longrightarrow & \mathrm{RTI}(Y|X) & \Longleftarrow & \mathrm{RCSI}(X,Y) \\
\Downarrow & & \Downarrow & & \Downarrow \\
\mathrm{LTD}(Y|X) & \Longrightarrow & \mathrm{PQD}(X,Y) & \Longleftarrow & \mathrm{RTI}(X|Y) \\
\Uparrow & & \Uparrow & & \Uparrow \\
\mathrm{LCSD}(X,Y) & \Longrightarrow & \mathrm{LTD}(X|Y) & \Longleftarrow & \mathrm{SI}(X|Y)
\end{array}$$

Table 1.2: Implications among dependence concepts

and the *lower tail dependence parameter* λ_L of (X, Y) is defined by

$$\lambda_L = \lim_{t \to 0^+} P\left(Y \le G^{[-1]}(t) \mid X \le F^{[-1]}(t)\right).$$

In particular, if $\lambda_U = 0$ (resp. $\lambda_L = 0$), then X and Y are said to be *asymptotically independent in the upper tail* (resp. *in the lower tail*).

Proposition 1.7.5. *Let X and Y be continuous r.v.'s with copula C. If the following limits exist and take values in $]0,1]$, then*

$$\lambda_L = \lim_{u \to 0^+} \frac{C(u,u)}{u} \quad \text{and} \quad \lambda_U = \lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1-u}.$$

Moreover, if δ_C is the diagonal section of C, we have:

$$\lambda_L = \delta'_C(0^+) \quad \text{and} \quad \lambda_U = 2 - \delta'_C(1^-).$$

1.8 Measures of Association

There are a variety of ways to measure the association (or dependence) between random variables and, as noted by Hoeffding, many such descriptions are "scale invariant" ([71]), that is they remain unchanged under stricly increasing transformations of r.v.'s. But, in the words of B. Schweizer and E.F. Wolff, "*it is precisely the copula which captures those properties of the joint distribution function which are invariant under almost surely strictly increasing transformations*" ([143]). Thus, Sklar's Theorem and Theorem 1.6.3(i) suggest that copulas are a powerful tool to measure dependence.

In this section, we give a representation of some known measures of association in terms of copula; for more details, see [114, chapter 5] and [143, 74, 50].

Theorem 1.8.1. *Let X and Y be continuous r.v.'s whose copula is C. Then the population version for Kendall's tau for X and Y is given by*

$$\tau_{X,Y} := 4 \int\int_{[0,1]^2} C(u,v)\, dC(u,v) - 1 = 1 - 4 \int\int_{[0,1]^2} \partial_1 C(u,v) \cdot \partial_2 C(u,v)\, du dv.$$

Theorem 1.8.2. *Let X and Y be continuous r.v.'s whose copula is C. Then the population version of Spearman's rho for X and Y is given by*

$$\rho_{X,Y} := 12 \int\int_{[0,1]^2} C(u,v)\, dudv - 3.$$

Theorem 1.8.3. *Let X and Y be continuous r.v.'s whose copula is C. Then the population version of Gini's measure of association for X and Y is given by*

$$\gamma_{X,Y} := 4\left[\int_0^1 C(u, 1-u)\, du - \int_0^1 (u - C(u,u))\, du\right].$$

Theorem 1.8.4. *Let X and Y be continuous r.v.'s whose copula is C. Then the medial correlation coefficient of X and Y (called also Blomqvist coefficient) is given by*

$$\beta_{X,Y} := 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1.$$

Theorem 1.8.5. *Let X and Y be continuous r.v.'s whose copula is C. Then the Spearman's footrule coefficient of X and Y is given by*

$$\varphi_{X,Y} := 6\int_0^1 C(u,u)\, du - 2.$$

On the definition of such measures for non–continuous random variables, we refer to the paper [124].

1.9 Multivariate Copulas

In this section, we consider copulas in the n–dimensional case ($n \geq 3$).

Definition 1.9.1. A function $C\colon [0,1]^n \to [0,1]$ is an n–copula if, and only if, it satisfies the following conditions:

(C1') $C(\mathbf{x}) = 0$ if at least one coordinate of $\mathbf{x}$ is 0, and $C(\mathbf{x}) = x_i$ if all coordinates of $\mathbf{x}$ are 1 except at most the i–th one;

(C2') C is n–increasing.

As a consequence, every copula is increasing in each place and satisfies the 1–Lipschitz condition, viz.

$$|C(x_1, x_2, \ldots, x_n) - C(x_1', x_2', \ldots, x_n')| \leq \sum_{i=1}^{n} |x_i - x_i'|$$

for all $(x_1, x_2, \ldots, x_n)$ and $(x_1', x_2', \ldots, x_n')$ in $[0,1]^n$.

For every n–copula C, we have

$$W_n(\mathbf{x}) \leq C(\mathbf{x}) \leq M_n(\mathbf{x}) \quad \text{for all } \mathbf{x} \in [0,1]^n,$$

where

$$W_n(\mathbf{x}) := \max\left\{\sum_{i=1}^{n} x_i - n + 1, 0\right\}, \qquad M_n(\mathbf{x}) := \min\{x_1, x_2, \dots, n_n\}.$$

These bounds are the best–possible. Notice that, for $n \geq 3$, W_n is not a copula. Another important n–copula is the product

$$\Pi_n(\mathbf{x}) := \Pi_{i=1}^{n} x_i.$$

The set of all n copulas will be denoted by $\mathcal{C}_n$.

For sake of completeness, we give the analogous of Sklar's Theorem.

Theorem 1.9.1. *Let $X_1, X_2, \dots, X_n$ be r.v.'s with joint d.f. H and marginal d.f.'s $F_1, F_2, \dots, F_n$. Then there exists an n–copula C such that, for all $x \in \overline{\mathbb{R}}^n$*

$$H(\mathbf{x}) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n)). \tag{1.24}$$

If $F_1, F_2, \dots, F_n$ are continuous, then C is unique; otherwise C is uniquely determined on $Ran\, F_1 \times Ran\, F_2 \times \cdots \times Ran\, F_n$.

Conversely, if C is an n–copula and $F_1, F_2, \dots, F_n$ are univariate d.f.'s, then the function H given by (1.24) is an n–d.f. with margins $F_1, F_2, \dots, F_n$.

In the case $n \geq 3$, Theorems 1.6.2 and 1.6.3 can be partially reformulated in this way:

Theorem 1.9.2. *Let $X_1, X_2, \dots, X_n$ be continuous r.v.'s with copula C.*

- ▷ *$X_1, X_2, \dots, X_n$ are independent if, and only if, $C = \Pi_n$.*
- ▷ *each of the r.v.'s $X_1, X_2, \dots, X_n$ is almost surely a strictly increasing function of any of the others if, and only if, $C = M_n$.*
- ▷ *If $\alpha_1, \alpha_2, \dots, \alpha_n$ are strictly increasing mapping, respectively, on $RanX_1$, $RanX_2$, $\dots$, $RanX_n$, then $C_{\alpha_1(X_1)\dots\alpha_n(X_n)} = C_{X_1\dots X_n}$.*

The following result gives an important class of multivariate copulas, called *multivariate Archimedean copulas* for their analogy with the bivariate case (see [114, 112]).

Theorem 1.9.3. *Let $\varphi : [0,1] \to [0,+\infty]$ be continuous and strictly decreasing function with $\varphi(0) = +\infty$ and $\varphi(1) = 0$. Let C be the function defined by*

$$C_\varphi(\mathbf{x}) := \varphi^{-1}\left(\varphi(x_1) + \varphi(x_2) + \cdots + \varphi(x_n)\right).$$

If, for all $t \in]0,+\infty[$ and $k \in \mathbb{N} \cup \{0\}$

$$(-1)^k \frac{d^k}{dt^k}(\varphi^{-1}(t)) \geq 0,$$

then C_φ is an n–copula, called Archimedean copula.

1.10 Quasi-copulas

Quasi–copulas were introduced by Alsina, Nelsen and Schweizer ([4]) in order to characterize operations on distribution functions that can, or cannot, be derived from operations on random variables (see [122] and [116]). The concept of quasi–copula, which will be defined shortly, is closely connected to that of copula.

Definition 1.10.1. An *n–track* is any subset B of $[0,1]^n$ that can be written in the form

$$B = \{(F_1(t), F_2(t), \dots, F_n(t)) : t \in [0,1]\},$$

where $F_1, F_2, \dots, F_n$ are some continuous and increasing functions such that $F_i(0) = 0$ and $F_i(1) = 1$ for $i = 1, 2, \dots, n$.

Definition 1.10.2. An *n–quasi–copula* is a function $Q : [0,1]^n \to [0,1]$ such that for any n–track B there exists an n–copula C_B that coincides with Q on B, namely, for all $\mathbf{x} \in B$, $Q(\mathbf{x}) = C_B(\mathbf{x})$.

Such a definition of quasi–copula is, however, of little practical use because it is hard to tell whether a function $Q : [0,1]^n \to [0,1]$ is, or is not, a quasi–copula according to it. In view of this purpose, quasi–copulas were characterized in a different way: see [64] for the bivariate case and [21] for the multivariate case.

Theorem 1.10.1. *A function $Q : [0,1]^n \to [0,1]$ is an n–quasi–copula if, and only if, it satisfies the following conditions:*

(Q1) $Q(\mathbf{x}) = x_i$ *if all coordinates of* $\mathbf{x}$ *are* 1 *except at most the* i*–th one;*

(Q2) Q *is increasing in each variable;*

(Q3) Q *satisfies the* 1*–Lipschitz condition, viz.*

$$|Q(x_1, x_2, \dots, x_n) - Q(x'_1, x'_2, \dots, x'_n)| \le \sum_{i=1}^{n} |x_i - x'_i|$$

for all $(x_1, x_2, \dots, x_n)$ *and* $(x'_1, x'_2, \dots, x'_n)$ *in* $[0,1]^n$.

The set of all n–quasi–copulas will be denoted by $\mathcal{Q}_n$. Since an n–copula is obviously also an n–quasi–copula, the set $\mathcal{C}_n$ of all n–copulas is (strictly) included in $\mathcal{Q}_n$. If Q belongs to $\mathcal{Q}_n \setminus \mathcal{C}_n$, then we say that it is a *proper* n–quasi–copula.

For every n–quasi–copula Q, we have

$$W_n(\mathbf{x}) \le C(\mathbf{x}) \le M_n(\mathbf{x}) \quad \text{for all } \mathbf{x} \in [0,1]^n,$$

and W_n is a quasi–copula.

The concept of quasi–copulas has important applications on finding of best–possible bounds on arbitrary sets of d.f.'s (see [121, 131]). In particular, if we restrict to the bivariate case, we have:

Theorem 1.10.2 ([123])**.** *A function* $Q : [0,1]^2 \to [0,1]$ *is a quasi–copula if, and only if, there exists a nonempty set* $\mathcal{B}$ *of copulas such that, for every* $(x, y) \in [0,1]^2$, $Q(x, y) := \sup\{C(x, y) : C \in \mathcal{B}\}$.

1.11 Aggregation operators

The aggregation of several input values into a single output is an indispensable tool not only in mathematics, but also in any other disciplines where the fusion of different pieces of information is of vital interest (see [12]). In a very intuitive sense, an *aggregation operator* associates a single value to a list of values, where a value is simply an element of a given class (e.g., numbers, functions, sets, etc.). Therefore, from a mathematical point of view, an aggregation operator is simply a function that, *a priori*, has a varying number of variables. Here, following [10], we restrict ourselves to aggregations of a finite number of input values that belong to the unit interval $[0,1]$ into an output value belonging to the same interval and we consider aggregation operators according to the following

Definition 1.11.1. Let $n \in \mathbb{N}$, $n \geq 2$. An *n–ary aggregation operator* (briefly, *n–agop*) is a function $A : [0,1]^n \to [0,1]$ satisfying

(A1) $A(0, 0, \dots, 0) = 0$ and $A(1, 1, \dots, 1) = 1$;

(A2) A is increasing in each variable.

We note that the above conditions seem quite natural with respect to the intuitive idea of aggregation: (A1) states that if we have only minimal (respectively, maximal) possible inputs, then we should obtain the minimal (respectively, maximal) possible output; (A2) ensures that the aggregation preserves the cartesian ordering on the inputs. The assumptions that inputs and outputs belong to $[0,1]$ is not restrictive: in fact, if they belong to some interval $[a, b] \subset \overline{\mathbb{R}}$, it is always possible to re–scale them on $[0,1]$.

Definition 1.11.2. A (*global*) *aggregation operator* is a family $\mathbf{A} = \{A_{(n)}\}_{n \in \mathbb{N}}$ of n–agops, with the convention that $\mathrm{id}_{[0,1]}$ is the only 1–agop.

Such a definition of global aggregation operator is very useful because, in general, the number of input values to be aggregated is not known. Notice that, given a global aggregation operator A, $A_{(n)}$ and $A_{(m)}$ need not be related for $n \neq m$.

Remark 1.11.1. In 2005, during the Summer School on Aggregation Operators, E.P. Klement suggested to use the term "aggregation function" instead of "aggregation operator", when we aggregate real numbers and not complex quantities. We agree with this point of view, but it is not adopted here for the sake of uniformity with the literature of this field.

As it is easily seen, copulas and quasi–copulas are special types of n–agops. In particular, they are in the class of 1–stable n–agops, as stated in the following

Definition 1.11.3. Let $n \in \mathbb{N}$, $n \geq 2$ and $p \in [1, +\infty]$. An n–agop A is *p–stable* if, for all $\mathbf{x}$ and $\mathbf{y}$ in $[0,1]^n$

$$|A(\mathbf{x}) - A(\mathbf{y})| \leq \|x - y\|_p, \tag{1.25}$$

where $\|\cdot\|_p$ is the standard L^p norm on $\mathbb{R}^n$.

The class of p–stable aggregation operators was introduced in [11] for controlling output errors in aggregation processes. In particular, a 1–stable 2–agop A, also called 1*–Lipschitz* 2*–agop* ([90]), satisfies

$$|A(x,y) - A(x',y')| \leq |x - x'| + |y - y'|, \quad \text{for every } x, x'y, y' \in [0,1];$$

and a ∞–stable 2–agop A, also called *kernel* 2*–agop* ([93]), satisfies

$$|A(x,y) - A(x',y')| \leq \max\{|x - x'|, |y - y'|\}, \quad \text{for every } x, x'y, y' \in [0,1].$$

In the sequel, if no confusion arises, we use the term agop to denote simply a binary aggregation operators.

For every agop $A : [0,1]^2 \to [0,1]$, we have

$$A_S(x,y) \leq A(x,y) \leq A_G(x,y) \qquad \text{for every } (x,y) \in [0,1]^2,$$

where

$$A_S(x,y) = \begin{cases} 1, & \text{if } (x,y) = (1,1); \\ 0, & \text{otherwise;} \end{cases} \qquad A_G(x,y) = \begin{cases} 0, & \text{if } (x,y) = (0,0); \\ 1, & \text{otherwise;} \end{cases}$$

are called, respectively, *the smallest* and *the greatest* agop .

Given an A, the *dual* of A is defined, for every point (x,y) in $[0,1]^2$, by $A^d(x,y) := 1 - A(1-x, 1-y)$.

Chapter 2

The new concept of semicopula

The focus of this chapter is on the notion of *semicopula*. To the best of our knowledge, this term was used for the first time by B. Bassan and F. Spizzichino ([7]) and arises from a statistical application: the study of multivariate aging through the analysis of the Schur–concavity of the survival distribution function. Specifically, in order to define some notions of aging from the univariate case to the bivariate case, B. Bassan and F. Spizzichino introduced the so–called *bivariate aging function*, which *"has all the formal properties of a copula, except possibly for the rectangle inequlity"* (see [6]). Therefore, they call "semicopula" a function of this type. As it will be seen shortly, this function generalizes the concept both of copula and of triangular norm.

However, this concept was already known, in different contexts, as *conjunctor*, a monotone extension of the Boolean conjunction with neutral element 1 ([26, 27]), *t–seminorm* ([154]), or *generalized copula* ([136]). Moreover, the class of semicopulas appeared also in [140, Definition 2], where it is used in order to characterize some operations on d.f.'s that are not derivable from any operation on r.v.'s.

In section 2.1, we give the basic properties and examples of semicopulas. Some characterizations of the semicopulas M, Π and W are given in section 2.2, where super– and sub– harmonic semicopulas are studied and their statistical interpretation is presented. The study of the class of semicopulas is the object of section 2.3. The extension of semicopulas to the multivariate case is presented in section 2.4, where an interesting connection to the theory of fuzzy measures is also given.

These results can be also found in [47, 42, 34, 45].

2.1 Definition and basic properties

Definition 2.1.1. A function $S : [0,1]^2 \to [0,1]$ is said to be a *semicopula* if, and only if, it satisfies the two following conditions:

(S1) $S(x,1) = S(1,x) = x$ for all x in $[0,1]$;

(S2) $S(x,y) \leq S(x',y')$ for all $x,x',y,y' \in [0,1]$, $x \leq x'$ and $y \leq y'$.

The class of semicopulas will be denoted by $\mathcal{S}$.

In other words, a semicopula is a binary aggregation operator with neutral element 1 and, consequently, annihilator 0, because

$$0 \leq S(x,0) \leq S(1,0) = 0,$$

and, analogously, $S(0,x) = 0$ for all $x \in [0,1]$.

The class $\mathcal{S}$ strictly includes the class $\mathcal{Q}$ of quasi–copulas and, if we denote by $\mathcal{S}_C$ the set of continuous semicopulas, $\mathcal{S}_C \subset Q$. Moreover, the set $\mathcal{S}_S$ of symmetric semicopulas is a proper subset of $\mathcal{S}$ and it strictly includes the set $\mathcal{T}$ of t–norms.

Example 2.1.1.

▷ The drastic t–norm Z is a semicopula, but it is not a quasi–copula, because it is not continuous.

▷ $S_1(x,y) = xy \max\{x,y\}$ is a continuous semicopula, but, because it is not associative, it is not a t–norm. Moreover, S_1 is not a quasi–copula, because

$$S_1(8/10, 9/10) - S_1(8/10, 8/10) = 136/1000 > 1/10.$$

▷ The following mapping S_2 is an associative semicopula that is not commutative

$$S_2(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1/2] \times [0,1[; \\ \min\{x,y\}, & \text{otherwise.} \end{cases}$$

Proposition 2.1.1. *If $S : [0,1]^2 \to [0,1]$ is a semicopula, then*

$$Z(x,y) \leq S(x,y) \leq M(x,y) \qquad \textit{for all } x \textit{ and } y \textit{ in } [0,1]. \tag{2.1}$$

Proof. If S is a semicopula, then, for all $x, y \in [0,1[$, we obtain

$$0 = S(x,0) \leq S(x,y) \leq S(x,1) = x.$$

Analogously,

$$0 = S(x,0) \leq S(x,y) \leq S(1,y) = y,$$

so that $S(x,y) \leq \min\{x,y\}$. □

It must be noticed that no assumption on the (left– or right–) continuity of a semicopula has hitherto been made and different types of continuity can be also considered in the class of semicopulas in the spirit of [88]; but, the next result can be useful (see, e.g., [95]).

Proposition 2.1.2. *Let $H : [0,1]^2 \to [0,1]$ be increasing in each variable. The following statements are equivalent:*

(a) *H is jointly (left–) continuous, in the sense that if $\{s_n\}$ and $\{t_n\}$ are two increasing sequences of points of $[0,1]$ that tend to s and t respectively, then*

$$\lim_{n\to+\infty} H(s_n,t_n) = H(s,t);$$

(b) *H is (left–) continuous in each place.*

Because of (S2), every semicopula has derivatives almost everywhere on $[0,1]^2$. In particular, some conditions on derivatives allow us to characterize the semicopulas that are also quasi–copulas. But, first, we give two technical lemmata (see, respectively, page 333 and 337 of [153]).

Lemma 2.1.1. *Let $f : [a,b] \to \mathbb{R}$ be given. If f is continuous on $[a,b]$ and differentiable except at countably many points of $[a,b]$, and f' is Lebesgue integrable on $[a,b]$, then f is absolutely continuous on $[a,b]$.*

Lemma 2.1.2. *Let $f : [a,b] \to \mathbb{R}$ be given. The following statements are equivalent:*

(a) *for some $k > 0$, we have*

$$|f(x) - f(y)| \le k|x-y| \quad \text{for all } x, y \in [a,b];$$

(b) *f is absolutely continuous on $[a,b]$ and $|f'(t)| \le k$ on $[a,b]$ for some $k > 0$.*

Proposition 2.1.3. *Let S be a semicopula such that all the horizontal and vertical sections of S are differentiable on $[0,1]$ except at countably many points. The following statements are equivalent:*

(a) *S is a quasi–copula;*

(b) *S satisfies the following two conditions:*

  (b1) *S is continuous;*

  (b2) *for every (x,y) in $[0,1]^2$ that admits first–order partial derivatives of S*

$$0 \le \partial_x S(x,y) \le 1 \quad \text{and} \quad 0 \le \partial_y S(x,y) \le 1.$$

Proof. Implication $(a) \implies (b)$ is trivial. In order to prove $(b) \implies (a)$, let $S_y(t)$ be the horizontal section of S at $y \in [0,1]$ and $S_x(t)$ be the vertical section of S at $x \in [0,1]$. The functions S_x and S_y are continuous and differentiable on $[0,1]$ except at countably many points and their derivatives are bounded. Therefore, from Lemma 2.1.1 it follows that they are absolutely continuous. But, again, if S_x and S_y are absolutely continuous and their derivatives are bounded from above by 1, then Lemma 2.1.2 ensures that S_x and S_y are Lipschitz with constant 1. Therefore, for every (x,y) and (x',y') in $[0,1]^2$, we have

$$\begin{aligned} |S(x,y) - S(x',y')| &\leq |S(x,y) - S(x',y)| + |S(x',y) - S(x',y')| \\ &\leq |S_y(x) - S_y(x')| + |S_{x'}(y) - S_{x'}(y')| \\ &\leq |x - x'| + |y - y'|, \end{aligned}$$

which is the desired assertion. □

Notice that there exists also a semicopula which is not Lebesgue measurable.

Example 2.1.2. Let J be a subset of $[0,1]$ that is not Lebesgue measurable. Define the function

$$S(x,y) = \begin{cases} 0, & (x+y<1) \quad \text{or} \quad (x+y=1 \text{ and } x \in J); \\ \min\{x,y\}, & \text{otherwise.} \end{cases}$$

Then S is a semicopula that is not Lebesgue measurable. In [79] there is an analogous example of a t–norm which is not Lebesgue measurable.

Given a semicopula S, its diagonal section δ satisfies the following properties:

(a) $\delta(1) = 1$;

(b) $\delta(t) \leq t$ for all $t \in [0,1]$;

(c) δ is increasing.

Conversely, given a function δ satisfying properties (a), (b) and (c), it is always possible to construct a semicopula whose diagonal section is δ; for instance:

$$S_\delta(x,y) := \begin{cases} \delta(x) \wedge \delta(y), & \text{if } (x,y) \in [0,1[^2; \\ x \wedge y, & \text{otherwise.} \end{cases}$$

A semicopula need not be uniquely determined by its diagonal. For example, if $\delta(t) = t^2$ for all $t \in [0,1]$, there are two different semicopulas, Π and S_δ with diagonal section equal to δ. The only semicopulas uniquely determined by their diagonal sections are M and Z, as asserted in the following

Proposition 2.1.4. *The only semicopula with diagonal section equal to $id_{[0,1]}$ is M.*

Proof. Suppose that $\delta(t) = t$ for all t in $[0,1]$. For all $x, y \in [0,1]$, if $x \geq y$, then

$$S(y,y) = y \leq S(x,y) \leq S(1,y) = y;$$

whereas if $x < y$, then

$$S(x,x) = x \leq S(x,y) \leq S(x,1) = x;$$

that is $S(x,y) = \min\{x,y\}$. □

Analogously, we can prove

Proposition 2.1.5. *The only semicopula with diagonal $\delta(t) = 0$ on $[0,1[$ is Z.*

The proof of the following result is immediate and will not be given.

Proposition 2.1.6. *Let $S = (\langle a_i, b_i, S_i \rangle)_{i \in I}$ be an ordinal sum of semicopulas. Then S is a semicopula.*

Another simple construction method for semicopulas is presented here.

Example 2.1.3 (Frame semicopula)**.** Let the points

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

partition the unit interval $[0,1]$ and let

$$0 = v_0 \leq v_1 \cdots \leq v_n < 1$$

be points in $[0,1]$ such that $v_i \leq t_i$ ($i \in \{1, 2, \ldots, n\}$). The *frame semicopula* S_f corresponding to $(t_0, t_1, \ldots, t_n)$ and $(v_0, v_1, \ldots, v_n)$ is defined by

$$S_f(x,y) := \begin{cases} v_{i-1}, & \text{if } (x,y) \in [t_{i-1}, 1[^2 \setminus [t_i, 1[^2; \\ x \wedge y, & \text{if } x \vee y = 1. \end{cases}$$

Moreover, if continuity questions arise, we may choose as the value taken on the side of each frame the value taken on the frame below.

2.2 Characterizations of some semicopulas

At a first glance, the definition of semicopula might appear somewhat more general than actually is. In this sense, it will be shown in this section that condition (S1) is quite restrictive and that it allows to characterize some basic semicopulas.

Proposition 2.2.1. *Let S be a semicopula. The following statements are equivalent:*

(a) *S is concave;*

(b) *S is super–homogeneous, viz.* $S(\lambda x, \lambda y) \geq \lambda\, S(x,y)$ *for all* x, y *and* λ *in* $[0,1]$*;*

(c) *S is idempotent, viz.* $S(x,x) = x$ *for every* $x \in [0,1]$*;*

(d) $S = M$*.*

Proof. If S is concave, then $S(\lambda x, \lambda y) = S(\lambda(x,y) + (1-\lambda)(0,0)) \geq \lambda S(x,y)$, and (b) holds. If S is super–homogeneous, then $S(x,x) \geq x S(1,1) = x$, which together with $S(x,x) \leq S(x,1) = x$, leads to (c). If S is idempotent, then Proposition 2.1.4 ensures that $S = M$. Finally, it is clear that M is concave. □

Proposition 2.2.2. *Let S be a semicopula. The following statements are equivalent:*

(a) *S is convex and 1–Lipschitz;*

(b) *S is a function of the sum of its arguments, i.e.* $S(x,y) = F(x+y)$ *for some function F from* $[0,2]$ *into* $[0,1]$*;*

(c) $S = W$*.*

Proof. (a) ⇒ (c): Suppose that S is convex and 1–Lipschitz. If $x + y \in\,]0,1]$, define $\lambda := y/(x+y)$, which is in $[0,1]$; then $(x,y) = \lambda(0, x+y) + (1-\lambda)(x+y, 0)$. Now, since S is convex,

$$0 \leq S(x,y) \leq \lambda S(0, x+y) + (1-\lambda) S(x+y, 0) = 0;$$

therefore, $S(x,y) = 0$. If $x + y \geq 1$, define $\lambda := (1-y)/[2-(x+y)]$, which is in $[0,1]$, in order to obtain $(x,y) = \lambda(1, x+y-1) + (1-\lambda)(x+y-1, 1)$. Again, since S is convex,

$$S(x,y) \leq \lambda S(1, x+y-1) + (1-\lambda) S(x+y-1, 1) = x+y-1,$$

and, since it is 1–Lipschitz,

$$S(1,1) - S(x,y) \leq 1 - x + 1 - y.$$

Therefore $S(x,y) = x+y-1$, and (c) holds.

(b) ⇒ (c): Suppose that there exists a function F from $[0,2]$ into $[0,1]$ such that $S(x,y) = F(x+y)$. If t is in $[0,1]$, then $F(t) = S(0,t) = 0$, and if t is in $[1,2]$, then $F(t) = S(1, t-1) = t-1$. Therefore, $F(t) = \max\{0, t-1\}$, and $S(x,y) = F(x+y) = \max\{x+y-1, 0\} = W(x,y)$.

Parts "(c)⇒(a)" and "(c)⇒(b)" can be easily proved. □

In particular, part (b) is equivalent to the fact that S is Schur–constant.

Proposition 2.2.3. *The following properties are equivalent for a semicopula S:*

(a) *S is positively homogeneous with respect to one variable, viz. for every x, y, λ in $[0,1]$, either $S(x,\lambda y)=\lambda S(x,y)$ or $S(\lambda x,y)=\lambda S(x,y)$;*

(b) *S has separate variables, viz. there exist two functions F_1 and F_2 defined from $[0,1]$ into $[0,1]$ such that $S(x,y)=F_1(x)\cdot F_2(y)$;*

(c) *S has linear section in both the variables;*

(d) $S=\Pi$.

Proof. Without loss of generality assume that S is homogeneous with respect to the first variable; then $S(x,y)=x\,S(1,y)=xy$; therefore (a) implies (b).

Now, suppose that (b) holds and let $S(x,y)=F_1(x)\cdot F_2(y)$ be a semicopula. It follows that $S(x,1)=F_1(x)\cdot F_2(1)=x$ and $S(1,x)=F_1(1)\cdot F_2(x)=x$. Therefore, for every $a\in[0,1]$, we have $S(x,a)=F_1(x)\cdot F_2(a)=(F_2(a)/F_2(1))\cdot x$, viz. the horizontal section of S at the point a is linear. The same result holds for the vertical section of S.

Finally, if S has linear sections in both the variables, then, fixed $a\in[0,1]$, we have $S(x,a)=\lambda_a x$ for a suitable $\lambda_a\in[0,1]$. But $S(1,a)=a$ and, hence, $\lambda_a=a$ and $S=\Pi$. Obviously, (d) implies (a). □

2.2.1 Harmonic semicopulas

Let Ω be an open subset of $\mathbb{R}^2$. A twice continuously differentiable function $F\colon\Omega\to\mathbb{R}$ is said to be *harmonic* if

$$\Delta F(x,y) := \frac{\partial^2 F(x,y)}{\partial x^2}+\frac{\partial^2 F(x,y)}{\partial y^2}=0 \quad \text{for all } (x,y)\in\Omega.$$

Moreover, such F is said to be *superharmonic* (resp. *subharmonic*) if $\Delta F\le 0$ (resp. $\Delta F\ge 0$). For more details on harmonic function theory, we refer to [5]. Here we recall two important results for harmonic functions.

Theorem 2.2.1 (Maximum–minimum principle for harmonic functions)**.** *Let Ω be a connected open subset of $\mathbb{R}^2$ and let F be a harmonic function on Ω. If F has either a maximum or a minimum on Ω, then F is constant on Ω.*

Theorem 2.2.2. *Let Ω be a connected open subset of $\mathbb{R}^2$ and let F be a superharmonic (respectively, subharmonic) function on Ω. If F has a minimum (respectively, a maximum) on Ω, then it is constant on Ω.*

Proposition 2.2.4. *The only harmonic semicopula is* Π.

Proof. It is easily shown that Π is harmonic. Suppose that there exists another harmonic semicopula F and let (x_0,y_0) be a point in $]0,1[^2$ such that $\Pi(x_0,y_0)\neq F(x_0,y_0)$. Now, $G:=F-\Pi$ is a harmonic function that vanishes on the boundary of $[0,1]^2$. Therefore, G has either a maximum or a minimum on $]0,1[^2$, and, in view of the maximum–minimum principle for harmonic functions, G is constant, and this constant is equal to zero, viz. $F = \Pi$. □

Proposition 2.2.5. *If S is a superharmonic (resp. subharmonic) semicopula, then $S \geq \Pi$ (resp. $S \leq \Pi$).*

Proof. If S is a superharmonic semicopula, then $G := S - \Pi$ is also superharmonic and it vanishes on the boundary of $[0,1]^2$. Therefore, $S(x,y) - \Pi(x,y) \geq 0$ for every (x,y) in $[0,1]^2$, because, otherwise, Theorem 2.2.2 would imply $S = \Pi$. A similar argument holds for subharmonic semicopulas. □

In the case of copulas, the following result holds.

Proposition 2.2.6. *Let (X,Y) be a continuous random pair with copula C. If C is superharmonic, then (X,Y) is positively quadrant dependent. Analogously, if C is subharmonic, then (X,Y) is negatively quadrant dependent.*

Proposition 2.2.7. *Let the copula C of a pair (X,Y) of continuous random variables be twice–differentiable.*

(a) *If Y is stochastically increasing in X and if X is stochastically increasing in Y, then C is superharmonic.*

(b) *If Y is stochastically decreasing in X and if X is stochastically decreasing in Y, then C is subharmonic.*

Proof. In view of Proposition 1.7.3, the property $SI(Y|X)$ is equivalent to the concavity of the function $x \mapsto C(x,y)$ for every $y \in [0,1]$, and $SI(X|Y)$ is equivalent to the concavity of the function $y \mapsto C(x,y)$ for every $x \in [0,1]$. Because C is twice differentiable, it follows that $\partial^2_{xx} C(x,y) \leq 0$ and $\partial^2_{yy} C(x,y) \leq 0$, from which $\Delta C(x,y) \leq 0$. The proof of part (b) is analogous. □

Therefore we can insert the concept of super– and sub– harmonicity in the scheme of dependence concepts (note that the converse implications in Table 2.1 are, in general, false).

SI(Y|X) & SI(X|Y)	$\Longrightarrow$	Superharmonicity	$\Longrightarrow$	PQD(X,Y)
SD(Y|X) & SD(X|Y)	$\Longrightarrow$	Subharmonicity	$\Longrightarrow$	NQD(X,Y)

Table 2.1: Superharmonicity and dependence concepts

Example 2.2.1. Let consider the class of copulas given by $C_{fg}(x,y) = xy + \lambda f(x)\, g(y)$, where f and g are suitable functions and $\lambda > 0$ (see [132]). We have

$$\Delta C_{fg}(x,y) = \lambda(f''(x)\, g(y) + f(x)\, g''(y)).$$

If $f(t) = t(1-t)^2$ and $g(t) = t(1-t)$, then C_{fg} is a PQD copula, but

$$\Delta C_{fg}(x,y) = \lambda \left[(6x-4)y(1-y) - 2x(1-x)^2\right]$$

is (strictly) positive on the set $\{(x,y) \in [0,1]^2 : x = 1\}$ and it is (strictly) negative on the set $\{(x,y) \in [0,1]^2 : 0 \le x < 2/3\}$; thus C_{fg} is neither superharmonic nor subharmonic.

Analogously, we can find two functions f and g such that C_{fg} is superharmonic, but f and g are not both concave and, thus, C_{fg} is not $SI(Y|X)$ and $SI(X|Y)$.

2.3 The class of semicopulas

Proposition 2.3.1. *If S_1 and S_2 are semicopulas, then for all $\theta \in [0,1]$ both the weighted arithmetic mean $(1-\theta)S_1 + \theta S_2$ and the weighted geometric mean $S_1^\theta\, S_2^{1-\theta}$ are semicopulas. In other words, the set $\mathcal{S}$ is convex and log–convex.*

Let $\mathcal{X}$ denote the set of all functions from $[0,1]^2$ to $[0,1]$ equipped with the product topology (which corresponds to pointwise convergence).

Theorem 2.3.1. *The class $\mathcal{S}$ of semicopulas is a compact subset of $\mathcal{X}$ (under the topology of pointwise convergence).*

Proof. Since $\mathcal{X}$ is a product of compact spaces, it is well known from Tychonoff Theorem (see, e.g., [76]) that $\mathcal{X}$ is compact. The proof is completed by showing that $\mathcal{S}$ is a closed subset of $\mathcal{X}$, viz. given a sequence $\{S_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}$, if S_n converges pointwise to S, then S belongs to $\mathcal{S}$. In fact, for all $x, x', y \in [0,1]$ and $n \in \mathbb{N}$,

$$S_n(x,1) = x \xrightarrow[n\to+\infty]{} x = S(x,1),$$

and, if $x \le x'$, $S_n(x,y) \le S_n(x',y)$ implies $S(x,y) \le S(x',y)$, which is the desired conclusion. □

A sequence $\{S_n\}_{n\in\mathbb{N}}$ of semicopulas is a Cauchy sequence with respect to pointwise convergence if, for every $\epsilon > 0$ and for every point (x,y) in $[0,1]^2$, there exists a natural number $n_0 = n_0(\epsilon, x, y)$ such that

$$|S_n(x,y) - S_m(x,y)| < \epsilon,$$

whenever $n, m \ge n_0$. As an immediate consequence, each Cauchy sequence of semicopulas converges pointwise to some semicopula; in other words $\mathcal{S}$ is complete. Notice that it is known that the class $\mathcal{T}$ of t–norms is neither a complete nor a compact subset of $\mathcal{S}$ ([83]).

Now, consider the set $\mathcal{S}$ equipped with the pointwise ordering. Obviously, $(\mathcal{S}, \leq)$ is partially ordered, and not all pairs of semicopulas are comparable: it is sufficient to consider the following example.

Example 2.3.1. Let S_1 and S_2 be, respectively, the two ordinal sums given by

$$S_1(x,y) = (\langle 0, 1/2, Z\rangle) = \begin{cases} 0, & \text{if } (x,y) \in [0,1/2[^2, \\ \min\{x,y\}, & \text{otherwise;} \end{cases}$$

and by

$$S_2(x,y) = (\langle 1/2, 1, Z\rangle) = \begin{cases} 1/2, & \text{if } (x,y) \in [1/2,1[^2; \\ \min\{x,y\}, & \text{otherwise.} \end{cases}$$

Then

$$0 = S_1(1/4, 1/4) < S_2(1/4,1/4) = 1/4,$$

but

$$3/4 = S_1(3/4,3/4) > S_2(3/4,3/4) = 1/2.$$

Proposition 2.3.2. *The set $\mathcal{S}$, equipped with the classical pointwise ordering, is a complete lattice.*

Proof. Let $\mathcal{B}$ be a nonempty subset of $\mathcal{S}$. For all $x, x', y \in [0,1]$ such that $x \leq x'$,

$$\vee\mathcal{B}(x,1) = \sup\{S(x,1) \colon S \in \mathcal{B}\} = x,$$

that is $\vee\mathcal{B}$ satisfies the condition $(S1)$ of Definition 2.1.1; moreover,

$$\vee\mathcal{B}(x,y) = \sup\{S(x,y) \colon S \in \mathcal{B}\} \leq \sup\{S(x',y) \colon S \in \mathcal{B}\} = \vee\mathcal{B}(x',y),$$

that is $\vee\mathcal{B}$ satisfies the condition $(S2)$ of Definition 2.1.1, and hence $\vee\mathcal{B}$ is a semicopula. Analogously $\wedge\mathcal{B}$ is a semicopula. □

In particular, the minimum (and the maximum) of two semicopulas is a semicopula. This result holds also for quasi–copulas, but neither for copulas nor for t-norms, as the following examples show (see, also, [123]).

Example 2.3.2. Consider the two copulas defined, for α and β in $]0,1[$ by

$$A_\alpha(x,y) := \begin{cases} \alpha \vee (x+y-1), & \text{if } (x,y) \in [\alpha,1]^2; \\ x \wedge y, & \text{otherwise;} \end{cases}$$

(this is the ordinal sum $(\langle \alpha, 1, W\rangle)$) and

$$B_\beta(x,y) := \begin{cases} \dfrac{xy}{\beta}, & \text{if } (x,y) \in [0,\beta]^2; \\ x \wedge y, & \text{otherwise;} \end{cases}$$

(this is the ordinal sum $(\langle 0, \beta, \Pi \rangle)$). Now, for $\alpha = 1/3$ and $\beta = 1/2$, the function $F : [0,1]^2 \to [0,1]$ defined by $F(x,y) := A_{(1/3)}(x,y) \wedge B_{(1/2)}(x,y)$ is not a copula. In fact, choose $s = t = 1/3$ and $s' = t' = 1/2$,

$$F(s',t') - F(s',t) - F(s,t') + F(s,t) = -1/9 < 0.$$

Moreover, $A_{(1/3)}$ and $B_{(1/2)}$ are t–norms, but the function F is not associative, because $F\left(F(1/2,1/2),1/3\right) = 2/9$, while $F\left(1/2, F(1/2,1/3)\right) = 1/3$.

Example 2.3.3. Consider the two copulas:

$$A_\lambda(x,y) = \begin{cases} y, & 0 \le y < \lambda x; \\ \lambda x, & \lambda x \le y < 1 - (1-\lambda)x; \\ x + y - 1, & \text{otherwise}; \end{cases}$$

and $B_\lambda = A^T$ the transpose of A. Then, for $\lambda = 1/2$, we have

$$\max\left\{A_{(1/2)}, B_{(1/2)}\right\}\left(\left[\frac{1}{3}, \frac{2}{3}\right]^2\right) = -\frac{1}{6} < 0.$$

Example 2.3.4. Consider the two t–norms:

$$T_1(x,y) = \begin{cases} x \wedge y, & x + y > 1; \\ 0, & \text{otherwise}; \end{cases}$$

and $T_2(x,y) = \Pi$. Then

$$T = \max\{T_1(x,y), T_2(x,y)\} = \begin{cases} x \wedge y, & x + y > 1; \\ xy, & \text{otherwise}; \end{cases}$$

is not associative. In fact,

$$T\left(T\left(\frac{4}{10}, \frac{5}{10}\right), \frac{7}{10}\right) = T\left(\frac{20}{100}, \frac{7}{10}\right) = \frac{14}{100},$$

but

$$T\left(\frac{4}{10}, T\left(\frac{5}{10}, \frac{7}{10}\right)\right) = T\left(\frac{4}{10}, \frac{5}{10}\right) = \frac{20}{100}.$$

2.3.1 Extremal semicopulas

Definition 2.3.1. A semicopula S is said to be *extremal* if it can not be expressed as a non–trivial convex sum of two semicopulas; in the sense that, if S admits the representation $S = \lambda\, A + (1-\lambda)\, B$ for A and B in $\mathcal{S}$ and $\lambda \in\,]0,1[$, then $S = A = B$.

By connecting Proposition 2.3.1 and Theorem 2.3.1, it follows that $\mathcal{S}$ is a compact and convex subset of $\mathcal{X}$; therefore, in view of the Krein–Millman Theorem (see, e.g., [32]), we have:

Proposition 2.3.3. *The class $\mathcal{S}$ of semicopulas is the convex hull of the set formed by extremal semicopulas.*

Next we show that the semicopulas Z and M are extremal.

Given the semicopula Z, suppose that there exist B and C in $\mathcal{S}$ and $\lambda \in]0,1[$ such that $Z(x,y) = \lambda\, B(x,y) + (1-\lambda)\, C(x,y)$ on $[0,1]^2$. For all $x, y \in [0,1[$, the equality

$$Z(x,y) = 0 = \lambda\, B(x,y) + (1-\lambda)\, C(x,y)$$

implies

$$B(x,y) = 0 = C(x,y),$$

so that $B = Z = C$ on $[0,1]^2$.

Using the same notations, we consider the semicopula M and suppose

$$M(x,y) = \lambda\, B(x,y) + (1-\lambda)\, C(x,y)$$

on $[0,1]^2$. In particular, for every $x \in [0,1]$ the equality

$$M(x,x) = x = \lambda\, B(x,x) + (1-\lambda)\, C(x,x)$$

implies

$$\delta_B(x) = \delta_C(x) = x,$$

which, in view of Proposition 2.1.4, yields $B = C = M$.

Extremal semicopulas can be easily constructed beginning from root sets. We recall that a *root set* $A \subset [0,1]^2$ is defined by the property:

$$(x,y) \in A \quad \text{implies} \quad (x',y') \in A \quad \text{for every } 0 \le x' \le x \text{ and } 0 \le y' \le y.$$

Thus, given a root set A, the semicopula S_A defined by

$$S_A(x,y) = \begin{cases} 0, & \text{if } (x,y) \in A; \\ x \wedge y, & \text{otherwise;} \end{cases}$$

is extremal, and this can be proved by the same arguments of the cases M and Z. Such S_A are called 1–*internal* semicopulas. Notice that M and Z are 1–internal semicopulas with root sets, respectively, $A_M = \emptyset$ and $A_Z = [0,1[^2$. Moreover, S_A is a t–norm if the set A is symmetric with respect to the main diagonal of the unit square.

Remark 2.3.1. For every semicopula S and for every $u \in [0,1]$, we can define the root set

$$A_u := \{(x,y) \in [0,1[^2 : S(x,y) < u\},$$

and we have

$$S(x,y) = \bigvee_{u \in [0,1]} S_{A_u}(x,y).$$

Thus every semicopula is the supremum of a set formed by 1–internal semicopulas.

Notice that the semicopula W is not extremal in $\mathcal{S}$. In fact, it suffices to consider the two semicopulas

$$S_1(x,y) = W(x,y)\,(2 - \max\{x,y\}) \text{ and } S_2(x,y) = W(x,y)\cdot \max\{x,y\}.$$

Then $W = (S_1 + S_2)/2$.

Analogously, Π is not extremal in $\mathcal{S}$ (and also in the class of copulas). In fact, $\Pi = (C_1 + C_2)/2$, where

$$C_1(x,y) = \begin{cases} \frac{xy}{2}, & (x,y) \in \left[0,\frac{1}{2}\right]^2; \\ \frac{3xy-x}{2}, & (x,y) \in \left[0,\frac{1}{2}\right] \times \left[\frac{1}{2},1\right]; \\ \frac{3xy-y}{2}, & (x,y) \in \left[\frac{1}{2},1\right] \times \left[0,\frac{1}{2}\right]; \\ \frac{xy+x+y-1}{2}, & (x,y) \in \left[\frac{1}{2},1\right]^2; \end{cases}$$

and

$$C_2(x,y) = \begin{cases} \frac{3xy}{2}, & (x,y) \in \left[0,\frac{1}{2}\right]^2; \\ \frac{xy+x}{2}, & (x,y) \in \left[0,\frac{1}{2}\right] \times \left[\frac{1}{2},1\right]; \\ \frac{xy+y}{2}, & (x,y) \in \left[\frac{1}{2},1\right] \times \left[0,\frac{1}{2}\right]; \\ \frac{3xy-x-y+1}{2}, & (x,y) \in \left[\frac{1}{2},1\right]^2; \end{cases}$$

and C_1 and C_2 are copulas.

2.4 Multivariate semicopulas

The notion of semicopula can be extended in a natural way to the n–dimensional case ($n \geq 3$).

Definition 2.4.1. A function $S : [0,1]^n \to [0,1]$ is said to be an *n–semicopula* if it satisfies the two following conditions:

(S1') $S(\mathbf{x}) = x_i$ if all coordinates of $\mathbf{x}$ are 1 except at most the i–th one;

(S2') S is increasing in each place.

Higher dimensional semicopulas are easily constructed from lower dimensional ones, in view of the following results, whose easy proofs will not be reproduced here.

Proposition 2.4.1. *Let H be a 2–semicopula and let S_m and S_n be, respectively, an m–semicopula and an n–semicopula ($m, n \in \mathbb{N}$). Then the function $S : [0,1]^{m+n} \to [0,1]$ defined by*

$$S(x_1,\ldots,x_{m+n}) := H\left(S_m(x_1,\ldots,x_m), S_n(x_{m+1},\ldots,x_{m+n})\right) \tag{2.2}$$

is an $(m+n)$–semicopula.

Aggregation operators of type (2.2) are called *double aggregation operators*; they allow to combine two input lists of information coming from different sources into a single output (see [13] for more details).

In the opposite direction we can construct lower dimensional semicopulas from higher dimensional ones.

Proposition 2.4.2. *Any m–marginal of an n–semicopula S_n, $2 \leq m < n$, is an m–semicopula, viz., if S_n is an n–semicopula, then the function $S_m : [0,1]^m \to [0,1]$ defined by*

$$S_m(x_1, x_2, \dots, x_m) = S_n(x_1, x_2, \dots, x_m, 1, 1, \dots, 1)$$

is an m–semicopula, and so any function obtained from it by permuting its arguments.

From Definition 2.4.1, it follows that all n–quasi–copulas are n–semicopulas. On the other hand, it is clear that an n–semicopula is a special n–ary aggregation operator.

In particular, a family of semicopulas $\{S_n : [0,1]^n \to [0,1]\}_{n\in\mathbb{N}}$ is, obviously, a global aggregation operator, but it need not have the neutral element property (in the sense of global agop), because, in general, $S_n(x_1, \dots, x_{n-1}, 1) \neq S_{n-1}(x_1, \dots, x_{n-1})$. Here we propose a possible definition of *global semicopula*.

Definition 2.4.2. A family of commutative semicopulas $\{S_n : [0,1]^n \to [0,1]\}_{n\in\mathbb{N}}$ is called a *global semicopula* if $S_1 = \mathrm{id}_{[0,1]}$ and, for every $n \geq 2$,

$$S_{n-1}(x_1, \dots, x_{n-1}) = S_n(x_1, \dots, x_{n-1}, 1).$$

Notice that, in this way, a global semicopula is a global aggregation operator with neutral element 1.

Analogously, we can define the concepts of *global quasi–copula* and *global copula*.

In practice, it is not difficult to construct a global semicopula. It suffices to take a commutative 2–semicopula S and construct the family $\{S_n : [0,1]^n \to [0,1]\}_{n\in\mathbb{N}}$ in such a way that $S_1 = \mathrm{id}_{[0,1]}$, and, for every $n \geq 2$,

$$S_n(x_1, \dots, x_n) := S(S_{n-1}(x_1, \dots, x_{n-1}), x_n).$$

This method can be used also for quasi–copulas, but not for copulas, where it is not immediate to construct a copula beginning from his margins (see [141] for more details).

Finally, we present a few comments on a possible use of global copulas in a probabilistic context.

Consider a stochastic process $\{X_n\}_{n\in\mathbb{N}}$ in which all the random variables (=r.v.'s) are continuous. In view of Sklar's Theorem, a (unique) k–dimensional copula C_k can be associated with any choice of k r.v.'s $X_{i_1}, \dots, X_{i_k}$. In particular, if the r.v.'s of the process are *exchangeable*, the copula C_k is commutative and it does not depend on the choice of the r.v.'s. Moreover, C_{k-1} is the $(k-1)$–margin copula of C_k.

Conversely, if $\{C_n : [0,1]^n \to [0,1]\}_{n\in\mathbb{N}}$ is a global copula, in view of the Kolmogorov compatibility Theorem (see [94]), we can construct an exchangeable stochastic process $\{X_n\}_{n\in\mathbb{N}}$ (where each r.v. X_n is uniformly distributed on $[0,1]$) such that, for every $n \in \mathbb{N}$, C_n is the copula associated with any choice of n r.v.'s of the process.

Thus we have established a one–to–one correspondence between global copulas and exchangeable stochastic processes.

2.4.1 Multivariate semicopulas and fuzzy measures

Here, we reformulate a result of M. Scarsini (see [136]) through the concept of multivariate semicopula. To this end, some basic notations will be useful (see [30, 16]).

For every $n \geq 2$, let $\mathcal{B}(\overline{\mathbb{R}}^n)$ be the class of Borel sets in $\overline{\mathbb{R}}^n$. A set function $\nu : \mathcal{B}(\overline{\mathbb{R}}^n) \to [0,1]$ is called *fuzzy measure* (or *capacity*) if it satisfies:

(a) $\nu(\emptyset) = 0$ and $\nu(\overline{\mathbb{R}}^n) = 1$;

(b) $\nu(A) \leq \nu(B)$ for all Borel sets A and B, $A \subseteq B$.

In particular, a fuzzy measure ν is called *supermodular* (or *convex*) if, for all Borel sets A and B

$$\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B).$$

Given a fuzzy measure ν, the *distribution function associated with* ν is the function $F_\nu : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$ given by

$$F_\nu(x_1, \dots, x_n) = \nu\left([-\infty, x_1] \times \cdots \times [-\infty, x_1]\right).$$

Moreover, we denote by F_{ν_i} the marginal d.f. associated with ν_i, where ν_i is the i–th projection of ν $(i = 1, 2, \dots, n)$. Notice that, due to lack of additivity, a fuzzy measure is not completely characterized by its distribution function.

Theorem 2.4.1 ([136])**.** *Let ν be a supermodular fuzzy measure on $(\overline{\mathbb{R}}^n, \mathcal{B}(\overline{\mathbb{R}}^n))$, F_ν its associated d.f., and F_{ν_i}, $(i = 1, 2, \dots, n)$, the marginal d.f.'s associated with the projections $\nu_1, \nu_2, \dots, \nu_n$ of ν. Then there exists a semicopula $S_\nu : [0,1]^n \to [0,1]$ such that*

$$\forall (x_1, \dots, x_n) \in \overline{\mathbb{R}}^n \qquad F_\nu(x_1, \dots, x_n) = S_\nu\left(F_{\nu_1}(x_1), \dots, F_{\nu_n}(x_n)\right).$$

The above result is a direct generalization of Sklar's Theorem to fuzzy measures; in fact, if ν is a probability measure, we obtain Thereom 1.9.1. Moreover, we stress the fact that as a copula links a joint d.f. to its margins so a semicopula joins the d.f. of a fuzzy measure to its one–dimensional marginal d.f.'s.

Chapter 3

2–increasing aggregation operators

The aim of this chapter is the study of the class of binary aggregation operators (agops, for short) satisfying the 2–increasing property, specifically, by recalling for sake of completeness the definitions already given, we are interested in the functions $A : [0,1]^2 \to [0,1]$ such that

- $A(0,0) = 0$ and $A(1,1) = 1$;
- $A(x,y) \le A(x',y')$ for $x \le x'$ and $y \le y'$;
- $V_A(R) \ge 0$ for every rectangle $R \subseteq [0,1]^2$.

One of the main reasons to study the class $\mathcal{A}_2$ of 2–increasing agops is that it contains, as a distinguished subclass, the restrictions to $[0,1]^2$ of all the bivariate distribution functions F such that $F(0,0) = 0$ and $F(1,1) = 1$; in particular copulas are in this class. On other hand, the 2–increasing property has a relevant connection with the theory of fuzzy measures, where it is also known as "supermodularity" (see [30]).

Notice that, we may limit ourselves to considering only 2–increasing agops because, if A is a 2–increasing agop, it is immediately seen that its dual A^d is 2–decreasing, and conversely. Therefore, analogous results for the 2–decreasing ones can be obtained by duality.

In section 3.1, we characterize some subclasses of 2–increasing agops and some construction methods are presented in section 3.2. Instead, section 3.3 presents the lattice structure of several subsets of $\mathcal{A}_2$. A method for generating a copula using 2–increasing agops is presented in section 3.4.

The results of this chapter are also contained in [38]

3.1 Characterizations of 2–increasing agops

In this section, some subclasses of agops satistying the 2–increasing property are characterized.

Proposition 3.1.1. *Let A be a 2–increasing agop. The following statements hold:*

(a) *the neutral element $e \in [0,1]$ of A, if it exists, is equal to 1;*

(b) *the annihilator $a \in [0,1]$ of A, if it exists, is equal to 0;*

(c) *if A is continuous on the border of $[0,1]^2$, then A is continuous on $[0,1]^2$.*

Proof. Let A be a 2–increasing agop.

If A has neutral element $e \in [0,1[$, then

$$A(1,1) + A(e,e) = 1 + A(e,e) \geq A(e,1) + A(1,e) = 1 + 1,$$

a contradiction. Therefore $e = 1$ (and, as a consequence, A is a copula).

If A has an annihilator $a \in [0,1]$, we assume, if possible, that $a > 0$. We have

$$A(a,a) - A(a,0) - A(0,a) + A(0,0) = -a \geq 0,$$

a contradiction; as a consequence, $a = 0$.

Let A be continuous on the border of $[0,1]^2$ and let (x_0, y_0) be a point in $]0,1[^2$ such that A is not continuous in (x_0, y_0). Suppose, without loss of generality, that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,1]$, $x_n \leq x_0$ for every $n \in \mathbb{N}$, which tends to x_0, and we have

$$\lim_{n\to+\infty} A(x_n, y_0) < A(x_0, y_0).$$

Therefore, there exists $\epsilon > 0$ and $n_0 \in \mathbb{N}$ such that $A(x_0, y_0) - A(x_n, y_n) > \epsilon$ for every $n \geq n_0$. But, because A is continuous on the border of the unit square, there exists $\overline{n} > n_0$ such that $A(x_0, 1) - A(x_{\overline{n}}, 1) < \epsilon$. But this violates the 2–increasing property, because, in this case,

$$V\left([x_{\overline{n}}, x_0] \times [y_0, 1]\right) < 0.$$

Thus the only possibility is that A is continuous on $[0,1]$. □

Remark 3.1.1. Note that, if $A : [0,1]^2 \to [0,1]$ is 2–increasing and has an annihilator element (which is necessarily equal to 0), then A is increasing in each place. In fact, because of the 2–increasing property, for every x_1, x_2 and y in $[0,1]$, $x_1 \leq x_2$, we have

$$A(x_2, y) - A(x_1, y) \geq A(x_2, 0) - A(x_1, 0) = 0.$$

But, in general, if $A : [0,1]^2 \to [0,1]$ is 2–increasing, then A need not be increasing in each place. Consider, for example, $A(x,y) = (2x-1)(2y-1)$.

Proposition 3.1.2. *Let M_f be a quasi–arithmetic mean, viz. let a continuous strictly monotone function $f : [0,1] \to \mathbb{R}$ exist such that*

$$M_f(x,y) := f^{-1}\left(\frac{f(x)+f(y)}{2}\right).$$

Then M_f is 2–increasing if, and only if, f^{-1} is convex.

Proof. Let s and t be real numbers and set $a := f^{-1}(s)$ and $b := f^{-1}(t)$. If M_f is 2–increasing, we have, because M_f is also commutative,

$$M_f(a,a) + M_f(b,b) \geq 2\, M_f(a,b),$$

which is equivalent to

$$f^{-1}(s) + f^{-1}(t) \geq 2\, f^{-1}\left(\frac{s+t}{2}\right).$$

This shows that f^{-1} is Jensen–convex and hence convex.

Conversely, let f^{-1} be convex; we have to prove that, whenever $x_1 \leq x_2$ and $y_1 \leq y_2$,

$$M_f(x_1,y_1) + M_f(x_2,y_2) \geq M_f(x_2,y_1) + M_f(x_1,y_2),$$

or, equivalently, that

$$f^{-1}(s_1) + f^{-1}(s_4) \geq f^{-1}(s_2) + f^{-1}(s_3),$$

where

$$s_1 := \frac{f(x_1)+f(y_1)}{2}, \qquad s_4 := \frac{f(x_2)+f(y_2)}{2},$$
$$s_2 := \frac{f(x_2)+f(y_1)}{2}, \qquad s_3 := \frac{f(x_1)+f(y_2)}{2}.$$

Assume now that f is (strictly) increasing; setting

$$\alpha := \frac{s_4 - s_2}{s_4 - s_1},$$

we obtain $\alpha \in [0,1]$ and

$$s_2 = \alpha\, s_1 + (1-\alpha)\, s_4, \qquad s_3 = (1-\alpha)\, s_1 + \alpha\, s_4.$$

Because f^{-1} is convex, we have

$$f^{-1}(s_2) + f^{-1}(s_3) \leq f^{-1}(s_1) + f^{-1}(s_4),$$

namely the assertion.

If, on the other hand, f is (strictly) decreasing, then we set

$$\alpha := \frac{s_1 - s_2}{s_1 - s_4}$$

in order to reach the same conclusion. □

Corollary 3.1.1. *If M_f is a 2–increasing quasi–arithmetic mean generated by f, then*

$$M_f(x,y) \le \frac{x+y}{2} \quad \textit{for every } (x,y) \in [0,1]^2.$$

Proof. In view of Proposition 3.1.2, M_f is 2–increasing if, and only if, f^{-1} is convex. But, if f is increasing, so is f^{-1}, and $M_f(x,y) \le \frac{x+y}{2}$ is equivalent to the fact that f is Jensen–concave and, thus, f^{-1} convex. Instead, if f is decreasing, so is f^{-1}, and $M_f(x,y) \le \frac{x+y}{2}$ is equivalent to the fact that f is Jensen–convex and, thus, f^{-1} convex. □

Proposition 3.1.3. *The Choquet integral–based agop, defined for a and b in $[0,1]$ by*

$$A_{Ch}(x,y) = \begin{cases} (1-b)x + by, & \textit{if } x \le y, \\ ax + (1-a)y, & \textit{if } x > y, \end{cases}$$

is 2–increasing if, and only if, $a + b \le 1$.

Proof. It is easily proved that A_{Ch} is 2–increasing on every rectangle contained either in Δ_+ or in Δ_-. Now, let $R := [s,t]^2$. Then, for all s and t such that $0 \le s < t \le 1$,

$$V_{A_{Ch}}([s,t]^2) = s + t - [(1-b)s + bt] - [at + (1-a)s] \ge 0$$

if, and only if, $a+b \le 1$. Now, the assertion follows directly from Proposition 1.6.1. □

Notice that, if $a+b=1$, A_{Ch} is the weighted arithmetic mean; and, if $a = b \le 1/2$, we have an OWA operator, $A_{Ch}(x,y) = (1-a)\min\{x,y\} + a\max\{x,y\}$ (see [159]).

Remark 3.1.2. The above proposition can be also proved by using some known results on fuzzy measures. In fact, following [30], it is known that a Choquet integral operator based on a fuzzy measure m is supermodular if, and only if, the fuzzy measure m is supermodular. But, in the case of 2 inputs, say $\mathbb{X}_2 := \{1,2\}$, we can define a fuzzy measure m on $2^{\mathbb{X}_2}$ by giving the values $m(\{1\}) = a$ and $m(\{2\}) = b$, where a and b are in $[0,1]$. Moreover, it is also known that m is supermodular if, and only if, $a + b \le 1$.

A special subclass of 2–increasing agops is that formed by modular agops, i.e. those A for which $V_A(R) = 0$ for every rectangle $R \subseteq [0,1]^2$. For these operators the following characterization holds.

Proposition 3.1.4. *For an agop A the following statements are equivalent:*

(a) *A is modular;*

(b) *increasing functions f and g from $[0,1]$ into $[0,1]$ exist such that $f(0) = g(0) = 0$, $f(1) + g(1) = 1$, and*

$$A(x,y) = f(x) + g(y). \tag{3.1}$$

Proof. If A is modular, set $f(x) := A(x, 0)$ and $g(y) := A(0, y)$. From the modularity of A

$$0 = V_A([0, x] \times [0, y]) = A(x, y) - f(x) - g(y) + A(0, 0),$$

which implies (b). Viceversa, it is clear that every function of type (3.1) is modular. □

3.2 Construction of 2–increasing agops

In the literature, there are a variety of construction methods for agops (see [10] and the references therein). In this section, some of these methods are used to obtain an agop satisfying the 2–increasing property.

Proposition 3.2.1. *Let f and g be increasing functions from $[0, 1]$ into $[0, 1]$ such that $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Let A be a 2–increasing agop. Then, the function defined by*

$$A_{f,g}(x, y) := A(f(x), g(y)) \tag{3.2}$$

is a 2–increasing agop.

Proof. It is obvious that $A_{f,g}(0, 0) = 0$, $A_{f,g}(1, 1) = 1$ and $A_{f,g}$ is increasing in each place, since it is the composition of increasing functions. Moreover, given a rectangle $R = [x_1, x_2] \times [y_1, y_2]$, we obtain

$$V_{A_{f,g}}(R) = V_A([f(x_1), f(x_2)] \times [g(y_1), g(y_2)]) \geq 0,$$

which is the desired assertion. □

Example 3.2.1. Let f and g be increasing functions from $[0, 1]$ into $[0, 1]$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Then

$$A_{f,g}(x, y) := f(x) \wedge g(y), \qquad B_{f,g}(x, y) := f(x) \cdot g(y),$$
$$C_{f,g}(x, y) := \max\{f(x) + g(y) - 1, 0\}.$$

are 2–increasing agops as a consequence of the previous proposition by taking, respectively, $A = M$, $B = \Pi$ and $C = W$.

Corollary 3.2.1. *The following statements are equivalent:*

(a) *H is the restriction to the unit square $[0, 1]^2$ of a bivariate d.f. on $[0, 1]^2$ with $H(0, 0) = 0$ and $H(1, 1) = 1$;*

(b) *there exist a copula C and increasing and left continuous functions f and g from $[0, 1]$ into $[0, 1]$, $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$, such that $H(x, y) := C(f(x), g(y))$.*

Proof. It is a direct consequence of Sklar's Theorem 1.6.1. □

Corollary 3.2.2. *If A is a 2–increasing and continuous agop with annihilator element 0, then there exist two increasing functions f and g from $[0,1]$ into $[0,1]$, $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$, such that $A_{f,g}$ defined by (3.2) is a copula.*

Proof. Let f and g be the functions given by

$$f(x) := \sup\{t \in [0,1] : A(t,1) = x\},$$
$$g(y) := \sup\{t \in [0,1] : A(1,t) = y\}.$$

Then f and g satisfy the assumptions of Proposition 3.2.1 and, hence, $A_{f,g}$ is 2–increasing. Moreover, it is easily proved that 1 is the neutral element of $A_{f,g}$ and, thus, $A_{f,g}$ is a copula. □

Example 3.2.2. Let B and C be copulas and consider the function $A(x,y) = B(x,y) \cdot C(x,y)$. As we will show in the sequel (see chapter 8), A is a continuous 2–increasing agop with annihilator 0. Moreover, we have

$$f(x) = g(x) = \sup\{t \in [0,1] : A(t,1) = x\} = \sqrt{x}.$$

Therefore, in view of Corollary 3.2.2 the function

$$A_{f,g}(x,y) = A(f(x), g(y)) = B(\sqrt{x}, \sqrt{y}) \cdot C(\sqrt{x}, \sqrt{y})$$

is a copula.

Proposition 3.2.2. *Let f be an increasing and convex function from $[0,2]$ into $[0,1]$ such that $f(0) = 0$ and $f(2) = 1$. Then the function*

$$A_f(x,y) := f(x+y) \tag{3.3}$$

is a 2–increasing agop.

Proof. It is obvious that $A_f(0,0) = 0$, $A_f(1,1) = 1$ and A_f is increasing in each place. Moreover, given a rectangle $R = [x_1, x_2] \times [y_1, y_2]$, we obtain

$$V_{A_f}(R) = f(x_2 + y_2) + f(x_1 + y_1) - f(x_2 + y_1) - f(x_1 + y_2).$$

By using an argument similar to the proof of Proposition 3.1.2, the convexity of f implies that $V_{A_f}(R) \geq 0$. □

Notice that the agop A_f given in (3.3) is Schur–constant.

Example 3.2.3. Consider the function $f : [0,2] \to [0,1]$, given for every $t \in [0,2]$ by $f(t) := \max\{t-1, 0\}$. Then the function A_f defined by (3.3) is W.

Sometimes, it is useful to construct an agop with specified values on its diagonal, horizontal or vertical section (see, for example, [91, 81]). Specifically, given a suitable function f, the problem is whether there is a 2–increasing agop with (diagonal, horizontal or vertical) section equal to f.

Proposition 3.2.3. *Let h, v and δ be increasing functions from $[0,1]$ into $[0,1]$, $\delta(0) = 0$ and $\delta(1) = 1$. The following statements hold:*

- *$A_\delta(x, y) = \delta(x)$ is a 2–increasing agop with diagonal section is δ;*
- *a 2–increasing agop with horizontal section at $b \in]0,1[$ equal to h is given by*

$$A_h(x,y) = \begin{cases} 1, & \textit{if } y = 1; \\ 0, & \textit{if } y = 0; \\ h(x), & \textit{otherwise}; \end{cases}$$

- *a 2–increasing agop with vertical section at $a \in]0,1[$ equal to v is given by*

$$A_v(x,y) = \begin{cases} 1, & \textit{if } x = 1; \\ 0, & \textit{if } x = 0; \\ v(y), & \textit{otherwise.} \end{cases}$$

Proof. The proof is a consequence of Proposition 3.1.4 because A_δ, A_h and A_v are all modular agops. ☐

In [107] (see also [10]), an ordinal sum construction for agops is given. Here, we modify that method in order to ensure that an ordinal sum of 2–increasing agops is again 2–increasing.

Consider a partition of the unit interval $[0,1]$ by the points $0 = a_0 < a_1 < \cdots < a_n = 1$ and let A_1, A_2,…,A_n be 2–increasing agops. For every $i \in \{1, 2, \dots, n\}$, consider the function $\widetilde{A}_i$ defined on the square $[a_i, a_{i+1}]^2$ by

$$\widetilde{A}_i(x,y) = a_i + (a_{i+1} - a_i) A_i \left(\frac{x - a_i}{a_{i+1} - a_i}, \frac{y - a_i}{a_{i+1} - a_i} \right).$$

Then we can easily prove that $\widetilde{A}_i$ is 2–increasing on $[a_i, a_{i+1}]^2$. Now, define, for every point (x, y) such that $a_i \le \min\{x, y\} < a_{i+1}$,

$$A_{1,n}(x,y) := \widetilde{A}_i \left(\min\{x, a_{i+1}\}, \min\{y, a_{i+1}\} \right) \tag{3.4}$$

(and $A_{1,n}(1,1) = 1$ by definition). Therefore, it is not difficult to prove that $A_{1,n}$ is also a 2–increasing agop, called the *ordinal sum* of the agops $\{A_i\}_{i=1,2,\dots,n}$; we write

$$A_{1,n} = (\langle a_i, A_i \rangle)_{i=1,2,\dots,n}.$$

Example 3.2.4. Consider a partition of $[0,1]$ by means of the points $0 = a_0 < a_1 < \cdots < a_n = 1$. Let $A_1, A_2, \dots, A_n$ be 2–increasing agops such that, for every index i, $A_i = A_S$, the smallest agop. Let $A_{1,n}$ be the ordinal sum $(\langle a_i, a_{i+1}, A_i\rangle)_{i=1,2,\dots,n}$. For every point (x,y) such that $a_i \le \min\{x,y\} < a_{i+1}$, $A_{1,n}(x,y) = a_i$. Note that $A_{1,n}$ is the smallest agop with idempotent elements $a_0, a_1, \dots, a_n$.

3.3 Bounds on sets of 2–increasing agops

Given a (2–increasing) agop A, it is obvious that

$$A_S(x,y) \le A(x,y) \quad \text{for every } (x,y) \text{ in } [0,1],$$

where A_S is the smallest agop defined in section 1.11. Because A_S is 2–increasing, it is also the best–possible lower bound in the set $\mathcal{A}_2$, because it is 2–increasing.

The best–possible upper bound in $\mathcal{A}_2$ is the greatest agop A_G. Notice that A_G is not 2–increasing, e.g. $V_{A_G}([0,1]^2) = -1$, but it is the pointwise limit of the sequence $\{A_n\}_{n\in\mathbb{N}}$ of 2–increasing agops, defined by

$$A_n(x,y) = \begin{cases} 1, & \text{if } (x,y) \in [1/n, 1]^2; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $(\mathcal{A}, \le)$ is not a complete lattice. But, the following result holds.

Proposition 3.3.1. *Every agop is the supremum of a suitable subset of $\mathcal{A}_2$.*

Proof. Let A be an agop; we may (and, in fact do) suppose that $A \ne A_G$, since this case has already been considered, and that A is not 2–increasing, this case being trivial. For every (x_0, y_0) in $[0,1]$, let $z_0 = A(x_0, y_0)$ and consider the following 2–increasing agop

$$\widehat{A}_{x_0,y_0} := \begin{cases} 1, & \text{if } (x,y) = (1,1); \\ z_0, & \text{if } (x,y) \in [x_0,1] \times [y_0,1] \setminus \{(1,1)\}; \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$A(x,y) = \sup\{\widehat{A}_{x_0,y_0} : (x_0,y_0) \in [0,1]^2\}.$$

□

The lattice structure of the class of copulas was considered in [123]. Here, other cases will be considered. The following result, for instance, gives the bounds on the subsets of 2–increasing agops with the same margins.

Proposition 3.3.2. *Let A be a 2–increasing agop with margins h_0, h_1, v_0 and v_1. Let*

$$A_*(x,y) := \max\{h_0(x)+v_0(y), h_1(x)+v_1(y)-1\} \tag{3.5}$$

and

$$A^*(x,y) := \min\{h_1(x)+v_0(y)-A(0,1), h_0(x)+v_1(y)-A(1,0)\}. \tag{3.6}$$

Then, for every (x,y) in $[0,1]$,

$$A_*(x,y) \le A(x,y) \le A^*(x,y). \tag{3.7}$$

Proof. Let A be a 2–increasing agop. Let (x,y) be a point in $]0,1[^2$. In view of the 2–increasing property, we have

$$A(x,y) \ge A(x,0)+A(0,y) = h_0(x)+v_0(y),$$
$$A(x,y) \ge A(x,1)+A(1,y)-1 = h_1(x)+v_1(y)-1,$$

which together yield the first of the inequalities (3.7). Analogously,

$$A(x,y) \le A(0,y)+A(x,1)-A(0,1) = h_1(x)+v_0(y)-A(0,1),$$
$$A(x,y) \le A(x,0)+A(1,y)-A(1,0) = h_0(x)+v_1(y)-A(1,0),$$

namely the second of the inequalities (3.7). □

It should be noticed that, in the special case of copulas, the bounds of (3.7) coincide with the usual Fréchet–Hoeffding bounds (1.13).

The subclasses of 2–increasing agops with prescribed margins have the smallest and the greatest element (in the pointwise ordering), as stated here.

Theorem 3.3.1. *For every 2–increasing agop A, the bounds A_* and A^* defined by (3.5) and (3.6) are 2–increasing agops.*

Proof. The functions A_* and A^* defined by (3.5) and (3.6), respectively, are obviously agops. Below we shall prove that they are also 2–increasing. To this end, let $R = [x,x'] \times [y,y']$ be any rectangle contained in the unit square.

Consider, first, the case of A^*. Then

$$A^*(x',y') := \min\{h_1(x')+v_0(y')-A(0,1), h_0(x')+v_1(y')-A(1,0)\},$$
$$A^*(x,y) := \min\{h_1(x)+v_0(y)-A(0,1), h_0(x)+v_1(y)-A(1,0)\},$$
$$A^*(x',y) := \min\{h_1(x')+v_0(y)-A(0,1), h_0(x')+v_1(y)-A(1,0)\},$$
$$A^*(x,y') := \min\{h_1(x)+v_0(y')-A(0,1), h_0(x)+v_1(y')-A(1,0)\}.$$

There are four cases to be considered.
Case 1. If

$$A^*(x',y') = h_1(x')+v_0(y')-A(0,1),\ A^*(x,y) = h_1(x)+v_0(y)-A(0,1),$$

then

$$A^*(x',y') + A^*(x,y) = h_1(x') + v_0(y) - A(0,1)$$
$$+ h_1(x) + v_0(y') - A(0,1) \geq A^*(x',y) + A^*(x,y').$$

Case 2. If

$$A^*(x',y') = h_0(x') + v_1(y') - A(1,0), \; A^*(x,y) = h_0(x) + v_1(y) - A(1,0),$$

then

$$A^*(x',y') + A^*(x',y') = h_0(x') + v_1(y) - A(1,0)$$
$$+ h_0(x) + v_1(y') - A(1,0) \geq A^*(x',y) + A^*(x,y').$$

Case 3. If

$$A^*(x',y') = h_1(x') + v_0(y') - A(0,1), \; A^*(x,y) = h_0(x) + v_1(y) - A(1,0),$$

then, since A is 2–increasing, we have $h_1(x') + h_0(x) \geq h_1(x) + h_0(x')$, so that

$$\begin{aligned} A^*(x',y') &+ A^*(x',y') \\ &= h_1(x') + h_0(x) - A(0,1) + v_0(y') + v_1(y) - A(1,0) \\ &\geq h_1(x) + v_0(y') - A(0,1) + h_0(x') + v_1(y) - A(0,1) \\ &\geq A^*(x',y) + A^*(x,y'). \end{aligned}$$

Case 4. If

$$A^*(x',y') = h_0(x') + v_1(y') - A(1,0), \; A^*(x,y) = h_1(x) + v_0(y) - A(0,1),$$

then, since A is 2–increasing, we have $v_1(y') + v_0(y) \geq v_1(y) + v_0(y')$, so that

$$\begin{aligned} A^*(x',y') &+ A^*(x',y') \\ &= h_0(x') + v_1(y') - A(1,0) + h_1(x) + v_0(y) - A(0,1) \\ &\geq h_0(x') + v_1(y) - A(1,0) + h_1(x) + v_0(y') - A(0,1) \\ &\geq A^*(x',y) + A^*(x,y'). \end{aligned}$$

This proves that A^* is 2–increasing.

A similar proof holds for A_*. Given a rectangle $R = [x,x'] \times [y,y']$ in the unit square, we have

$$\begin{aligned} A_*(x',y') &:= \max\{h_0(x') + v_0(y'), h_1(x') + v_1(y') - 1\}, \\ A_*(x,y) &:= \max\{h_0(x) + v_0(y), h_1(x) + v_1(y) - 1\}, \\ A_*(x',y) &:= \max\{h_0(x') + v_0(y), h_1(x') + v_1(y) - 1\}, \\ A_*(x,y') &:= \max\{h_0(x) + v_0(y'), h_1(x) + v_1(y') - 1\}. \end{aligned}$$

Here, again, four cases will be considered.
Case 1. If

$$A_*(x',y) = h_0(x') + v_0(y), \quad A_*(x,y') = h_0(x) + v_0(y'),$$

then

$$\begin{aligned} A_*(x',y) + A_*(x,y') &= h_0(x) + v_0(y) + h_0(x') + v_0(y') \\ &\leq A_*(x',y') + A_*(x,y). \end{aligned}$$

Case 2. If

$$A_*(x',y) = h_0(x') + v_0(y), \quad A_*(x,y') = h_1(x) + v_1(y') - 1,$$

then, since A is 2–increasing, we have $h_0(x') + h_1(x) \leq h_1(x') + h_0(x)$ so that

$$\begin{aligned} A_*(x',y) + A_*(x,y') &= h_0(x') + v_0(y) + h_1(x) + v_1(y') - 1 \\ &\leq h_1(x') + v_1(y') - 1 + h_0(x) + v_0(y) \\ &\leq A_*(x',y') + A_*(x,y). \end{aligned}$$

Case 3. If

$$A_*(x',y) = h_1(x') + v_1(y) - 1, \quad A_*(x,y') = h_0(x) + v_0(y'),$$

then, since A is 2–increasing, we have $v_1(y) + v_0(y') \leq v_1(y') + v_0(y)$, so that

$$\begin{aligned} A_*(x',y) + A_*(x,y') &= h_1(x') + v_1(y) - 1 + h_0(x) + v_0(y') \\ &\leq h_1(x') + v_1(y') - 1 + h_0(x) + v_0(y) \\ &\leq A_*(x',y') + A_*(x,y). \end{aligned}$$

Case 4. If

$$A_*(x',y) = h_1(x') + v_1(y) - 1, \; A_*(x,y') = h_1(x) + v_1(y') - 1,$$

then

$$\begin{aligned} A_*(x',y) + A_*(x,y') &= h_1(x') + v_1(y') - 1 + h_1(x) + v_1(y) - 1 \\ &\leq A_*(x',y') + A_*(x,y). \end{aligned}$$ □

The following result gives a necessary and sufficient condition that ensures $A_* = A^*$ in the case of a symmetric agop A.

Proposition 3.3.3. *For a symmetric and 2–increasing agop A, the following statements are equivalent:*

(a) $A_* = A^*$;

(b) *there exists an interval* $I \subseteq [0,1]$, $0 \in I$, *and* $a \in [0,1]$ *such that*

$$h_1(t) = \begin{cases} h_0(t) + a, & \text{if } t \in I, \\ h_0(t) + (1-a), & \text{if } t \in [0,1] \setminus I. \end{cases} \tag{3.8}$$

Proof. If A is a symmetric agop, then $h_0 = v_0$ and $h_1 = v_1$. Set $a := A(0,1) = A(1,0)$, $a \leq 1/2$. Therefore

$$A_*(x,y) := \max\{h_0(x) + h_0(y), h_1(x) + h_1(y) - 1\}$$

and

$$A^*(x,y) := \min\{h_1(x) + h_0(y) - a, h_0(x) + h_1(y) - a\}.$$

If $A = A^*$, then $A(x,x) = h_1(x) + h_0(x) - a$. Now, from $A = A_*$, we obtain that either $A(x,x) = 2h_0(x)$ or $A(x,x) = 2h_1(x) - 1$. Therefore, either

$$h_1(x) - h_0(x) = a, \tag{3.9}$$

or

$$h_1(x) - h_0(x) = 1 - a. \tag{3.10}$$

If $a = 1/2$, then $h_1(x) = h_0(x) + a$ on $[0,1]$. Otherwise, note that (3.9) holds at the point $x = 0$ and (3.10) holds at the point $x = 1$. Moreover, if (3.9) does not hold at a point x_1, then (3.9) does not hold also for every $x_2 > x_1$. In fact, for the 2–increasing property, we obtain

$$h_1(x_2) - h_0(x_2) \geq h_1(x_1) - h_0(x_1) = 1 - a > 1/2.$$

Thus h_1 has the form (3.8), where I is an interval. The converse is just a matter of straightforward verification. □

Note that if $A = A^* = A_*$, then $A = 2aB + (1-2a)C$, where B is a symmetric and modular agop, and $C = 1_{I^2}$ is the indicator function of the set I^2.

Example 3.3.1. Consider the arithmetic mean $A(x,y) := (x+y)/2$, which is obviously 2–increasing. Then, we easily evaluate $A_* = A^* = A$.

Consider the 2–increasing agop given by the geometric mean $G(x,y) := \sqrt{xy}$. We have

$$G_*(x,y) = \max\{0, \sqrt{x} + \sqrt{y} - 1\} \qquad \text{and} \qquad G^*(x,y) = \min\{\sqrt{x}, \sqrt{y}\},$$

both of which are 2–increasing.

Remark 3.3.1. In the general case of a 2–increasing agop A such that $A = A_* = A^*$, as above it can be proved that one among the following four equalities holds:

- $h_1(x) - h_0(x) = A(0,1)$;

- $h_1(x) - h_0(x) = 1 - A(1,0)$;
- $v_1(y) - v_0(y) = 1 - A(0,1)$;
- $v_1(y) - v_0(y) = A(1,0)$.

However, one need not have explicit conditions as in the symmetric case for $h_1(x) - h_0(x)$ and $v_1(y) - v_0(y)$.

Let h, v and δ be increasing functions from $[0,1]$ into $[0,1]$, $\delta(0) = 0$ and $\delta(1) = 1$. Denote by $\mathcal{A}_h$, $\mathcal{A}_v$ and $\mathcal{A}_\delta$, respectively, the subclasses of 2–increasing agops with horizontal section at $b \in]0,1[$ equal to h, vertical section at $a \in]0,1[$ equal to v, diagonal section equal δ, respectively. Notice that the sets $\mathcal{A}_h$, $\mathcal{A}_v$ and $\mathcal{A}_\delta$ are not empty, in view of Proposition 3.2.3. The following results give the best–possible bounds in these subclasses.

Proposition 3.3.4. *Let $h : [0,1] \to [0,1]$ be an increasing function. For every A in $\mathcal{A}_h$ we obtain*

$$(A_h)_* \leq A(x,y) \leq (A_h)^*, \tag{3.11}$$

where

$$(A_h)_*(x,y) := \begin{cases} 1, & \text{if } (x,y) = (1,1); \\ 0, & \text{if } 0 \leq y < b; \\ h(x), & \text{otherwise}; \end{cases}$$

$$(A_h)^*(x,y) := \begin{cases} 0, & \text{if } (x,y) = (0,0); \\ 1, & \text{if } b < y \leq 1; \\ h(x), & \text{otherwise}. \end{cases}$$

Moreover,

$$(A_h)_*(x,y) = \bigwedge_{A \in \mathcal{A}_h} A(x,y) \quad \text{and} \quad (A_h)^*(x,y) = \bigvee_{A \in \mathcal{A}_h} A(x,y),$$

where $(A_h)_*$ is a 2–increasing agop and $(A_h)^*$, while it is still an agop, is not necessarily 2–increasing.

Proof. For all $(x,y) \in [0,1]^2$ and $A \in \mathcal{A}_h$, $A(x,y) \geq 0$ for every $y \in [0,b[$ and $A(x,y) \geq h(x)$ for every $y \in [b,1]$, viz. $A(x,y) \geq (A_h)_*(x,y)$ on $[0,1]^2$. Analogously, $A(x,y) \leq h(x)$ for every $y \in [0,b]$ and $A(x,y) \leq 1$ for every $y \in]b,1]$, viz. $A(x,y) \leq (A_h)^*(x,y)$ on $[0,1]^2$. Both $(A_h)_*$ and $(A_h)^*$ are agops, as is immediately seen; it is also immediate to check that $(A_h)_*$ is 2–increasing and, therefore, that $(A_h)_* = \bigwedge_{A \in \mathcal{A}_h} A$. Now, suppose that B is any agop greater than, or at least equal to,

$\bigvee_{A\in\mathcal{A}_h} A$. Then $B(x,y) \geq A_1(x,y)$, where A_1 is the 2–increasing agop given by

$$A_1(x,y) := \begin{cases} 0, & \text{if } y = 0; \\ h(x), & \text{if } 0 < y \leq b; \\ 1, & \text{if } b < y \leq 1; \end{cases}$$

and $B(x,y) \geq A_2(x,y)$, where A_2 is the 2–increasing agop given by

$$A_2(x,y) := \begin{cases} 0, & \text{if } x = 0; \\ h(x), & \text{if } x \neq 0 \text{ and } 0 < y \leq b; \\ 1, & \text{if } x \neq 0 \text{ and } b < y \leq 1; \end{cases}$$

therefore $B(x,y) \geq \max\{A_1(x,y), A_2(x,y)\} = (A_h)^*(x,y)$ on $[0,1]^2$ and we obtain $(A_h)^* = \bigvee_{A\in\mathcal{A}_h} A$. However $(A_h)^*$ need not be 2–increasing; in fact,

$$V_{(A_h)^*}\left([0,1]\times[b,1]\right) = h(0) - h(1),$$

and thus $(A_h)^*$ is 2–increasing if, and only if, $h = 0$. □

Analogously, we prove the following result for the class $\mathcal{A}_v$.

Proposition 3.3.5. *Let $v : [0,1] \to [0,1]$ be an increasing function. For every A in $\mathcal{A}_v$ we obtain*

$$(A_v)_* \leq A(x,y) \leq (A_v)^*, \tag{3.12}$$

where

$$(A_v)_*(x,y) := \begin{cases} 1, & \textit{if } (x,y) = (1,1); \\ 0, & \textit{if } 0 \leq x < a; \\ v(y), & \textit{otherwise}; \end{cases}$$

$$(A_v)^*(x,y) := \begin{cases} 0, & \textit{if } (x,y) = (0,0); \\ 1, & \textit{if } a < x \leq 1; \\ v(y), & \textit{otherwise}. \end{cases}$$

Moreover,

$$(A_v)_*(x,y) = \bigwedge_{A\in\mathcal{A}_v} A(x,y) \quad \textit{and} \quad (A_v)^*(x,y) = \bigvee_{A\in\mathcal{A}_v} A(x,y),$$

where $(A_v)_*$ is a 2–increasing agop and $(A_v)^*$, while it is still an agop, is not necessarily 2–increasing.

Proposition 3.3.6. *Let δ be an increasing function with $\delta(0) = 0$ and $\delta(1) = 1$. For every A in $\mathcal{A}_\delta$, we obtain*

$$(A_\delta)_* := \min\{\delta(x), \delta(y)\} \leq A(x,y) \leq (A_\delta)^* := \max\{\delta(x), \delta(y)\}. \tag{3.13}$$

Moreover, $(A_\delta)_*$ and $(A_\delta)^*$ are the best–possible bounds, in the sense that

$$(A_\delta)_*(x,y) = \bigwedge_{A \in \mathcal{A}_\delta} A(x,y) \quad \text{and} \quad (A_\delta)^*(x,y) = \bigvee_{A \in \mathcal{A}_\delta} A(x,y),$$

where $(A_\delta)_*$ is a 2–increasing agop and $(A_\delta)^*$, while it is still an agop, is never 2–increasing.

Proof. For all $(x,y) \in [0,1]^2$ and $A \in \mathcal{A}_\delta$,

$$A(x,y) \geq A(x \wedge y, x \wedge y) = \min\{\delta(x), \delta(y)\}$$

and

$$A(x,y) \leq A(x \vee y, x \vee y) = \max\{\delta(x), \delta(y)\}.$$

This proves (3.13). Both $(A_\delta)_*$ and $(A_\delta)^*$ are agops, as is immediately seen; it is also immediate to check that $(A_\delta)_*$ is 2–increasing (because of Proposition 3.2.1) and, therefore, that $(A_\delta)_* = \bigwedge_{A \in \mathcal{A}_\delta} A$. Now, suppose that B is any agop greater than, or at least equal to, $\bigvee_{A \in \mathcal{A}_\delta} A$. Then $B(x,y) \geq A_1(x,y) := \delta(x)$ and $B(x,y) \geq A_2(x,y) := \delta(y)$, where A_1 and A_2 are 2–increasing agops. Thus, $B(x,y) \geq (A_\delta)^*$ so that $(A_\delta)^* = \bigvee_{A \in \mathcal{A}_\delta} A$. This proves that $(A_\delta)^*$ is the best possible upper bound for the set $\mathcal{A}_\delta$. However $(A_\delta)^*$ is never 2–increasing, in fact

$$V_{(A_\delta)^*}\left([0,1]^2\right) = \delta(0) - \delta(1) = -1 < 0. \qquad \square$$

Corollary 3.3.1. *Let δ be an increasing function with $\delta(0) = 0$ and $\delta(1) = 1$. For every symmetric agop A in $\mathcal{A}_\delta$, we obtain*

$$(A_\delta)_* := \min\{\delta(x), \delta(y)\} \leq A(x,y) \leq \frac{\delta(x) + \delta(y)}{2},$$

where $(\delta(x) + \delta(y))/2$ is the maximal element in the subclass of the symmetric agops in $\mathcal{A}_2$.

Proof. If A is symmetric and 2–increasing, we have, for every x, y in $[0,1]$,

$$\delta(x) + \delta(y) = A(x,x) + A(y,y) \geq 2\ A(x,y). \qquad \square$$

3.4 A construction method for copulas

The main result of this section is to give a simple method of constructing a copula from a 2–increasing and 1–Lipschitz agop.

Theorem 3.4.1. *For every 2–increasing and 1–Lipschitz agop A, the function*

$$C(x,y) := \min\{x, y, A(x,y)\}$$

is a copula.

Proof. First, in order to prove that C is a copula, we note that C has neutral element 1 and annihilator 0; in fact, for every $x \in [0,1]$, we have

$$|A(1,1) - A(x,1)| \leq 1 - x$$

and thus $A(x,1) \geq x$. Consequently, we have

$$C(x,1) = \min\{A(x,1), x\} = x, \quad C(x,0) = \min\{A(x,0), 0\} = 0,$$

and, similarly, $C(1,x) = x$ and $C(0,x) = 0$. Then, we prove that C is 2–increasing by using Proposition 1.6.1.

For every rectangle $R := [s,t] \times [s,t]$ on $[0,1]^2$, set

$$V_C(R) = \min\{A(s,s), s\} + \min\{A(t,t), t\} - \min\{A(s,t), s\} - \min\{A(t,s), s\}.$$

We have to prove that $V_C(R) \geq 0$ and several cases are considered.

If $A(s,s) \geq s$, then also $A(s,t)$, $A(t,s)$ and $A(t,t)$ are greater than s, because A is increasing in each variable, and thus

$$V_C(R) = \min\{A(t,t), t\} - s \geq 0.$$

If $A(s,s) < s$, then we distinguish:

- if $A(t,t) < t$, since A is 2–increasing, we have

$$A(s,s) + A(t,t) \geq A(s,t) + A(t,s) \geq \min\{A(s,t), s\} + \min\{A(t,s), s\},$$

  viz. $V_C(R) \geq 0$;

- if $A(t,t) \geq t$, since A is 1–Lipschitz, we have

$$\min\{A(t,s), s\} - \min\{A(s,s), s\} \leq t - s \leq t - \min\{A(t,s), s\},$$

  and thus $V_C(R) \geq 0$.

Now, let $R = [x_1, x_2] \times [y_1, y_2]$ be a rectangle contained in Δ_+. Then $V_C(R)$ is given by

$$\begin{aligned} V_C(R) = &\min\{A(x_1,y_1), y_1\} + \min\{A(x_2,y_2), y_2\} \\ &- \min\{A(x_2,y_1), y_1\} - \min\{A(x_1,y_2), y_2\}. \end{aligned}$$

If $A(x_1,y_1) \geq y_1$, then also $A(x_2,y_1)$, $A(x_1,y_2)$ and $A(x_2,y_2)$ are greater than y_1, because A is increasing in each variable, and thus

$$V_C(R) = \min\{A(x_2,y_2), y_2\} - y_1 \geq 0.$$

If $A(x_1,y_1) < y_1$, then we distinguish:

- if $A(x_2, y_2) < y_2$, since A is 2–increasing, we have

$$\begin{aligned} A(x_2, y_2) + A(x_1, y_1) &\geq A(x_2, y_1) + A(x_1, y_2) \\ &\geq \min\{A(x_2, y_1), y_1\} + \min\{A(x_1, y_2), y_2\}, \end{aligned}$$

  viz. $V_C(R) \geq 0$;

- if $A(x_2, y_2) \geq y_2$, we have

$$V_C(R) = A(x_1, y_1) + y_2 - A(x_1, y_2) - \min\{A(x_2, y_1), y_1\},$$

  and, since A is 1–Lipschitz,

$$A(x_1, y_2) \leq y_2 - y_1 + A(x_1, y_1) \leq y_2,$$

  moreover, from the fact that

$$A(x_1, y_2) - A(x_1, y_1) \leq y_2 - y_1 \leq y_2 - \min\{A(x_2, y_1), y_1\},$$

  it follows that $V_C(R) \geq 0$.

Finally, let $R = [x_1, x_2] \times [y_1, y_2]$ be a rectangle contained in Δ_-. Then $V_C(R)$ is given by

$$\begin{aligned} V_C(R) = &\min\{A(x_1, y_1), x_1\} + \min\{A(x_2, y_2), x_2\} \\ &- \min\{A(x_2, y_1), x_2\} - \min\{A(x_1, y_2), x_1\}. \end{aligned}$$

If $A(x_1, y_1) \geq x_1$, then, because A is increasing in each variable,

$$V_C(R) = \min\{A(x_2, y_2), x_2\} - x_1 \geq 0.$$

If $A(x_1, y_1) < x_1$, then we distinguish:

- if $A(x_2, y_2) < x_2$, since A is 2–increasing, we have

$$\begin{aligned} A(x_2, y_2) + A(x_1, y_1) &\geq A(x_2, y_1) + A(x_1, y_2) \\ &\geq \min\{A(x_2, y_1), x_1\} + \min\{A(x_1, y_2), x_2\}, \end{aligned}$$

  viz. $V_C(R) \geq 0$;

- if $A(x_2, y_2) \geq x_2$, we have

$$V_C(R) = A(x_1, y_1) + x_2 - \min\{A(x_1, y_2), x_1\} - A(x_2, y_1),$$

  and, since A is 1–Lipschitz

$$A(x_2, y_1) \leq x_2 - x_1 + A(x_1, y_1) \leq x_2;$$

  moreover, from the inequality

$$A(x_2, y_1) - A(x_1, y_1) \leq x_2 - x_1 \leq x_2 - \min\{A(x_1, y_2), x_1\},$$

  it follows that $V_C(R) \geq 0$. □

Notice that agops satisfying the assumptions of Theorem 3.4.1 are stable under convex combinations. Thus, many examples can be provided by using, for examples, copulas, quasi–arithmetic means bounded from above by the arithmetic mean, and their convex combinations.

Example 3.4.1. Let A be the modular agop $A(x,y) = (\delta(x) + \delta(y))/2$, where $\delta : [0,1] \to [0,1]$ is an increasing and 2–Lipschitz function with $\delta(0) = 0$ and $\delta(1) = 1$. Then A satisfies the assumptions of Theorem 3.4.1 and it generates the following copula

$$C_\delta(x,y) = \min\left\{x, y, \frac{\delta(x)+\delta(y)}{2}\right\}.$$

Copulas of this type were introduced in [56] and are called *diagonal copulas*.

Example 3.4.2. Let consider the following 2–increasing and 1–Lipschitz agop

$$A(x,y) = \lambda B(x,y) + (1-\lambda)\frac{x+y}{2},$$

defined for every $\lambda \in [0,1]$ and for every copula B. This A satisfies the assumptions of Theorem 3.4.1 and, therefore, the following class of copulas is obtained

$$C_\lambda(x,y) := \min\left\{x, y, \lambda B(x,y) + (1-\lambda)\frac{x+y}{2}\right\}.$$

Example 3.4.3. Let A be a 2–increasing agop of the form $A(x,y) = f(x) \cdot g(y)$. If A is 1–Lipschitz, then A satisfies the assumptions of Theorem 3.4.1. Consider, for instance, either $f(x) = x$ and $g(y) = (y+1)/2$, or $f(x) = (x+1)/2$ and $g(y) = y$, which yield, respectively, the following copulas

$$C_1(x,y) = \min\left\{y, \frac{x(y+1)}{2}\right\}, \quad C_2(x,y) = \min\left\{x, \frac{y(x+1)}{2}\right\}.$$

Chapter 4

A new family of PQD copulas

In this chapter we introduce a new class of bivariate copulas, depending on a univariate function, that includes some already known families. This class is characterized in section 4.1, where a probabilistic interpretation is given, and its properties (dependence, measures of association, symmetries, associativity, absolute continuity) are studied in detail in section 4.2. Section 4.3 is devoted to the introduction of a similar class in the set of quasi–copulas.

The contents of this chapter can be also found in [36, 42, 43].

4.1 Characterization of the new class

Let f be a mapping from $[0, 1]$ into $[0, 1]$. Consider the function C_f given, for every $x, y \in [0, 1]$, by

$$C_f(x, y) := (x \wedge y)\, f(x \vee y). \tag{4.1}$$

It is obvious that every C_f is symmetric and the copulas Π and M are of this type: it suffices to take, respectively, $f(t) = t$ and $f(t) = 1$ for all $t \in [0, 1]$. Our aim is to study under which conditions on f, C_f is a copula. Notice that, in view of the properties (1.9) and (1.10) of a copula, it is quite natural to require that f is increasing and continuous and, then, simple considerations of real analysis imply that f is differentiable almost everywhere on $[0, 1]$ and the left and right derivatives of f exist for every $x \in [0, 1]$ and assume finite values. We aim to characterize the copulas of type (4.1).

Lemma 4.1.1. *Let $f : [0, 1] \to [0, 1]$ be a continuous and increasing function, differentiable except at finitely many points. The following statements are equivalent:*

(a) *for every $s, t \in\,]0, 1]$, with $s \le t$, $sf(s) + tf(t) - 2sf(t) \ge 0$;*

(b) *the function $t \mapsto f(t)/t$ is decreasing on $]0, 1]$.*

Proof. $(a) \Rightarrow (b)$: Let s_i $(i = 1, 2, \dots, n)$ be the points in $[0, 1]$ such that $f'(s_i^+) \neq f'(s_i^-)$. Set $s_0 := 0$ and $s_{n+1} := 1$. For every $i \in \{0, 1, \dots, n\}$, let s and t be in $]s_i, s_{i+1}[$, $s < t$. The inequality

$$sf(s) + tf(t) - 2sf(t) \geq 0$$

is equivalent to

$$\frac{f(t)}{s} \geq \frac{f(t) - f(s)}{t - s}.$$

In the limit $t \downarrow s$, we have $f(s) \geq sf'(s)$. It follows that

$$\left(\frac{f(s)}{s}\right)' = \frac{sf'(s) - f(s)}{s^2} \leq 0,$$

viz. $t \mapsto f(t)/t$ is decreasing in each interval $]s_i, s_{i+1}[$, $(i = 0, 1, \dots, n)$. But $f(t)/t$ is continuous and, therefore, it is decreasing on the whole $]0, 1]$.
$(b) \Rightarrow (a)$: Let s, t be in $]0, 1]$, with $s < t$. Then

$$\frac{f(s)}{s} \geq \frac{f(t)}{t}$$

is equivalent to

$$\frac{f(s)}{s} \geq \frac{f(t) - f(s)}{t - s},$$

and, because f is increasing,

$$\frac{f(t)}{s} \geq \frac{f(t) - f(s)}{t - s},$$

viz. condition (a). □

Theorem 4.1.1. *Let $f : [0, 1] \to [0, 1]$ be a differentiable function (except at finitely many points). Let C_f be the function defined by* (4.1). *Then C_f is a copula if, and only if, the following statements hold:*

(i) $f(1) = 1$;

(ii) f *is increasing;*

(iii) *the function $t \mapsto f(t)/t$ is decreasing on* $]0, 1]$.

Proof. It is immediate that C_f satisfies the boundary conditions (C1) if, and only if, $f(1) = 1$. We now prove that C_f is 2–increasing if, and only if, (ii) and (iii) hold. Let x, x', y, y' be in $[0, 1]$ with $x \leq x'$ and $y \leq y'$. First, we suppose that the rectangle $[x, x'] \times [y, y']$ is a subset of Δ_+ (see notations (1.12)). Then

$$V_C([x, x'] \times [y, y']) = (y' - y)\,(f(x') - f(x)) \geq 0$$

if, and only if, f is increasing. Analogously, the 2–increasing property is equivalent to (ii) for rectangles contained in Δ_-. If, instead, the diagonal of $[x, x'] \times [y, y']$ lies on the diagonal $\{(x,y)\in[0,1]^2 : y=x\}$ of the unit square, then $x=y$ and $x'=y'$ and, in view of Lemma 4.1.1,

$$V_C([x,x']\times[x,x']) = xf(x)+x'f(x')-2xf(x')\geq 0$$

if, and only if, (iii) holds. Now, the assertion follows from Proposition 1.6.1. □

A function f that satisfies the assumptions of Theorem 4.1.1 is called *generator* of a copula of type (4.1). In particular, the class of generators is convex and, because of condition (iii), it has minimal element $\mathrm{id}_{[0,1]}$ and maximal element the constant function equal to 1. Note that $f:[0,1]\to[0,1]$ satisfies condition (iii) of Theorem 4.1.1 if, and only if, f is *star–shaped*, i.e., $f(\alpha x)\geq \alpha f(x)$ for all $\alpha\in[0,1]$. Moreover, every concave function satisfies (iii) (these results can also be found in [103, Chap. 16]). Now, we give a probabilistic interpretation of the generators.

Proposition 4.1.1. *Let U and V be r.v.'s uniformly distributed on $[0,1]$ with copula C_f of type (4.1). Then*

$$f(t) = P\left(\max\{U,V\}\leq t \mid U\leq t\right).$$

Proof. For every t in $[0,1]$, we have

$$C(t,t) = tf(t) = P\left(U\leq t, V\leq t\right),$$

and

$$P\left(\max\{U,V\}\leq t \mid U\leq t\right) = \frac{P\left(U\leq t, V\leq t\right)}{P(U\leq t)} = f(t),$$

namely the assertion. □

In the sequel we give some sub–classes of copulas $\{C_\alpha\}$ of type (4.1) generated by a one–parameter family $\{f_\alpha\}$.

Example 4.1.1 (Fréchet copulas)**.** Given $f_\alpha(t):=\alpha t+(1-\alpha)$ $(\alpha\in[0,1])$, we obtain $C_\alpha=\alpha\Pi+(1-\alpha)M$, which is a convex sum of Π and M and, therefore, is a member of the Fréchet family of copulas (see Example 1.6.2) (see, also, family (B11) in [74]). Notice that $C_0=M$ and $C_1=\Pi$.

Example 4.1.2 (Cuadras–Augé copulas)**.** Given $f_\alpha(t):=t^\alpha$ $(\alpha\in[0,1])$, C_α is defined by

$$C_\alpha(x,y) = (x\wedge y)(x\vee y)^\alpha = \begin{cases} xy^\alpha, & \text{if } x\leq y;\\ x^\alpha y, & \text{if } x>y.\end{cases}$$

Then C_α describes the Cuadras–Augé family of copulas (see Example 1.6.4). Notice that $C_0=M$ and $C_1=\Pi$.

Example 4.1.3. Given $f_\alpha(t) := \min(\alpha t, 1)$ $(\alpha \geq 1)$, C_α is defined by

$$C_\alpha(x,y) = (x \wedge y)\min\{\alpha(x \vee y), 1\} = \begin{cases} \alpha xy, & \text{if } (x,y) \in [0,1/\alpha]^2\,; \\ x \wedge y, & \text{otherwise;} \end{cases}$$

viz. C_α is the ordinal sum $(\langle 0, 1/\alpha, \Pi\rangle)$. Notice that $C_1 = \Pi$ and $C_\infty = M$, where, if $g(x) = \lim f_\alpha(x)$ as $\alpha \to +\infty$ and $x \in]0,1]$, $C_\infty := C_g$.

Example 4.1.4. Given the function $f_\alpha(t) := c\exp(t^\alpha/\alpha)$, where $\alpha > 0$ and $c = \exp(-1/\alpha)$, we obtain the following family

$$C_\alpha(x,y) = \begin{cases} cx\exp(y^\alpha/\alpha), & \text{if } x \leq y; \\ cy\exp(x^\alpha/\alpha), & \text{if } x > y. \end{cases}$$

Example 4.1.5. The function $f_\alpha(t) := \frac{1}{\sin\alpha}\sin(\alpha t)$ $(\alpha \in]0, \pi/2])$ is increasing with $f_\alpha(t)/t$ decreasing on $]0,1]$, as is easily proved. Therefore, Theorem 4.1.1 ensures that

$$C_\alpha(x,y) = \begin{cases} \frac{x}{\sin\alpha}\sin(\alpha y)\,, & \text{if } x \leq y; \\ \frac{y}{\sin\alpha}\sin(\alpha x)\,, & \text{if } x > y. \end{cases}$$

is a copula.

For a copula C_f of type (4.1) the following result holds (see [100] for details).

Theorem 4.1.2. *If C_f is the copula given by* (4.1) *and $H(x,y) = C_f(F_1(x), F_2(y))$ for univariate d.f.'s F_1 and F_2, then the following statements are equivalent:*

(a) *random variables X and Y with joint d.f. H have a representation of the form*

$$X = \max\{R, W\} \qquad \textit{and} \qquad Y = \max\{S, W\}$$

where R, S and W are independent r.v.'s;

(b) *H has the form $H(x,y) = F_R(x)F_S(y)F_W(x \wedge y)$, where F_R, F_S and F_W are univariate d.f.'s.*

4.2 Properties of the new class

In this section we give the most important properties of a copula C_f of type (4.1).

4.2.1 Concordance order

Proposition 4.2.1. *Let C_f and C_g be two copulas of type* (4.1). *Then $C_f \leq C_g$ if, and only if, $f(t) \leq g(t)$ for all $t \in [0,1]$.*

In particular, for every copula C_f, $\Pi \le C_f \le M$ and, therefore, every C_f is positively quadrant dependent.

Example 4.2.1. Consider the family $\{f_\alpha\}$ $(\alpha \ge 1)$, given by $f_\alpha(t) := 1-(1-t)^\alpha$. It is easily proved by differentation that every f_α is increasing with $f_\alpha(t)/t$ decreasing on $]0,1]$. Therefore, this family generates a family of copulas C_α, that is positively ordered, with $C_1 = \Pi$ and $C_\infty = M$.

Example 4.2.2. Consider the family of copulas generated by the function $f_\alpha(t) := (1+\alpha)t/(\alpha t+1)$ for every $\alpha \ge 0$. This family is positively ordered with $C_0 = \Pi$ and $C_\infty = M$.

4.2.2 Dependence concepts

Theorem 4.2.1. *Let (X,Y) be a continuous random pair with copula C_f. Then*

(a) *Y is left tail decreasing in X;*

(b) *Y is stochastically increasing in X if, and only if, f' is decreasing a.e. on $[0,1]$;*

(c) *X and Y are left corner set decreasing.*

Proof. In order to prove $LTD(Y|X)$, according to Proposition 1.7.2 it suffices to notice that, for every $(x,y) \in [0,1]^2$

$$\frac{C_f(x,y)}{x} = \begin{cases} f(y), & \text{if } x \le y; \\ \dfrac{yf(x)}{x}, & \text{if } x > y; \end{cases}$$

is decreasing in x.

Property $SI(Y|X)$ follows from Proposition 1.7.3, observing that $\partial_x C_f$ is decreasing in the first place if, and only if, f' is decreasing a.e. on $[0,1]$.

In order to prove (c), because of Proposition 1.7.4, it suffices to prove that, for all x, x', y, y' in $[0,1]$, with $x \le x'$ and $y \le y'$,

$$C_f(x,y)C_f(x',y') \ge C_f(x,y')C_f(x',y) \ge 0. \tag{4.2}$$

Because $f(t)/t$ is decreasing and C_f is symmetric, inequality (4.2) follows easily from simple calculations on rectangles $[x,x'] \times [y,y']$ that have 4, 3 or 2 vertices in the set Δ_+. For instance, if $[x,x'] \times [y,y']$ has only two vertices, say (x,y) and (x',y) in Δ_+, then (4.2) holds if, and only if, $x'f(x) \ge xf(x')$, viz. $f(t)/t$ is decreasing. □

The following result for the tail dependence holds.

Proposition 4.2.2. *Let C_f be a copula of type (4.1). Then, the lower tail dependence of C_f is $f(0^+)$ and the upper tail dependence of C_f is $1 - f'(1^-)$.*

Proof. The diagonal section of C_f is $\delta_{C_f}(t) = tf(t)$. Therefore, from Proposition 1.7.5, we have $\lambda_L = \delta'_C(0^+) = f(0^+)$ and $\lambda_U = 2 - \delta'_C(1^-) = 1 - f'(1^-)$. □

Remark 4.2.1. As noted, a copula of type (4.1) is PQD and, therefore, it is suitable to describe positive dependence of a random vector (X, Y). However, it is very simple to introduce a copula to describing, for example, the (negative) dependence of the random vector $(X, -Y)$. It suffices to consider the copula $C^f_{0,1}$ given by

$$C^f_{0,1}(x,y) := x - C(x, 1-y) = \begin{cases} x(1 - f(1-y)), & \text{if } x + y \leq 1; \\ x - (1-y)f(x), & \text{otherwise.} \end{cases}$$

4.2.3 Measures of association

Theorem 4.2.2. *The values of several measures of association of C_f are, respectively, given by*

$$\tau_C = 4\int_0^1 xf^2(x)\,dx \, - 1, \qquad \rho_C = 12\int_0^1 x^2 f(x)\,dx \, - 3,$$
$$\gamma_C = 4\left(\int_0^{1/2} x\,[f(x) + f(1-x)]\,dx + \int_{1/2}^1 f(x)\,dx\right) - 2,$$
$$\beta_C = 2f(1/2) - 1, \qquad \varphi_C = 6\int_0^1 xf(x)\,dx \, - 2.$$

Proof. In view of Theorem 1.8.1, the Kendall's tau of C_f is given by

$$\tau_C = 1 - 4\int_0^1 \int_0^1 \partial_x C(x,y)\partial_y C(x,y)\,dx\,dy.$$

Now, we have

$$\begin{aligned} &\int_0^1 \int_0^1 \partial_x C(x,y)\partial_y C(x,y)\,dx\,dy \\ =& \int_0^1 dy \int_0^y xf(y)f'(y)\,dx + \int_0^1 dx \int_0^x yf(x)f'(x)\,dy \\ =& \int_0^1 x^2 f(x)f'(x)\,dx = \frac{1}{2} - \int_0^1 xf^2(x)\,dx, \end{aligned}$$

where the last equality is obtained through integration by parts. Then

$$\tau_C = 4\int_0^1 xf^2(x)\,dx \, - 1.$$

From Theorem 1.8.2, Spearman's rho is given by:

$$\begin{aligned} \rho_C &= 12\int_0^1 \int_0^1 C(x,y)\,dx\,dy \, - 3 \\ &= 12\int_0^1 dy \int_0^y xf(y)\,dx + \int_0^1 dx \int_0^x yf(x)\,dy \, - 3 \\ &= 12\int_0^1 x^2 f(x)\,dx \, - 3. \end{aligned}$$

Following Theorem 1.8.3, we have

$$\begin{aligned}
\gamma_C &= 4\left(\int_0^1 C(x,1-x)\,dx - \int_0^1 (x - C(x,x))\ dx\right) \\
&= 4\left(\int_0^{1/2} xf(1-x)\,dx - \int_0^{1/2} [x - xf(x)]\ dx\right) \\
&\qquad + \int_{1/2}^1 (1-x)f(x) - \int_{1/2}^1 [x - xf(x)]\ dx \\
&= 4\left(\int_0^{1/2} x\,[f(x)+f(1-x)]\ dx + \int_{1/2}^1 f(x)\,dx\ - \frac{1}{2}\right) \\
&= 4\left(\int_0^{1/2} x\,[f(x)+f(1-x)]\ dx + \int_{1/2}^1 f(x)\,dx\right) - 2.
\end{aligned}$$

The expressions of β_C and φ_C follow easily from Theorems 1.8.4 and 1.8.5. □

As an application of Theorem 4.2.2, the measures of association for the copulas in Examples 1.6.2 and 1.6.4 can be easily given:

– If C is a copula of the Fréchet family, then

$$\tau_C = \frac{(\alpha-1)(\alpha-3)}{3}, \qquad \rho_C = 1-\alpha = \gamma_C = \varphi_C.$$

– If C is a Cuadras–Augé copula, then

$$\tau_C = \frac{1-\alpha}{1+\alpha}, \qquad \rho_C = \frac{3-3\alpha}{3+\alpha}, \qquad \varphi_C = \frac{2-2\alpha}{2+\alpha}.$$

4.2.4 Symmetry properties

Theorem 4.2.3. *Let (X,Y) be continuous r.v.'s with copula C_f.*

(a) *If X and Y are identically distributed, then X and Y are exchangeable.*

(b) *If X and Y are symmetric about a and b, respectively $(a,b\in\mathbb{R})$, then (X,Y) is radially symmetric about (a,b) if, and only if, $C_f = \alpha\Pi + (1-\alpha)M$ for some $\alpha\in[0,1]$.*

(c) *If X and Y are symmetric about a and b, respectively $(a,b\in\mathbb{R})$, then (X,Y) is jointly symmetric about (a,b) if, and only if, $C_f = \Pi$.*

Proof. Statement (a) is a consequence of the symmetry of C_f. From Proposition 1.6.3, statement (b) holds if, and only if, C_f satisfies the following functional equation:

$$\forall x,y\in[0,1] \qquad C_f(x,y) = x+y-1+C_f(1-x,1-y). \tag{4.3}$$

But, equality (4.3) is equivalent to

$$(x\wedge y)f(x\vee y) = x+y-1+[1-(x\vee y)]\,f\,[1-(x\wedge y)]\,;$$

in particular, for all $y \in [x, 1[$, we have

$$\begin{aligned} & xf(y) = x + y - 1 + (1-y)f(1-x) \\ & \Longrightarrow x\,(1 - f(y)) + (1-y)f(1-x) = 1 - y \\ & \Longrightarrow x \cdot \frac{1-f(y)}{1-y} + f(1-x) = 1 \Longrightarrow f(1-x) = 1 - x \cdot \frac{f(y)-1}{y-1}. \end{aligned}$$

In the limit $y \uparrow 1$, we can derive

$$\frac{1-f(y)}{1-y} \longrightarrow f'(1^-),$$

where $f'(1^-)$ is a real number in $[0, 1]$. Thus $f(1-x) = 1-cx$, i.e. $f(x) = cx+(1-c)$, which corresponds to the family $C_f = c\Pi + (1-c)M$.

From Proposition 1.6.3, (X, Y) is jointly symmetric about (a, b) if, and only if, for all $(x, y) \in [0, 1]^2$

$$C_f(x, y) = x - C_f(x, 1-y) \qquad \text{and} \qquad C_f(x, y) = y - C_f(1-x, y). \tag{4.4}$$

In particular, for $x = y$, we obtain

$$\forall x \in [0,1] \qquad xf(x) = x - [x \wedge (1-x)]\, f\,[x \vee (1-x)]\,,$$

which implies

$$\begin{aligned} \forall x \in [1/2, 1] \qquad & xf(x) = x - (1-x)f(x), \\ \forall x \in [0, 1/2] \qquad & xf(x) = x - xf(1-x), \end{aligned}$$

viz. $f(x) = x$ on $[0, 1]$, which corresponds to $C_f = \Pi$. □

4.2.5 Associativity

Lemma 4.2.1. *Let C_f be a copula of type* (4.1). *Then C_f is Archimedean if, and only if, $C_f = \Pi$.*

Proof. If C_f is an Archimedean copula, then, there exists a convex function $\varphi : [0, 1] \to [0, +\infty]$, which is continuous and strictly increasing, $\varphi(1) = 0$, such that $C_f(x, y) = \varphi^{[-1]}\,(\varphi(x) + \varphi(y))$. In view of Theorem 1.6.8,

$$\varphi'(x)\ \frac{\partial C_f(x,y)}{\partial y} = \varphi'(y)\ \frac{\partial C_f(x,y)}{\partial x} \qquad \text{a.e. on } [0,1]^2.$$

In particular, if $x = y$, we obtain $\varphi'(x) \cdot xf'(x) = \varphi'(x) \cdot f(x)$, which leads to $xf'(x) = f(x)$. In the class of the generators of a copula of type (4.1), this differential equation has as unique solution the function $f(x) = x$, viz. $C_f = \Pi$. □

Theorem 4.2.4. *Let C_f be a copula of type* (4.1). *Then C_f is associative if, and only if, C_f is an ordinal sum of type $(\langle 0, a, \Pi \rangle)$ with $a \in [0, 1]$.*

Proof. First, notice that every ordinal sum of type $(\langle 0, a, \Pi\rangle)$ is associative and it is generated by the function $f(t) = \min\{t/a, 1\}$.

Conversely, let C_f be an associative copula. As asserted in Theorem 1.6.9, the representation of C_f depends on the set I_D of idempotent elements of C_f, given by $I_D := \{0\} \cup [a, 1]$, where $a := \inf\{t \in [0,1] : f(t) = 1\}$. If $I_D = \{0, 1\}$, then C_f is Archimedean and, therefore, Lemma 4.2.1 ensures that $C_f = \Pi = (\langle 0, 1, \Pi\rangle)$. If $I_D = [0, 1]$, then $C_f = M = (\langle 0, 0, \Pi\rangle)$. Otherwise, C_f is an ordinal sum of type $(\langle 0, a, D\rangle)$ for a suitable Archimedean copula D. Therefore, if φ is a generator of D, for all x, y in $[0, a]$,

$$C_f\,(x,y) = a\,\varphi^{[-1]}\left(\varphi\left(\frac{x}{a}\right) + \varphi\left(\frac{y}{a}\right)\right).$$

Hence, applying the chain rule to $\varphi(C_f(x,y)/a) = \varphi(x/a) + \varphi(y/a)$, we obtain

$$\varphi'\left(\frac{C_f(x,y)}{a}\right)\frac{\partial C_f(x,y)}{\partial x} = \varphi'\left(\frac{x}{a}\right), \quad \varphi'\left(\frac{C_f(x,y)}{a}\right)\frac{\partial C_f(x,y)}{\partial y} = \varphi'\left(\frac{y}{a}\right).$$

Therefore, a.e. on $[0,1]^2$, we have

$$\varphi'\left(\frac{x}{a}\right)\ \frac{\partial C_f(x,y)}{\partial y} = \varphi'\left(\frac{y}{a}\right)\ \frac{\partial C_f(x,y)}{\partial x}.$$

An argument similar to the proof of Lemma 4.2.1 gives $D = \Pi$, as asserted. □

4.2.6 Absolute continuity

Proposition 4.2.3. *The only absolutely continuous copula of type* (4.1) *is* Π.

Proof. Let C_f be a copula of type (4.1). If C_f is absolutely continuous, then

$$1 = C_f(1,1) = \int_0^1\int_0^1 \frac{\partial^2 C}{\partial x \partial y}\,dx\,dy = \int_0^1\int_0^1 f'(x \vee y)\,dx\,dy.$$

It follows that

$$\frac{1}{2} = \int_0^1 ds \int_0^s f'(s)\,dt = \int_0^1 s f'(s)\,ds;$$

integrating by parts, we have

$$\int_0^1 f(x)\,dx = \frac{1}{2}.$$

The function $f(x) = x$ is a solution of the above equation and, because all functions generating a copula of type (4.1) are greater than $\mathrm{id}_{[0,1]}$, it follows that $\mathrm{id}_{[0,1]}$ is the only solution in this class. □

Remark 4.2.2. Let C_f be a copula of type (4.1), $C \neq \Pi$. Consider the first derivative of C_f

$$\partial_1 C_f(x,y) = \begin{cases} f(y), & \text{if } x < y; \\ y \cdot f'(x), & \text{otherwise.} \end{cases}$$

For a fixed y_0, the mapping $t \mapsto \partial_1 C_f(t, y_0)$ has a jump discontinuity in y_0, and, thus, C_f has a singular component along the main diagonal of the unit square. By using [74, Theorem 1.1], the mass of this singular component is given by

$$m = \int_0^1 (f(x) - xf'(x))\, dx = 2 \cdot \int_0^1 f(x)\, dx - 1.$$

This m has a graphical interpretation if f admits an inverse: in fact, m is the area of the region of the unit square between the graph of f and the graph of f^{-1}.

4.3 A similar new class of quasi–copulas

Given a function $f : [0, 1] \to [0, 1]$, we are also interested in studying under which conditions on f, the following function

$$Q_f(x, y) := (x \wedge y)\, f(x \vee y), \quad \text{for all } (x, y) \in [0, 1]^2\,, \tag{4.5}$$

is a quasi–copula. The following result provides a characterization.

Theorem 4.3.1. *Let $f : [0, 1] \to [0, 1]$ be a continuous function and let Q_f be defined by* (4.5)*. Then Q_f is a quasi–copula if, and only if, the three following statements hold:*

(i) $f(1) = 1$;

(ii) *f is increasing;*

(iii) $x_1 \cdot \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \leq 1$ *for every $x_1, x_2 \in [0, 1]$, with $x_1 < x_2$.*

Proof. First, observe that Q_f satisfies (Q1) if, and only if, $f(1) = 1$ and Q_f satisfies (Q2) if, and only if, (ii) holds. In order to prove that Q_f satisfies (Q3), let x_1, x_2 and y be three points in $[0, 1]$ with $x_1 < x_2$. We distinguish three cases. If $x_1 < x_2 \leq y$, then

$$Q_f(x_2, y) - Q_f(x_1, y) = x_2 f(y) - x_1 f(y) \leq x_2 - x_1$$

because $f \leq 1$. If $y \leq x_1 < x_2$, then

$$Q_f(x_2, y) - Q_f(x_1, y) = y \cdot (f(x_2) - f(x_1)) \leq \frac{y}{x_1} \cdot (x_2 - x_1) \leq x_2 - x_1$$

if, and only if, (iii) holds. Finally, if $x_1 \leq y \leq x_2$, in view of the two above cases we obtain

$$\begin{aligned} Q_f(x_2, y) - Q_f(x_1, y) &= (Q_f(x_2, y) - Q_f(y, y)) + (Q_f(y, y) - Q_f(x_1, y)) \\ &\leq (x_2 - x_1) \end{aligned}$$

if, and only if, (iii) holds. In every case, (iii) is a necessary and sufficient condition that ensures that Q_f satisfies (1.10). □

Corollary 4.3.1. *Let* $f : [0,1] \to [0,1]$ *be a differentiable function and let* Q_f *be defined by* (4.5). *Then* Q_f *is a quasi–copula if, and only if, the three following statements hold:*

(i) $f(1) = 1$;

(ii) f *is increasing;*

(iii) $xf'(x) \leq 1$ *for every* $x \in [0,1]$.

Notice that if Q_f is a copula, then $t \mapsto f(t)/t$ is decreasing and

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_1)}{x_1}$$

for every $x_1, x_2 \in [0,1]$, with $x_1 < x_2$, from which the condition (iii) of Theorem 4.3.1 follows, viz. Q_f is a quasi–copula. The converse implication need not be true, as the following example shows.

Example 4.3.1. Consider the function $f(t) := t + t^2 - t^3$ on $[0,1]$. So, f satisfies the assumptions of Theorem 4.3.1, viz. $f'(t) \leq 1/t$ on $[0,1]$, but $f(t)/t$ is increasing on $[0,1/2]$. So Q_f is a proper quasi–copula. Another (not everywhere) differentiable function g, which leads to a proper quasi–copula, is given by

$$g(x) = \begin{cases} x, & \text{if } x \in [0,1/4]\,; \\ 2x - 1/4, & \text{if } x \in \,]1/4, 1/2[\,; \\ (x+1)/2, & \text{if } x \in [1/2, 1]\,. \end{cases}$$

We have $g'(x) \leq 1/x$ and thus Q_g is a quasi-copula; however, $h(x) := g(x)/x$ is not decreasing (e.g. $h(1/4) = 1$ but $h(1/2) = 3/2$).

Chapter 5

A family of copulas with given diagonal section

Given a copula C, its diagonal δ satisfies the following properties:

(D1) $\delta(1) = 1$;

(D2) $\delta(t) \leq t$ for all $t \in [0, 1]$;

(D3) δ is increasing;

(D4) $|\delta(t) - \delta(s)| \leq 2|t - s|$ for all $t, s \in [0, 1]$.

We recall that $\mathcal{D}$ denotes the set of functions $\delta : [0, 1] \to [0, 1]$ satisfying (D1)–(D3) and $\mathcal{D}_2$ denotes the subset of $\mathcal{D}$ of the functions satisfying also (D4). In literature the question of determining a copula beginning from a function $\delta \in \mathcal{D}_2$ has been already studied, as showed in subsection 1.6.3.

In this chapter, we give another class of copulas that can be derived from the diagonal section. Specifically, we are interested on copulas C satisfying the functional equation:

$$C(x, y) + |x - y| = C(x \vee y, x \vee y) \qquad \text{whenever } C(x, y) > 0.$$

In other words, we analyse under which conditions on $\delta \in \mathcal{D}_2$ the function

$$D_\delta(x, y) := \max\{0, \delta(x \vee y) - |x - y|\} \qquad \text{for all } x, y \in [0, 1], \tag{5.1}$$

is a copula. Notice that t–norms of type (5.1) were already studied by G. Mayor and J. Torrens ([105]), who obtained the following characterization.

Theorem 5.0.2. *Let T be a continuous t–norm with diagonal section δ. Then T satisfies the functional equation*

$$T(x, y) + |x - y| = \delta(x \vee y),$$

whenever $T(x, y) > 0$, if, and only if, T belongs to the Mayor–Torrens family of t–norms presented in Example 1.4.1.

For these reasons, we shall use the prefix MT to indicate a function of type (5.1) (e.g. MT–copula, MT–quasicopula, MT–semicopula), where "MT" stands for "à la Mayor and Torrens".

MT–copulas are characterized in section 5.1 and their properties are studied in section 5.2. Section 5.3 is devoted to the study of a simple procedure to generate an aggregation operator with additional properties (Lipschitz, 2–increasing, etc.) beginning from two aggregation operators of the same type and with the same diagonal section.

The results of this chapter are also contained in [39, 40].

5.1 Characterization of MT–copulas

In order to characterize MT–copulas, first, we establish an analogous characterization for semicopulas of the same type.

Lemma 5.1.1. *The following statements are equivalent:*

(a) *$\delta \in \mathcal{D}$ and there exists $a \in [0, 1[$ such that $\delta(x) = 0$ on $[0, a]$ and the function $x \mapsto (\delta(x) - x)$ is increasing on $[a, 1]$.*

(b) *D_δ is an MT–semicopula;*

Proof. $(a) \Longrightarrow (b)$: For all $t \in [0, 1]$

$$D_\delta(t, 1) = \max\{0, \delta(1) - |t - 1|\} = t = D_\delta(1, t).$$

In order to ensure that D_δ is increasing in each variable, consider $x, x', y \in [0, 1]$ with $x \leq x'$ such that $D_\delta(x, y) > 0$ and $D_\delta(x', y) > 0$. If $y \geq x'$, then

$$D_\delta(x, y) = \delta(y) - y + x \leq \delta(y) - y + x' = D_\delta(x', y).$$

If $y \leq x$, then, since $t \mapsto (\delta(t) - t)$ is increasing,

$$D_\delta(x, y) = \delta(x) - x + y \leq \delta(x') - x' + y = D_\delta(x', y). \tag{5.2}$$

Finally, if $x < y \leq x'$, then, again since $t \mapsto (\delta(t) - t)$ is increasing,

$$D_\delta(x, y) = \delta(y) - y + x \leq \delta(x') - x' + y = D_\delta(x', y). \tag{5.3}$$

$(b) \Longrightarrow (a)$: Set $a := \sup\{t \in [0,1] : D_\delta(t,t) = 0\}$ that satisfies the required conditions. The isotony of $(\delta(t) - t)$ is established in the same way of the proof of $(a) \Rightarrow (b)$ (see inequalities (5.2) and (5.3)). □

Theorem 5.1.1. *The following statements are equivalent:*

(a') *$\delta \in \mathcal{D}_2$ and there exists $a \in [0, 1/2]$ such that $\delta(x) = 0$ on $[0, a]$ and the function $x \mapsto (\delta(x) - x)$ is increasing on $[a, 1]$;*

(b') *D_δ is a copula.*

Proof. $(a') \Longrightarrow (b')$: In view of Proposition 1.6.1, it suffices to prove that

$$D_\delta(x', y') + D_\delta(x, y) - D_\delta(x', y) - D_\delta(x, y') \geq 0 \tag{5.4}$$

in three cases: on a rectangle $R := [x, x'] \times [y, y']$ contained in Δ_- or in Δ_+, and on a rectangle $R := [x, y] \times [x, y]$.

In the first case, put $F(x, y) := \delta(y) - y + x$. Then

$$F(x', y') + F(x, y) = F(x, y') + F(x', y)$$

and $D_\delta(x, y) = \max\{0, F(x, y)\}$. If two terms on the left hand side of (5.4) are equal to 0, then inequality (5.4) follows from the monotony of δ. If one of the terms in the left hand side of (5.4) is 0, then it is necessarily the value on the left–lower corner of the rectangle R and, on the remaining three corners, the values of D_δ are equal to those of F. Then $F(x, y) \leq D_\delta(x, y)$ implies

$$0 = V_F(R) \leq V_{D_\delta}(R).$$

If R is contained on Δ_+, the proof follows from the commutativity of D_δ.

In the third case, it suffices to prove that, for every $x \leq y$, $\delta(x) + \delta(y) \geq 2(\delta(y) - y + x)$, i.e. $(\delta(y) - y) - (\delta(x) - x) \leq y - x$. However, this inequality follows from (D4) because, if $\delta \in \mathcal{D}_2$, then $(\delta(t) - t)$ is 1-Lipschitz.
$(b') \Longrightarrow (a')$: It follows directly from Lemma 5.1.1, by observing that, because of the Fréchet–Hoeffding bounds (1.13), we have $a \in [0, 1/2]$. □

Corollary 5.1.1. *The following statements are equivalent:*

(a') *$\delta \in \mathcal{D}_2$ and there exists $a \in [0, 1/2]$ such that $\delta(x) = 0$ on $[0, a]$ and the function $x \mapsto (\delta(x) - x)$ is increasing on $[a, 1]$;*

(c') *D_δ is a quasi–copula.*

In other words, no proper MT–quasi–copula exists.

Proof. As in Theorem 5.1.1, we prove that $(c') \Longrightarrow (a')$. The assertion follows directly, since every copula is a quasi–copula. □

5.2 Properties of MT–copulas

In this section, we denote by D an MT–copula and by δ its diagonal satisfying the assumptions of Theorem 5.1.1.

Proposition 5.2.1. *Every MT–copula D is a simple Bertino copula.*

Proof. First, observe that $D(x, y) = 0$ if, and only if, $x \vee y \leq a$. In fact, if there exist $x, y \in [0, 1]$ such that $D(x, y) = 0$ with $x \vee y > a$, we have $\delta(x \vee y) - |x - y| \leq 0$, from which, for all $x > a \vee y$, $\delta(x) - x \leq -y \leq \delta(y) - y$: a contradiction, because $(\delta(x) - x)$ is increasing on $[a, 1]$.

Let x, y be in $[0, 1]$ such that $D(x, y) > 0$ so that x and y both belong to $[a, 1]$. By Theorem 5.1.1, $x \mapsto (x - \delta(x))$ is decreasing on $[a, 1]$. If $x \geq y$, we have

$$D(x, y) = \delta(x) - x + y = \min\{x, y\} - \min\{x - \delta(x), y - \delta(y)\}.$$

In the other case $x < y$, the proof is analogous. □

As a consequence, the following statistical characterization of MT–copulas can be formulated ([57, Corollary 3.2])

Corollary 5.2.1. *Let U and V be r.v.'s uniformly distributed on $[0, 1]$ whose joint distribution function is the copula D. Then, for each $(x, y) \in [0, 1]^2$, either*

$$P(U \leq x, V \leq y) = P(\max\{U, V\} \leq \min\{x, y\})$$

or

$$P(U > x, V > y) = P(\min\{U, V\} > \max\{x, y\}).$$

Moreover, since $t \mapsto (t - \delta(t))$ has slope 1 in the interval $[0, a]$ on which it is strictly increasing, in view of [57, Theorem 4.1], it follows

Proposition 5.2.2. *Every MT–copula D is extremal, in the sense that, if there exist two copulas A and B such that $D = \alpha A + (1 - \alpha)B$, with $\alpha \in]0, 1[$, then $D = A = B$.*

Remark 5.2.1. The support of D contains the part of the main diagonal of the unit square corresponding to the union of the intervals on which $\delta > 0$ and $\delta' < 2$ and a line which is the boundary of its zero region (see also [57, Theorem 2.2]).

Remark 5.2.2. Observing that the family T_α of Theorem 5.0.2 is an ordinal sum of W, $T_\alpha = (\langle 0, \alpha, W \rangle)$, and thus it is a copula for every α in $[0, 1]$, we have that, as a consequence of Theorem 5.0.2, the only associative MT–copulas are of this type.

Now, we present a result on symmetries.

Proposition 5.2.3. *Let X and Y be continuous r.v.'s with copula D. If X and Y are symmetric about α and β, respectively ($\alpha, \beta \in \mathbb{R}$), then (X, Y) is radially symmetric about (α, β) if, and only if, there exists $a \in [0, 1/2]$ such that D is a member of the family of copulas given by*

$$C_a(x, y) = \max\{W(x, y), M(x, y) - a\}. \tag{5.5}$$

Proof. Let D be an MT–copula with diagonal δ. From Proposition 1.6.3, it suffices to show that $D = \widehat{D}$, viz. for every $(x, y) \in [0, 1]^2$

$$\max\{0, \delta(x \vee y) - |x - y|\} = x + y - 1 + \max\{0, \delta(1 - x \wedge y) - |x - y|\}, \tag{5.6}$$

which is equivalent to

$$\delta(t) = 2t - 1 + \delta(1 - t) \qquad \text{for every } t \in [0, 1].$$

For some $a \in [0, 1/2]$, $\delta(t) = 0$ on $[0, a]$ and $D = \widehat{D}$ implies that $\delta(t) = 2t - 1$ on $[1-a, 1]$. Since $\delta(a) - a = -a = \delta(1-a) - (1-a)$ and, from Theorem 5.1.1, $(\delta(x) - x)$ is increasing on $[a, 1]$, this latter function must necessarily be a constant, which can only be equal to $-a$ on $[a, 1 - a]$, so that $\delta(t) = t - a$ on $[a, 1 - a]$. Thus we have that there exists $a \in [0, 1/2]$ such that

$$\delta(t) = \begin{cases} 0, & \text{if } t \in [0, a]; \\ t - a, & \text{if } t \in [a, 1 - a]; \\ 2t - 1, & \text{if } t \in [1 - a, 1]; \end{cases}$$

and D coincides with C_a. □

Notice that the copula C_a is a shuffle of Min, as showed in Example 1.6.8.

It is known from [57] that the Bertino copulas are the weakest (in the pointwise ordering) copulas with given diagonal section. Moreover, the following result is easily derived.

Proposition 5.2.4. *Let D_δ and D_γ be two MT–copulas with diagonals δ and γ, respectively. Then $D_\delta \le D_\gamma$ if, and only if, $\delta(t) \le \gamma(t)$ for all $t \in [0, 1]$.*

Thus, the concordance order on MT–copulas depends on the pointwise ordering of their diagonals. In the same way, the diagonal of an MT–copula describes the most common non–parametric measures of association between random variables.

Theorem 5.2.1. *Let D be the MT–copula associated with the random pair (X, Y). The values of the measures of association between X and Y are given, respecively, by*

$$\tau_D = 8 \int_0^1 \delta(x)\, dx \; - 3, \qquad \rho_D = 12 \cdot \int_0^1 \delta^2(x)\, dx \; - 3,$$

$$\gamma_D = 4 \cdot \left[3 \int_{1/2}^1 \delta(x) dx + \int_0^{1/2} \delta(x) dx - 1 \right],$$

$$\beta_D = 4 \cdot \delta(1/2) - 1, \qquad \varphi_D = 6 \int_0^1 \delta(x)\, dx - 2.$$

Proof. Let D be an MT–copula and let Ω, Ω_+ and Ω_- be the three subsets of the unit square defined by:

$$\Omega := \{(x,y) \in [0,1]^2 : D(x,y) > 0\};$$
$$\Omega_+ := \Delta_+ \cap \Omega; \qquad \Omega_- := \Delta_- \cap \Omega.$$

In view of Theorem 1.8.1, we have

$$\tau_D = 1 - 4 \int\int_{[0,1]^2} \partial_x D(x,y) \cdot \partial_y D(x,y)\, dx\, dy,$$

where

$$\partial_x D(x,y) \cdot \partial_y D(x,y) = \begin{cases} \delta'(x) - 1, & \text{if } (x,y) \in \Omega_+; \\ \delta'(y) - 1, & \text{if } (x,y) \in \Omega_-; \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned} &\int\int_{[0,1]^2} \partial_x D(x,y) \cdot \partial_y D(x,y)\, dx\, dy \\ &= \int\int_{\Omega_+} \partial_x D(x,y) \cdot \partial_y D(x,y)\, dx\, dy + \int\int_{\Omega_-} \partial_x D(x,y) \cdot \partial_y D(x,y)\, dx\, dy \\ &= 2 \cdot \int_0^1 (\delta'(x) - 1)\, dx \int_{x-\delta(x)}^{x} dy = 1 - 2 \int_0^1 \delta(x) dx. \end{aligned}$$

Simple calculations lead to the value of τ_D.

By using Theorem 1.8.2, ρ_D is given by:

$$\begin{aligned} \rho_D &= 12 \int\int_{[0,1]^2} D(x,y) dx dy\ - 3 = 24 \int\int_{\Omega_+} (\delta(x) - x + y)\, dx dy\ - 3 \\ &= 24 \cdot \int_0^1 (\delta(x) - x)\, dx \int_{x-\delta(x)}^{x} dy + 24 \cdot \int_0^1 dx \int_{x-\delta(x)}^{x} y\, dy\ - 3 \\ &= 24 \cdot \int_0^1 \left(\delta^2(x) - x\delta(x)\right) dx + 12 \cdot \int_0^1 \left(-\delta^2(x) + 2x\delta(x)\right)\, dx\ - 3 \\ &= 12 \cdot \int_0^1 \delta^2(x)\, dx\ - 3. \end{aligned}$$

In the same manner, from Theorem 1.8.3

$$\gamma_D = 4 \left[\int_0^1 D(x, 1-x) dx - \int_0^1 (x - D(x,x))\, dx \right].$$

For all $x \in [0,1]$

$$D(x, 1-x) = \begin{cases} \max(0, \delta(1-x) - 1 + 2x) & \text{if } x \le 1/2; \\ \max(0, \delta(x) - 2x + 1) & \text{if } x > 1/2. \end{cases}$$

It is easy to show that

$$\int_0^1 D(x, 1-x)dx = 2\int_{1/2}^1 \delta(x)dx - 1/2$$

and

$$\int_0^1 (x - \delta(x))\,dx = 1/2 - \int_0^1 \delta(x)dx,$$

from which we have the asserted value of γ_D.

The expressions of β_D and φ_D follow directly from Theorems 1.8.4 and 1.8.5. □

5.3 A construction method

Let Δ_+ and Δ_- be the two subsets of the unit square given in (1.12). For two binary aggregation operators A and B, we introduce the function $F_{A,B} : [0,1]^2 \to [0,1]$ given, for all x, y in $[0,1]$, by

$$F_{A,B}(x,y) := A(x,y)\,1_{\Delta_+}(x,y) + B(x,y)\,1_{\Delta_-}(x,y).$$

In other words, if we divide the unit square by means of its diagonal, then $F_{A,B}$ is equal to A in the lower triangle and equal to B in the upper one.

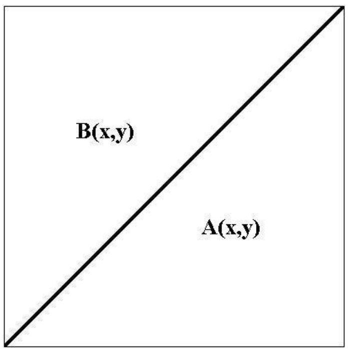

Figure 5.1: The function $F_{A,B}$

Proposition 5.3.1. *If A and B are agops with the same diagonal section, then $F_{A,B}$ is an agop. Moreover, if A and B are semicopulas, so is $F_{A,B}$.*

Proof. It is obvious that $F_{A,B}(0,0) = 0$ and $F_{A,B}(1,1) = 1$. Moreover, $F_{A,B}$ is increasing in each place because $\delta_A = \delta_B$. Finally, notice that, if A and B are semicopulas, then

$$F_{A,B}(x,1) = B(x,1) = x, \quad F_{A,B}(1,x) = A(1,x) = x,$$

for every $x \in [0,1]$. Therefere $F_{A,B}$ has neutral element 1. □

Notice that, if A and B are agops such that $\delta_A \neq \delta_B$, then $F_{A,B}$ need not be increasing. For example, if $A = M$ and $B = \Pi$, then

$$F_{A,B}(0.5, 0.4) = 0.4 > 0.3 = F_{A,B}(0.5, 0.6).$$

In the following results we consider the case in which A and B are copulas or quasi–copulas.

Proposition 5.3.2. *If A and B are quasi–copulas with the same diagonal section, then $F_{A,B}$ is a quasi–copula.*

Proof. In view of Proposition 5.3.1 we have to prove only that $F_{A,B}$ is 1–Lipschitz. Let x, x', y, y' be points in $[0, 1]$. If (x, y) and (x', y') are both in Δ_+ (or Δ_-), then $F_{A,B}$ is obviously 1–Lipschitz. Therefore, suppose that, for example, $(x, y) \in \Delta_+$ and $(x', y') \in \Delta_-$ and, without loss of generality, $x > x'$ and $y < y'$.

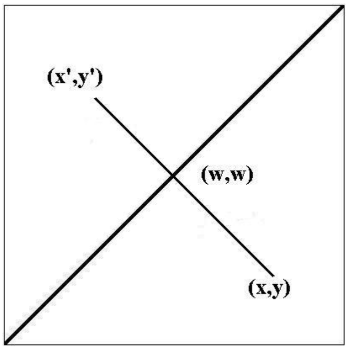

Figure 5.2: Proof of Proposition 5.3.2

Let (w, w) be the point of intersection between the segment line joining (x, y) and (x', y') and the diagonal section of the unit square. We have

$$\begin{aligned}
|F_{A,B}(x, y) - F_{A,B}(x', y')| &\leq |F_{A,B}(x, y) - F_{A,B}(w, w)| + |F_{A,B}(w, w) - F_{A,B}(x', y')| \\
&\leq |A(x, y) - A(w, w)| + |B(w, w) - B(x', y')| \\
&\leq (x - w) + (w - y) + (w - x') + (y' - w) \\
&\leq |x - x'| + |y - y'|.
\end{aligned}$$

The other cases can be proved in an analogous manner. □

Corollary 5.3.1. *If A and B are 1–Lipschitz agops with the same diagonal section, then $F_{A,B}$ is a 1–Lipschitz agop.*

Proposition 5.3.3. *Let A and B be copulas with the same diagonal section. If A and B are symmetric, then $F_{A,B}$ is a copula.*

Proof. In view of Proposition 5.3.1, we have to proof only the 2–increasing property for $F_{A,B}$. On the rectangles entirely contained in either Δ_+ or Δ_-, the rectangular inequality (C2) follows directly from the 2–increasing property of A and B. Therefore, in view of Proposition 1.6.1, it suffices to show that, for all $s, t \in [0,1]$ with $s < t$,

$$V_{F_{A,B}}([s,t]^2) := A(s,s) + A(t,t) - B(s,t) - A(t,s) \geq 0.$$

Because A and B are both symmetric, we have

$$V_{F_{A,B}}([s,t]^2) := \frac{1}{2}\left(V_A([s,t]^2) + V_B([s,t]^2)\right) \geq 0,$$

which concludes the proof. □

Remark 5.3.1. In Proposition 5.3.3, the assumption of the symmetry of the copulas A and B is essential. If, for example, A is a non–symmetric copula, then $F_{A,B}$ need not be a copula. We consider, for example, the copula A given by

$$A(x,y) = \begin{cases} \max\left(x + \frac{1}{2}(y-1), 0\right), & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ \\ \min\left(x + \frac{1}{2}(y-1), y\right), & \text{if } x \in \left]\frac{1}{2}, 1\right]; \end{cases}$$

and the copula B given by

$$B(x,y) := \min\left\{x, y, \frac{\delta_A(x) + \delta_A(y)}{2}\right\}.$$

If $R := [1/3, 2/3]^2$, we have

$$V_{F_{A,B}}(R) = -1/12 < 0,$$

viz. $F_{A,B}$ is not a copula. Specifically, because of Proposition 5.3.2, $F_{A,B}$ is a proper quasi–copula.

Remark 5.3.2. In [108], a general method was described to symmetrize an agop. Specifically, let A be an agop (generally, non-symmetric), for every $x, y \in [0,1]$, the *symmetrized version of* A is defined by

$$\widetilde{A}(x,y) = \begin{cases} A(x,y), & \text{if } x \geq y; \\ A(y,x), & \text{if } x < y. \end{cases} \tag{5.7}$$

Since it is a clear that, if A is an agop (quasi–copula), then the transpose A^T is an agop (quasi–copula), it follows from Proposition 5.3.1 (Proposition 5.3.2) that $\widetilde{A}$ is an agop (quasi–copula). Notice that, given a copula C, $\widetilde{C}$ need not be a copula. We consider, for example, the copula C_λ ($\lambda \in]0,1[$) defined by

$$C_\lambda(x,y) = \begin{cases} y, & \text{if } y \le \lambda x; \\ \lambda x, & \text{if } \lambda x < y \le 1-(1-\lambda)x; \\ x+y-1, & \text{if } 1-(1-\lambda)x < y \le 1. \end{cases}$$

For a fixed $\epsilon \in \left]0, \frac{1}{2}\right[$, we have

$$V_{\widetilde{C_\lambda}}\left(\left[\frac{1}{2}, \frac{1}{2}+\epsilon\right] \times \left[\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon\right]\right) = \frac{\lambda}{2} - \lambda(\frac{1}{2}+\epsilon) < 0.$$

A similar construction method can also be introduced for agops that have the same values in some fixed linear sections ([80]), for example, with the same opposite diagonal sections. In this case, let Γ_+ and Γ_- be the two subsets of the unit square defined by

$$\Gamma_+ := \{(x,y) \in [0,1]^2 : x+y \le 1\}, \qquad \Gamma_- := \{(x,y) \in [0,1]^2 : x+y > 1\}.$$

Given the agops A and B, we introduce the function $F^{A,B} : [0,1]^2 \to [0,1]$ given, for all $x, y \in [0,1]$, by

$$F^{A,B}(x,y) := A(x,y)\, 1_{\Gamma_+}(x,y) + B(x,y)\, 1_{\Gamma_-}(x,y).$$

As above, we have

Proposition 5.3.4. *If A and B are agops with the same opposite diagonal section, then $F^{A,B}$ is an agop. Moreover, if A and B are quasi–copulas, then $F^{A,B}$ is a quasi–copula too.*

Theorem 5.3.1. *Let A and B be copulas with the same opposite diagonal section. If $B(x,y) \ge A(x,y)$ for every $(x,y) \in \Gamma_-$, then $F^{A,B}$ is a copula.*

Chapter 6

A generalization of Archimedean copulas

In this chapter, we introduce and study a class of bivariate copulas that depend on two univariate functions. This new family, which contains the Archimedean copulas (see section 1.6.4), is presented in section 6.1. Several examples are then provided in section 6.2. Section 6.3 is devoted to the study of the concordance order in our class. The same circle of ideas will also enable us to construct and characterize a new family of quasi–copulas (section 6.4).

The contents of this chapter can be also found in the papers [41, 42].

6.1 The new family

We denote by Φ the class of all functions $\varphi : [0,1] \to [0,+\infty]$ that are continuous and strictly decreasing, and by Φ_0 the subset of Φ formed by the functions φ that satisfy $\varphi(1) = 0$. Moreover, we denote by Ψ the class of all functions $\psi : [0,1] \to [0,+\infty]$ that are continuous, decreasing and such that $\psi(1) = 0$. Notice that $\Phi_0 \subset \Psi$.

For all $(\varphi,\psi) \in \Phi \times \Psi$, we introduce the function $C_{\varphi,\psi} : [0,1]^2 \to [0,1]$ defined by

$$C_{\varphi,\psi}(x,y) := \varphi^{[-1]}\left(\varphi(x \wedge y) + \psi(x \vee y)\right). \tag{6.1}$$

Evidently, $C_{\varphi,\psi}$ is symmetric and, by using the properties (1.1) of the pseudo–inverse of a function, it is easily proved that, for all $x \in [0,1]$,

$$C_{\varphi,\psi}(x,1) = \varphi^{[-1]}\left(\varphi(x)\right) = x = C_{\varphi,\psi}(1,x)$$

and

$$0 \le C_{\varphi,\psi}(x,0) = \varphi^{[-1]}\left(\varphi(0) + \psi(x)\right) = C_{\varphi,\psi}(0,x) \le \varphi^{[-1]}\left(\varphi(0)\right) = 0,$$

viz. $C_{\varphi,\psi}$ satisfies the boundary conditions (C1).

Below we shall investigate under which conditions on φ and ψ, the function $C_{\varphi,\psi}$ defined by (6.1) is a copula.

Theorem 6.1.1. *Let φ and ψ belong to Φ and to Ψ, respectively, and let $C = C_{\varphi,\psi}$ be the function defined by* (6.1)*. If φ is convex and $(\psi - \varphi)$ is increasing in $[0,1]$, then C is a copula.*

Proof. Since C satisfies the boundary conditions (C1), it suffices to show that C is 2–increasing. Let $R = [x_1, x_2] \times [y_1, y_2]$ be a rectangle contained in the unit square. We distinguish three cases. If $R \subset \Delta_+$, then

$$\begin{aligned} V_C(R) =& \varphi^{[-1]}\left(\varphi(y_1) + \psi(x_1)\right) + \varphi^{[-1]}\left(\varphi(y_2) + \psi(x_2)\right) \\ & - \varphi^{[-1]}\left(\varphi(y_2) + \psi(x_1)\right) - \varphi^{[-1]}\left(\varphi(y_1) + \psi(x_2)\right). \end{aligned}$$

Set

$$\begin{aligned} s_1 &:= \varphi(y_1) + \psi(x_1), \qquad & s_2 &:= \varphi(y_2) + \psi(x_2), \\ t_1 &:= \varphi(y_2) + \psi(x_1), \qquad & t_2 &:= \varphi(y_1) + \psi(x_2). \end{aligned}$$

Then $(t_1, t_2) \prec (s_1, s_2)$ and $\varphi^{[-1]}$ is convex, because φ is convex. Thus Theorem 1.2.2 implies $V_C(R) \geq 0$.

Since C is symmetric, the same argument yields $V_C(R) \geq 0$ if the rectangle R is entirely contained in the region Δ_-.

Next, consider the case in which the diagonal of R lies on the diagonal of the unit square, viz. $x_1 = y_1$ and $x_2 = y_2$. If $x_1 = 0$, then $V_C(R) = \varphi^{[-1]}\left(\varphi(x_2) + \psi(x_2)\right) \geq 0$. Assume then that $x_1 > 0$. We obtain

$$\begin{aligned} V_C(R) =& \varphi^{[-1]}\left(\varphi(x_1) + \psi(x_1)\right) + \varphi^{[-1]}\left(\varphi(x_2) + \psi(x_2)\right) \\ & - \varphi^{[-1]}\left(\varphi(x_1) + \psi(x_2)\right) - \varphi^{[-1]}\left(\varphi(x_1) + \psi(x_2)\right). \end{aligned}$$

Now, set

$$s_1 := \varphi(x_1) + \psi(x_1), \quad s_2 := \varphi(x_2) + \psi(x_2), \quad t_1 := \varphi(x_1) + \psi(x_2) =: t_2.$$

Since $t \mapsto (\psi(t) - \varphi(t))$ is increasing in $[0,1]$, we obtain

$$\min\{t_1, t_2\} = \varphi(x_1) + \psi(x_2) \geq \varphi(x_2) + \psi(x_2) = \min\{s_1, s_2\}$$

and

$$t_1 + t_2 \geq s_1 + s_2.$$

Therefore $(t_1, t_2) \prec^w (s_1, s_2)$, and since $\varphi^{[-1]}$ is convex and decreasing, from Tomic's Theorem 1.2.3 we have $V_C(R) \geq 0$. By using the Proposition 1.6.1, we have the desired assertion. □

Remark 6.1.1. Notice that, since $t \mapsto (\psi(t) - \varphi(t))$ is increasing, then $\varphi(t) \geq \psi(t)$ for all $t \in [0,1]$. In fact, if there existed $x_0 \in (0,1)$ such that $\varphi(x_0) < \psi(x_0)$, then

$$0 < \psi(x_0) - \varphi(x_0) \leq \psi(1) - \varphi(1) = -\varphi(1) \leq 0,$$

which is a contradiction.

If (φ, ψ) is a pair of functions that generate a copula C of type (6.1), then, for any $c > 0$, also $(c\varphi, c\psi)$ generates C.

Given a copula C of type (6.1) generated by φ and ψ, let h and k be the two functions given by $h(t) := \exp(-\varphi(t))$ and $k(t) := \exp(-\psi(t))$. Then, we have

$$\begin{aligned} C(x,y) &= \varphi^{[-1]}\left(-\ln(h(x \wedge y)) - \ln(k(x \vee y))\right) \\ &= h^{[-1]}\left(\exp\left[\ln(h(x \wedge y)) + \ln(k(x \vee y))\right]\right) \\ &= h^{[-1]}\left(h(x \wedge y) \cdot k(x \vee y)\right). \end{aligned}$$

In particular, Theorem 6.1.1 can be easily reformulated in a multiplicative form.

Theorem 6.1.2. *Let h, k be two continuous and increasing functions from $[0,1]$ into $[0,1]$ such that $k(1) = 1$. If h is log–concave and $t \mapsto h(t)/k(t)$ is increasing, then*

$$C_{h,k}(x,y) := h^{[-1]}\left(h(x \wedge y) \cdot k(x \vee y)\right) \tag{6.2}$$

is a copula.

6.2 Examples

The most important family of copulas of type (6.1) is the Archimedean one. Specifically, given a function φ in Φ_0, $C_{\varphi,\varphi}$ is an Archimedean copula with additive generator φ. In particular, the copulas Π and W are of this type. On account of this fact, a copula of type (6.1) is called *generalized Archimedean copula* (briefly *GA–copula*).

Notice that also the copula M is of type (6.1): it suffices to take $\psi = 0$ and $\varphi \in \Phi$. As a consequence, the family of copulas of type (6.1) is comprehensive, viz. M, Π and W are GA–copulas.

Example 6.2.1. Given an increasing and differentiable function $f : [0,1] \to [0,1]$, let $\varphi(t) = -\ln t$ and $\psi(t) = -\ln f(t)$ be two functions satisfying the assumptions of Theorem 6.1.1. The corresponding copula of type (6.1) is given by

$$C_{\varphi,\psi}(x,y) := (x \wedge y) f(x \vee y),$$

which is a member of the family of copulas studied in chapter 4. In fact, notice that, if $\psi - \varphi$ is increasing, then we can deduce that $tf'(t) \leq f(t)$ on $[0,1]$ and, therefore, f satisfies the assumptions of Theorem 4.1.1.

Example 6.2.2. Let $\delta : [0,1] \to [0,1]$ be in the class $\mathcal{D}_2$ of the diagonals of a copula. Take $\varphi(t) = 1 - t$ and $\psi(t) = t - \delta(t)$. If ψ is decreasing, then $(\psi - \varphi)$ is increasing and Theorem 6.1.1 ensures that the pair (φ, ψ) generates a copula $C_{\varphi,\psi} = C_\delta$ given by

$$C_\delta(x, y) := \max\{0, \delta(x \vee y) - |x - y|\} \qquad \text{for all } x, y \in [0,1].$$

Thus C_δ is a member of the family of MT–copulas, characterized and studied in chapter 5.

Example 6.2.3. Take $\varphi \in \Phi_0$ and $\psi(t) = \alpha\varphi$ for $\alpha \in [0,1]$. Then $\psi \in \Psi$ and the corresponding copula, $C_{\varphi,\psi} = C_\alpha$, is given by

$$\begin{aligned} C_\alpha(x, y) &= \varphi^{[-1]}\left(\varphi(x \wedge y) + \alpha\varphi(x \vee y)\right) \\ &= \varphi^{[-1]}\left((\varphi(x) + \varphi(y)) \cdot A\left(\frac{\varphi(x)}{\varphi(x) + \varphi(y)}\right)\right), \end{aligned}$$

where

$$A(t) = \begin{cases} 1 - (1 - \alpha)t, & t \in [0, 1/2]\,; \\ \alpha + (1 - \alpha)t, & t \in [1/2, 1]\,. \end{cases}$$

Therefore C_α belongs to the Archimax family of copulas presented in Example 1.6.9.

Example 6.2.4. Take $\varphi(t) = -\alpha t + \alpha$ $(\alpha \geq 1)$ and $\psi(t) = 1 - t$. Then the pair (φ, ψ) belongs to $\Phi \times \Psi$ and satisfies the assumptions of Theorem 6.1.1. The corresponding copula, $C_{\varphi,\psi} = C_\alpha$, is given by

$$\begin{aligned} C_\alpha(x, y) &= \max\left\{0, x \wedge y - \frac{1}{\alpha}(1 - x \vee y)\right\} \\ &= \begin{cases} \dfrac{\alpha(x \wedge y) + (x \vee y) - 1}{\alpha}, & \alpha(x \wedge y) + (x \vee y) \geq 1; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The copula C_α has a probability mass $\frac{2}{\alpha+1}$ uniformly distributed on the two segments connecting the point $\left(\frac{1}{\alpha+1}, \frac{1}{\alpha+1}\right)$ with $(0,1)$ and $(1,0)$, respectively, and a probability mass $\frac{\alpha-1}{\alpha+1}$ uniformly distributed on the segment joining the point $\left(\frac{1}{\alpha+1}, \frac{1}{\alpha+1}\right)$ to $(1,1)$ (see also page 57 of [114]). In particular, we obtain $C_1 = W$ and $C_\infty = M$. Notice that this class of copulas has been also used in [29].

Example 6.2.5. Take $\varphi(t) = 1 - t$ and, for $\alpha \in [0,1]$,

$$\psi(t) = \begin{cases} \alpha/2, & \text{if } t \in [0, \alpha/2]\,; \\ \alpha - t, & \text{if } t \in [\alpha/2, \alpha]\,; \\ 0, & \text{if } t \in [\alpha, 1]\,. \end{cases}$$

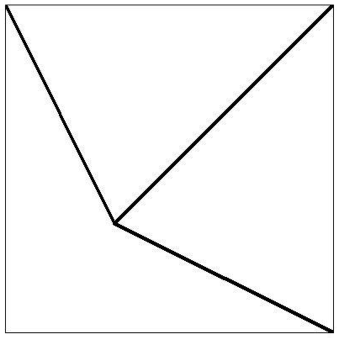

Figure 6.1: Support of the copula C_α ($\alpha = 2$) in Example 6.2.4

Then the pair (φ, ψ) belongs to $\Phi \times \Psi$ and satisfies the assumptions of Theorem 6.1.1. The corresponding copula, $C_{\varphi,\psi} = C_\alpha$, is given by

$$C_\alpha(x,y) := \begin{cases} \max\{0, x+y-\alpha\} & \text{if } (x,y) \in [0,\alpha]^2\,; \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$

Therefore, C_α spreads uniformly the mass on the two segments connecting, respectively, the points $(1,1)$ with (α,α) and $(\alpha,1)$ with $(1,\alpha)$. Notice that C_α is a member of the Mayor–Torrens family (1.5).

Note that copulas of type (6.1) that are ordinal sums of copulas are characterized in the following

Proposition 6.2.1. *The only (non trivial) ordinal sum of copulas that can be expressed in the form* (6.1) *is the ordinal sum* $(\langle 0, a, C\rangle)$, *where* C *is a suitable copula and* $a \in \,]0,1[$.

Proof. It suffices to observe that the set of idempotent elements of C is given by $\{0\} \cup [a,1]$, where $a := \inf\{t \in [0,1] : \psi(t) = 0\}$. In fact, given the copula C, let δ be its diagonal section given by $\delta(t) := C(t,t) = \varphi^{[-1]}\left(\varphi(t) + \psi(t)\right)$. In particular, for all $t \in \,]0,1[$ we have $\delta(t) < t$ if, and only if, $\min\{\varphi(t) + \psi(t), \varphi(0)\} > \varphi(t)$, which is equivalent to $\psi(t) > 0$. Since ψ is decreasing and $\psi(1) = 0$, we have $\delta(t) < t$ if, and only if, t is in $]0,a[$ where $a := \inf\{t \in [0,1] : \psi(t) = 0\}$. □

Theorem 6.1.1 highlights the importance of finding generators in order to construct GA–copulas. To this purpose the following result provides useful methods (the proofs are the same given in [61]).

Theorem 6.2.1. *Let* (φ, ψ) *be a pair in* $\Phi \times \Psi$. *The following statements hold:*

(a) *if* $f : [0,1] \to [0,1]$ *is an increasing and concave bijection, then* $(\varphi \circ f, \psi \circ f)$ *is in* $\Phi \times \Psi$;

(b) *if* $f : [0, +\infty[\to [0, +\infty[$ *is an increasing convex function such that* $f(0) = 0$, *then* $(f \circ \varphi, f \circ \psi)$ *is in* $\Phi \times \Psi$;

(c) *if* $0 < \alpha < 1$, *then* $(\varphi(\alpha t) - \varphi(\alpha), \psi(\alpha t) - \psi(\alpha))$ *is in* $\Phi \times \Psi$.

Notice that additive generators of Archimedean copulas can be combined together in order to construct copulas of the type (6.1). In fact, let φ and ψ belong to Φ_0; in view of Theorem 6.1.1, the convexity of φ and the condition that $(\psi - \varphi)$ be increasing ensure that $C_{\varphi,\psi}$ is a copula. Consider, for instance, the functions $\alpha(t) := 1 - t$, $\beta(t) := -\ln t$ and $\gamma(t) := 1/t - 1$, which are, respectively, the additive generators of the Archimedean copulas W, Π and the *Hamacker copula*

$$\frac{\Pi}{\Sigma - \Pi}(x, y) := \frac{xy}{x + y - xy};$$

then we obtain the following copulas:

$$\begin{aligned} C_{\beta,\alpha}(x, y) &= (x \wedge y) \exp\big((x \vee y) - 1\big), \\ C_{\gamma,\alpha}(x, y) &= \frac{x \wedge y}{1 + (x \wedge y) - xy}, \\ C_{\gamma,\beta}(x, y) &= \frac{x \wedge y}{1 - (x \wedge y) \ln(x \vee y)}. \end{aligned}$$

6.3 Concordance order

The concordance order between two GA–copulas is determined only by the properties of their generators.

Theorem 6.3.1. *Let* C *and* D *be two GA–copulas generated, respectively, by the pairs* (φ, ψ) *and* (γ, η). *Let* $\alpha := \varphi \circ \gamma^{[-1]}$ *and* $\beta := \psi \circ \eta^{[-1]}$. *Then* $C \leq D$ *if, and only if,*

$$\alpha(a + b) \leq \alpha(a) + \beta(b) \qquad \text{for all } a, b \in [0, +\infty]\,. \tag{6.3}$$

Proof. Let x and y be in $[0, 1]$ and suppose, first, that $x \leq y$. Then $C \leq D$ if, and only if,

$$\varphi^{[-1]}\left(\varphi(x) + \psi(y)\right) \leq \gamma^{[-1]}\left(\gamma(x) + \eta(y)\right).$$

Let $\gamma(x) = a$ and $\eta(y) = b$, then the above inequality is equivalent to

$$\varphi^{[-1]}\left(\varphi \circ \gamma^{[-1]}(a) + \psi \circ \eta^{[-1]}(b)\right) \leq \gamma^{[-1]}\left(a + b\right).$$

Applying the function γ to both sides, we obtain

$$\alpha^{[-1]}\left(\alpha(a) + \beta(b)\right) \geq a + b,$$

viz. condition (6.3).

If $x > y$, the proof can be completed by using the same arguments. □

Notice that, if C and D are Archimedean copulas generated, respectively, by φ and γ, then $\alpha = \beta$ and condition (6.3) is equivalent to the subadditivity of α, as stated in Theorem 4.4.2 of [114].

In two particular cases, the concordance order can be expressed in a form simpler than (6.3).

Corollary 6.3.1. *Let C and D be two copulas of type* (6.1) *generated, respectively, by the pairs (φ, ψ) and (γ, η). Let $\alpha := \varphi \circ \gamma^{[-1]}$ and $\beta := \psi \circ \eta^{[-1]}$.*

(a) *If $\varphi = \gamma$ is a strictly decreasing function with $\varphi(0) = +\infty$, then $C \leq D$ if, and only if, $\psi(t) \geq \eta(t)$ for every $t \in [0, 1]$.*

(b) *If $\psi = \eta$ is a strictly decreasing function with $\psi(0) = +\infty$, then $C \leq D$ if, and only if, α is 1–Lipschitz.*

Proof. Since $\varphi = \gamma$ admits an inverse, $\alpha(t) = t$. Therefore condition (6.3) is equivalent to

$$b \leq \beta(b) = \psi \circ \eta^{[-1]}(b) \qquad \text{for all } a, b \in [0, +\infty] .$$

Taking $b := \eta(t)$, we have $C \leq D$ if, and only if, $\psi(t) \geq \eta(t)$ for every $t \in [0, 1]$.

Analogously, for (b), since $\psi = \eta$ admits an inverse, we have $\beta(t) = t$, and (6.3) is equivalent to

$$\alpha(a + b) - \alpha(a) \leq b \qquad \text{for all } a, b \in [0, +\infty] ,$$

as asserted. □

6.4 A similar new class of quasi–copulas

It is interesting to ascertain under which conditions the function $C_{\varphi,\psi}$ defined by (6.1) is a quasi–copula; the following theorem provides a characterization of quasi–copulas in the class $\{C_{\varphi,\psi} : \varphi \in \Phi, \psi \in \Psi\}$.

Theorem 6.4.1. *Let φ and ψ belong to Φ and to Ψ, respectively. Let $C_{\varphi,\psi}$ be the function defined by* (6.1). *Then $C_{\varphi,\psi}$ is a quasi–copula if, and only if, both the following statements hold:*

(a) *for all $r \leq s$ and $t \in \left[0, \left(\psi \circ \varphi^{[-1]}\right)(r)\right]$*

$$\varphi^{[-1]}(r + t) - \varphi^{[-1]}(s + t) \leq \varphi^{[-1]}(r) - \varphi^{[-1]}(s);$$

(b) *for all $r \leq s$ and $t \geq \left(\varphi \circ \psi^{[-1]}\right)(r)$*

$$\varphi^{[-1]}(r + t) - \varphi^{[-1]}(s + t) \leq \psi^{[-1]}(r) - \psi^{[-1]}(s).$$

Proof. We already know that, when φ and ψ belong to Φ and to Ψ respectively, the function $C := C_{\varphi,\psi}$ given by (6.1) satisfies condition (Q1) for a quasi–copula. That both $x \mapsto C(x, y)$ and $y \mapsto C(x, y)$ are increasing functions for every $y \in [0, 1]$ and for every $x \in [0, 1]$, respectively, is an obvious consequence of the fact that both φ and ψ are decreasing functions. Therefore, in order to complete the proof, it suffices to show that the assumptions are equivalent to the Lipschitz condition (Q3) for C.

Assume, first, that $x_1 < x_2 \leq y$. The inequality

$$C(x_2, y) - C(x_1, y) \leq x_2 - x_1 \tag{6.4}$$

is equivalent to

$$\begin{aligned}\varphi^{[-1]}\left(\varphi(x_2) + \psi(y)\right) &- \varphi^{[-1]}\left(\varphi(x_1) + \psi(y)\right) \\ &\leq x_2 - x_1 = \varphi^{[-1]}(\varphi(x_2)) - \varphi^{[-1]}(\varphi(x_1)).\end{aligned}$$

By setting $r := \varphi(x_2)$, $s := \varphi(x_1)$ and $t := \psi(y)$, we obtain that t belongs to $\left[0, \left(\psi \circ \varphi^{[-1]}\right)(r)\right]$ and $r \leq s$; moreover, the last inequality is equivalent to (a).

Next assume $y \leq x_1 < x_2$. The inequality (6.4) is equivalent to

$$\begin{aligned}\varphi^{[-1]}\left(\varphi(y) + \psi(x_2)\right) - \varphi^{[-1]}\left(\varphi(y) + \psi(x_1)\right) &\leq x_2 - x_1 \\ &= \psi^{[-1]}(\psi(x_2)) - \psi^{[-1]}(\psi(x_1)).\end{aligned}$$

By setting $r := \psi(x_2)$, $s := \psi(x_1)$, $t := \varphi(y)$, we have $t \geq \left(\varphi \circ \psi^{[-1]}\right)(s)$ and $r \leq s$. For the arbitrariness of $s \geq r$, it follows that $t \geq \left(\varphi \circ \psi^{[-1]}\right)(r)$ and the last inequality is equivalent to condition (b).

The final case, $x_1 \leq y \leq x_2$, follows from the two previous cases, since

$$\begin{aligned}C(x_2, y) - C(x_1, y) &= C(x_2, y) - C(y, y) + C(y, y) - C(x_1, y) \\ &\leq x_2 - y + y - x_1 = x_2 - x_1,\end{aligned}$$

which concludes the proof. □

Although Theorem 6.4.1 characterizes quasi–copulas of the type (6.1), conditions (a) and (b) may be somewhat impractical. However, these conditions are equivalent to the convexity of φ, when $\varphi = \psi$, as is shown in the following

Corollary 6.4.1. *Let φ belong to Φ_0 and let $C_{\varphi,\varphi}$ be a function of the type* (6.1). *Then $C_{\varphi,\varphi}$ is a quasi–copula if, and only if, φ is convex.*

Proof. By Theorem 6.4.1, $C_{\varphi,\varphi}$ is a quasi–copula if, and only if, for every $r \leq s$ and for every $t \geq 0$, we have

$$\varphi^{[-1]}(r + t) - \varphi^{[-1]}(s + t) \leq \varphi^{[-1]}(r) - \varphi^{[-1]}(s),$$

which can be written in the form

$$\varphi^{[-1]}(r + t) + \varphi^{[-1]}(s) \leq \varphi^{[-1]}(s + t) + \varphi^{[-1]}(r). \tag{6.5}$$

If φ is convex, so is $\varphi^{[-1]}$, and therefore (6.5) follows directly from Theorem 1.2.2, observing that $(r+t,s) \prec (s+t,r)$. Conversely, if (6.5) holds, for all $a,b \geq 0$ we can put

$$r = a, \qquad t = \frac{b-a}{2} \qquad s = \frac{a+b}{2}.$$

Then, we have

$$2\varphi^{[-1]}\left(\frac{a+b}{2}\right) \leq \varphi^{[-1]}(a) + \varphi^{[-1]}(b),$$

viz. $\varphi^{[-1]}$ is mid–convex and, because $\varphi^{[-1]}$ is continuous, it follows that $\varphi^{[-1]}$ is convex, and hence so is φ. □

The following result provides a sufficient condition for $C_{\varphi,\psi}$ to be a quasi–copula.

Proposition 6.4.1. *Let φ and ψ belong to Φ and to Ψ, respectively. If φ is convex, then, for the function $C_{\varphi,\psi}$ defined by* (6.1)*, the following statements are equivalent:*

(a) *$C_{\varphi,\psi}$ is a quasi–copula;*

(b) *for every $\lambda \in [\varphi(1), \varphi(0)]$ the function $\rho_\lambda : \left[\varphi^{[-1]}(\lambda), 1\right] \to \mathbb{R}$ given by*

$$\rho_\lambda(t) := \varphi^{[-1]}\left(\lambda + \psi(t)\right) - t$$

is decreasing.

Proof. From Theorem 6.4.1, it suffices to show that C satisfy the 1–Lipschitz condition (Q3). Assume, first, that $x_1 \leq x_2 \leq y$. The inequality

$$C(x_2, y) - C(x_1, y) \leq x_2 - x_1 \tag{6.6}$$

is equivalent to

$$\varphi^{[-1]}\left(\varphi(x_2) + \psi(y)\right) + \varphi^{[-1]}(\varphi(x_1)) \leq \varphi^{[-1]}\left(\varphi(x_1) + \psi(y)\right) + \varphi^{[-1]}(\varphi(x_2)).$$

By setting $s_1 := \varphi(x_2) + \psi(y)$, $s_2 := \varphi(x_1)$, $t_1 := \varphi(x_1) + \psi(y)$ and $t_2 := \varphi(x_2)$, we have $(s_1, s_2) \prec (t_1, t_2)$ and therefore, since $\varphi^{[-1]}$ is convex, Theorem 1.2.2 ensures that (6.6) is satisfied. In this case, the Lipschitz condition is a consequence of the convexity of φ alone.

Next assume $y \leq x_1 < x_2$. The inequality (6.6) is equivalent to

$$\varphi^{[-1]}\left(\varphi(y) + \psi(x_2)\right) - \varphi^{[-1]}\left(\varphi(y) + \psi(x_1)\right) \leq x_2 - x_1;$$

viz. condition (b).

The final case, $x_1 \leq y \leq x_2$, follows from the two previous cases. □

Example 6.4.1. Take the functions

$$\varphi(t) := -\ln t \quad \text{and} \quad \psi(t) := -\ln\left(t + t^2 - t^3\right).$$

For every $\lambda \in [0, +\infty]$ the function $\rho_\lambda : [\exp(-\lambda), 1] \to \mathbb{R}$ is given by

$$\rho_\lambda(t) := \exp(-\lambda)\left(t + t^2 - t^3\right) - t.$$

Now, (φ, ψ) is in $(\Phi \times \Psi)$ and ρ_λ is decreasing, therefore Theorem 6.4.1 ensures that the function $C_{\varphi,\psi}$, given by (6.1) is a quasi–copula. Notice that $C_{\varphi,\psi}$ is not a copula, as shown in Example 4.3.1. This implies that the family $\{C_{\varphi,\psi} : \varphi \in \Phi, \psi \in \Psi\}$, where φ and ψ satisfy conditions (a) and (b) of Theorem 6.4.1, contains proper quasi–copulas.

Chapter 7

Binary operations on bivariate d.f.'s

Let H be a binary operation on $[0, 1]$ and let Δ^2 be the set of bivariate d.f.'s. A binary operation η on Δ^2 is said to be *induced pointwise* by H if, for all A and B in Δ^2 and for all $(x, y) \in \overline{\mathbb{R}}^2$,

$$\eta(A, B)(x, y) = H(A(x, y), B(x, y)). \tag{7.1}$$

The function $\eta(A, B) : [0, 1]^2 \to [0, 1]$ given by (7.1) is called *composition* of A and B via H.

The major result of this chapter is the characterization of the induced pointwise operations on the set Δ^2 (section 7.2). A similar operation has been studied, in the univariate case, by C. Alsina *et al.* ([4]) in order to solve some problems arising in the theory of probabilistic metric spaces. However, in the bivariate case, the characterization is quite different and involves the new notion of "P–increasing function", a generalization of the 2–increasing functions, here introduced and studied (section 7.1). Section 7.3 is devoted mainly to questions related to the Fréchet classes and the convergence of d.f.'s. We conclude with some remarks of this problem on the class of copulas (section 7.4). These results can be also found in [45, 48, 38].

7.1 P–increasing functions

The focus of this section is on the new concept of P–increasing function, which will be needed for the characterization of induced pointwise operations on bivariate d.f.'s.

Definition 7.1.1. A function $H : [0,1]^2 \to [0,1]$ is said to be P–*increasing* (i.e. *probabilistically increasing*) if, and only if,

$$H(s_1,t_1) + H(s_4,t_4) \geq \max\left[H(s_2,t_2) + H(s_3,t_3), H(s_3,t_2) + H(s_2,t_3)\right], \tag{7.2}$$

for all $s_i, t_i \in [0,1]$ ($i \in \{1,2,3,4\}$) such that

$$s_1 \leq s_2 \wedge s_3 \leq s_2 \vee s_3 \leq s_4, \qquad t_1 \leq t_2 \wedge t_3 \leq t_2 \vee t_3 \leq t_4, \tag{7.3}$$

$$s_1 + s_4 \geq s_2 + s_3, \qquad t_1 + t_4 \geq t_2 + t_3. \tag{7.4}$$

Here we present a geometric interpretation of the P–increasing property.

Given s_i, t_i ($i \in \{1,2,3,4\}$) as in Definition 7.1.1, let

$$u_1 := s_2 \wedge s_3, \qquad u_4 := s_2 \vee s_3, \quad v_1 := t_2 \wedge t_3, \qquad v_4 := t_2 \vee t_3.$$

Set

$$\begin{array}{llll} \mathbf{p} = (s_1,t_1), & \mathbf{q} = (s_4,t_1), & \mathbf{r} = (s_4,t_4), & \mathbf{s} = (s_1,t_4) \\ \mathbf{p'} = (u_1,v_1), & \mathbf{q'} = (u_4,v_1), & \mathbf{r'} = (u_4,v_4), & \mathbf{s'} = (u_1,v_4) \end{array}$$

Consider the rectangle R_1 with vertices **p**, **q**, **r** and **s**, and the rectangle R_2 with vertices **p'**, **q'**, **r'** and **s'**. Hence $R_2 \subseteq R_1$ and conditions (7.3) and (7.4) imply that the centre of R_2 lies below and to the left of the centre of R_1 (unless $R_1 = R_2$).

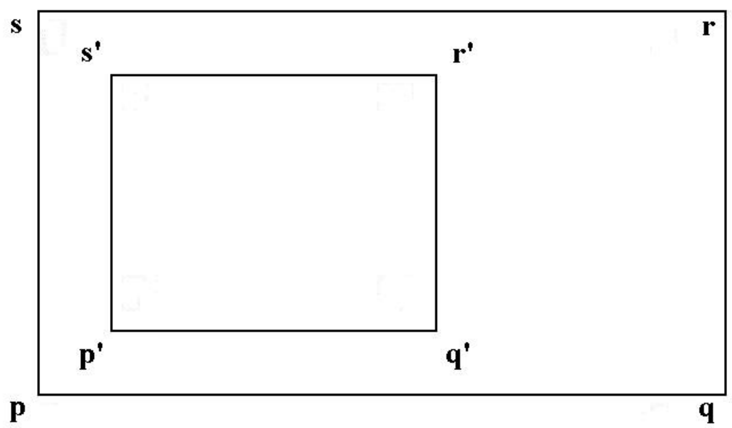

Figure 7.1: Geometric interpretation of the P–increasing property

Now, there are four choices for (u_1, v_1) – namely (s_2,t_2), (s_2,t_3), (s_3,t_2) and (s_3,t_3) – each leading to corresponding choices for the other vertices of R_2. For example, if

$(u_1, v_1) = (s_2, t_2)$ then $(u_4, v_4) = (s_3, t_3)$, and so on. In each case, (7.2) yields the two inequalities

$$H(\mathbf{p}) + H(\mathbf{r}) \geq H(\mathbf{p'}) + H(\mathbf{r'}),$$
$$H(\mathbf{p}) + H(\mathbf{r}) \geq H(\mathbf{q'}) + H(\mathbf{s'}).$$

In particular, when $R_1 = R_2$, the above inequalities establish that the P–increasing property implies the 2–increasing property.

Remark 7.1.1. Notice that conditions (7.3) and (7.4) on the points s_i and t_i ($i = 1, 2, 3, 4$) ensure that $(s_2, s_3) \prec_w (s_1, s_4)$ and $(t_2, t_3) \prec_w (t_1, t_4)$.

Remark 7.1.2. In the sequel, in order to prove that a function H is P–increasing, we restrict ourselves to showing that, for all s_i, t_i as in Definition 7.1.1,

$$H(s_1, t_1) + H(s_4, t_4) \geq H(s_2, t_2) + H(s_3, t_3), \tag{7.5}$$

instead of inequality (7.2) that can be easily obtained by means of a relabelling of the points. In fact, this was the primary definition of P–increasing function (see [45]). The equivalent definition given above was suggested by A. Sklar in a personal communication and it is adopted here because of its straightforward geometrical interpretation.

The P–increasing property is connected with the property of being directionally convex ([147, 111, 99]). We recall that a function $H : [0,1]^2 \to [0,1]$ is called *directionally convex* if, for all s_i, t_i ($i \in \{1, 2, 3, 4\}$) in $[0,1]$ such that (7.3) holds together with the condition, stronger than (7.4),

$$s_1 + s_4 = s_2 + s_3, \qquad t_1 + t_4 = t_2 + t_3, \tag{7.6}$$

we have

$$H(s_1, t_1) + H(s_4, t_4) \geq H(s_2, t_2) + H(s_3, t_3).$$

Theorem 7.1.1. *For a function $H : [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(a) *H is P–increasing;*

(b) *H is directionally convex and increasing in each place.*

Proof. (a) $\Longrightarrow$ (b): Given a P–increasing function H, it suffices to show that H is increasing in each place. Consider $b \in [0,1]$ and, for all $i \in \{1, 2, 3, 4\}$, take s_i and t_i as in Definition 7.1.1, but satisfying the further conditions $s_1 = s_2$ and $t_i = b$. Hence

$$H(s_4, b) - H(s_3, b) - H(s_2, b) + H(s_2, b) \geq 0,$$

from which $H(s_4, b) \geq H(s_3, b)$, viz. $t \mapsto H(t, b)$ is increasing. The isotony of H in the other variable is established in an analogous manner.

(b) $\Longrightarrow$ (a): Let the s_1's and the t_i's ($i \in \{1, 2, 3, 4\}$) be as in Definition 7.1.1 and choose v_4 and w_4 in $[0, 1]$ such that $v_4 \in [s_2 \vee s_3, s_4]$, $w_4 \in [t_2 \vee t_3, t_4]$ and

$$s_1 + v_4 = s_2 + s_3, \qquad t_1 + w_4 = t_2 + t_3.$$

Hence

$$H(s_2, t_2) + H(s_3, t_3) \leq H(s_1, t_1) + H(v_4, w_4) \leq H(s_1, t_1) + H(s_4, t_4),$$

which is the desired conclusion. □

In particular, by using a characterization of the directionally convex functions ([111, Theorem 2.5]), we can obtain the following

Theorem 7.1.2. *A function $H : [0, 1]^2 \to [0, 1]$ is P–increasing if, and only if, the following statements hold:*

(a) *H is 2–increasing;*

(b) *H is increasing in each place;*

(c) *H is convex in each place.*

Note that the convex combinations of two P–increasing functions are P–increasing.

Corollary 7.1.1. *Let $H : [0, 1]^2 \to [0, 1]$ be P–increasing. The following statements hold:*

(a) *H is jointly continuous on $[0, 1[^2$;*

(b) $H \leq \Pi$.

Proof. (a): By classical properties of convex functions, it follows that every P–increasing function $H : [0, 1]^2 \to [0, 1]$ is continuous in each variable on $[0, 1[$ and then, in view of Proposition 2.1.2, it is jointly continuous on $[0, 1[^2$.

(b) If there exists (x_0, y_0) in $]0, 1[$ such that $H(x_0, y_0) > x_0 y_0$, then the horizontal section of H at y_0 is not be convex and, thus, H is not be P–increasing. □

Corollary 7.1.2. *Let $H : [0, 1]^2 \to \mathbb{R}$ be twice differentiable. Then H is P–increasing if, and only if, all the derivatives of the first and the second order of H are greater than (or equal to) 0 on $[0, 1]^2$.*

Example 7.1.1. The copulas Π and W are P–increasing, and so is their convex sum $C_\alpha = \alpha\Pi + (1 - \alpha)W$. But, the copula M is not P–increasing; in fact, if we consider s_i and t_i in $[0, 1]$ ($i \in \{1, 2, 3, 4\}$) such that

$$s_1 = 2/10 \leq s_2 = 3/10 = s_3 \leq s_4 = 5/10,$$
$$t_1 = 0 \leq t_2 = 3/10 = t_3 \leq t_4 = 1,$$

then

$$M(2/10, 0) - M(3/10, 3/10) - M(3/10, 3/10) + M(5/10, 1) = -1/10 < 0.$$

Notice that P–increasing copulas are associated with a random pair (X, Y) that is both SD$(X|Y)$ and SD$(Y|X)$ (see Proposition 1.7.3). For example, we can consider the family of copulas given, for every $\alpha \in]-1, 0]$, by

$$C_\alpha(x, y) = xy + \alpha xy(1-x)(1-y),$$

which is a subclass of the FGM class (see Example 1.6.3).

Important examples of P–increasing functions are given by the following result.

Proposition 7.1.1. *Let f and g be increasing and convex functions from $[0, 1]$ into $[0, 1]$. Let $H : [0, 1]^2 \to [0, 1]$ be P–increasing. Then, the function $H_{f,g}$ defined by*

$$H_{f,g}(x, y) := H(f(x), g(y))$$

is P–increasing.

Proof. From Proposition 3.2.1, it follows that the function $H_{f,g}$ is a 2–increasing agop. Moreover, every horizontal (resp., vertical) section of H is convex, because it is composition of the convex and increasing horizontal (resp., vertical) section of A with f (resp. g). Now, the desired assertion follows from Theorem 7.1.2. □

Example 7.1.2. For every $\alpha, \beta \geq 1$, $\Lambda_{\alpha,\beta}(x, y) := \lambda x^\alpha + (1-\lambda) y^\beta$ ($\lambda \in [0, 1]$) and $\Pi_{\alpha,\beta}(x, y) := x^\alpha \cdot y^\beta$ are P–increasing. In particular, the weighted arithmetic mean is P–increasing, but it is not the case of the weighted geometric mean. Consider, for instance, s_i and t_i in $[0, 1]$ ($i \in \{1, 2, 3, 4\}$) given by

$$s_1 = 0 < s_2 = \frac{4}{10} = s_3 < s_4 = \frac{8}{10}, \qquad t_1 = \frac{4}{10} < t_2 = \frac{7}{10} = t_3 < t_4 = 1,$$

then

$$\sqrt{s_1\, t_1} + \sqrt{s_4\, t_4} - \sqrt{s_2\, t_2} - \sqrt{s_3\, t_3} = \frac{\sqrt{80}}{10} - \frac{\sqrt{112}}{10} < 0.$$

7.2 Induced pointwise operations on d.f.'s

Here we characterize the induced pointwise operations on Δ^2.

Lemma 7.2.1. *If H is a 2–increasing agop, then, for all s, s', t, t' in $[0, 1]$, it satisfies the condition*

$$|H(s', t') - H(s, t)| \leq |H(s', 1) - H(s, 1)| + |H(1, t') - H(1, t)|\,.$$

Family	Parameters
$\Lambda_{\alpha,\beta}(x,y) := \lambda x^\alpha + (1-\lambda)y^\beta$	$\alpha,\beta \geq 1$
$\Pi_{\alpha,\beta}(x,y) := x^\alpha \cdot y^\beta$	$\alpha,\beta \geq 1$
$F_\alpha(x,y) := \alpha xy + (1-\alpha)\max\{x+y-1,0\}$	$\alpha \in [0,1]$
$G_\alpha(x,y) := xy + \alpha xy(1-x)(1-y)$	$\alpha \in [-1,0]$
$S_\alpha(x,y) := xy + \alpha \frac{\sin \pi x}{x} \frac{\sin \pi y}{y}$	$\alpha \in [-1,0]$
$M_\alpha(x,y) := xy + \alpha \min\{x,1-x\}\min\{y,1-y\}$	$\alpha \in [-1,0]$

Table 7.1: Family of P–increasing functions

Proof. Let s and s' be in $[0,1]$ with $s \leq s'$. Then, for every $t \in [0,1]$,

$$H(s',1) - H(s,1) \geq H(s',t) - H(s,t).$$

Similarly, for all $s \in [0,1]$ and for t and t' in $[0,1]$, with $t \leq t'$,

$$H(1,t') - H(1,t) \geq H(s,t') - H(s,t).$$

Therefore, for all s, s', t, t' in $[0,1]$, we have

$$\begin{aligned}|H(s',t') - H(s,t)| &\leq |H(s',t') - H(s,t')| + |H(s,t') - H(s,t)| \\ &\leq |H(s',1) - H(s,1)| + |H(1,t') - H(1,t)|.\end{aligned}$$

□

Theorem 7.2.1. *For a function $H : [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(a) *H induces pointwise a binary operation η on Δ^2;*

(b) *H fulfils the conditions*

(b.1) *$H(0,0) = 0$ and $H(1,1) = 1$,*

(b.2) *H is P–increasing,*

(b.3) *H is left–continuous in each place.*

Proof. (a) $\Longrightarrow$ (b): Let H induce pointwise the binary operation η on Δ^2, viz. for all A and B in Δ^2 and $(x,y) \in \overline{\mathbb{R}}^2$, the function

$$\eta(A,B)(x,y) := H(A(x,y), B(x,y))$$

is in Δ^2. For all 2–d.f.'s A and B we have

$$H(0,0) = H\left(A(x,-\infty), B(x,-\infty)\right) = \eta(A,B)(x,-\infty) = 0$$

and

$$H(1,1) = H\left(A(+\infty,+\infty), B(+\infty,+\infty)\right) = \eta(A,B)(+\infty,+\infty) = 1.$$

Let s_i and t_i be in $[0,1]$ ($i \in \{1,2,3,4\}$) such that (7.3) and (7.4) hold. Hence, there exist two d.f.'s A and B in Δ^2 and four points x_1, x_2, y_1, y_2 in $\overline{\mathbb{R}}$, with $x_1 \le x_2$ and $y_1 \le y_2$, such that

$$\begin{aligned} s_1 &= A(x_1,y_1), \quad s_2 = A(x_1,y_2), \quad s_3 = A(x_2,y_1), \quad s_4 = A(x_2,y_2), \\ t_1 &= B(x_1,y_1), \quad t_2 = B(x_1,y_2), \quad t_3 = B(x_2,y_1), \quad t_4 = B(x_2,y_2). \end{aligned}$$

Since $\eta(A,B)$ is 2–increasing,

$$\eta(A,B)(x_1,y_1) + \eta(A,B)(x_2,y_2) - \eta(A,B)(x_1,y_2) - \eta(A,B)(x_2,y_1) \ge 0,$$

which, with the above positions, is equivalent to

$$H(s_1,t_1) + H(s_4,t_4) \ge H(s_2,t_2) + H(s_3,t_3).$$

But we may exchange s_2 and s_3 and find a bivariate d.f. A' such that

$$s_1 = A'(x_1,y_1), \quad s_3 = A'(x_1,y_2), \quad s_2 = A'(x_2,y_1), \quad s_4 = A'(x_2,y_2).$$

Hence, with B unchanged, we have

$$H(s_1,t_1) + H(s_4,t_4) \ge H(s_3,t_2) + H(s_2,t_3),$$

from which it follows (7.2).

In order to prove (b.3), let s be any point in $[0,1]$ and let $\{s_n\}$ be any sequence in $[0,1]$ that increases to s, $s_n \uparrow s$. Let A and B be in Δ^2 such that (i) the margin $F(x) := A(x,+\infty)$ of A is continuous and strictly increasing and (ii) the margin $G(x) := B(x,+\infty)$ of B is constant on $\mathbb{R}$ and equal to t, $G(x) = t$ for all $x \in \mathbb{R}$. Thus the sequence $\{x_n\}$, where $x_n := F^{-1}(s_n)$ for all $n \in \mathbb{N}$, converges to $x := F^{-1}(s)$, $x_n \uparrow x$. Now, for all $t \in [0,1]$

$$\begin{aligned} H(s_n,t) &= H\left(F(x_n), G(x_n)\right) = H\left(A(x_n,+\infty), B(x_n,+\infty)\right) \\ &= \eta(A,B)(x_n,+\infty) \xrightarrow{n\to+\infty} \eta(A,B)(x,+\infty) \\ &= H\left(A(x,+\infty), B(x,+\infty)\right) = H\left(F(x), G(x)\right) = H(s,t). \end{aligned}$$

In an analogous manner, the function $t \mapsto \eta(A,B)(s,t)$ is proved to be left–continuous for all $s \in [0,1]$.

(b) $\Longrightarrow$ (a): Let H satisfy conditions (b.1) through (b.3) and define an operation η on Δ^2 via

$$\eta(A,B)(x,y) := H(A(x,y), B(x,y)) \qquad \text{for all } A, B \in \Delta^2.$$

It is a straightforward matter to verify that $\eta(A,B)$ thus defined satisfies the boundary conditions $\eta(A,B)(+\infty,+\infty)=1$, and $\eta(A,B)(t,-\infty)=0=\eta(A,B)(-\infty,t)$ for all $t\in\mathbb{R}$. Moreover, given x,x',y,y' in $\mathbb{R}$ with $x\le x'$ and $y\le y'$, we have

$$\begin{aligned}\eta(A,B)(x',y')-\eta(A,B)(x',y)-\eta(A,B)(x,y')+\eta(A,B)(x,y)\\ =H(A(x',y'),B(x',y'))-H(A(x',y),B(x',y))\\ -H(A(x,y'),B(x,y'))+H(A(x,y),B(x,y)).\end{aligned}$$

Now, take

$$\begin{aligned}&s_1=A(x,y),\quad s_2=A(x',y),\quad s_3=A(x,y'),\quad s_4=A(x',y')\\ &t_1=B(x,y),\quad t_2=B(x',y),\quad t_3=B(x,y'),\quad t_4=B(x',y');\end{aligned}$$

then s_i and t_i ($i\in\{1,2,3,4\}$) satisfy (7.3) and (7.4) and, because H is P–increasing, it follows that $\eta(A,B)$ is 2-increasing. Thus it remains to verify that $\eta(A,B)$ is left–continuous in each variable. Let x be in $\mathbb{R}$, let y be any point in $\overline{\mathbb{R}}$, and let $\{x_n\}$ be a sequence of reals such that $x_n\uparrow x$. Hence

$$\begin{aligned}&|\eta(A,B)(x_n,y)-\eta(A,B)(x,y)|\\ &\qquad=|H\left(A(x_n,y),B(x_n,y)\right)-H\left(A(x,y),B(x,y)\right)|\xrightarrow[n\to+\infty]{}0,\end{aligned}$$

since $s\mapsto A(s,y)$ and $s\mapsto B(s,y)$ are left–continuous and Proposition 2.1.2 holds. In an analogous manner, $t\mapsto\eta(A,B)(x,t)$ is proved to be left–continuous for all $x\in\overline{\mathbb{R}}$. This completes the proof. □

The class of all functions that induce pointwise a binary operation on Δ^2 shall be denoted by $\mathcal{P}$. In particular, notice that if H is in $\mathcal{P}$, then H is a binary aggregation operator.

Theorem 7.2.1 is similar to the characterization of induced pointwise operations on Δ, which is reproduced here (see [4]).

Theorem 7.2.2. *For a function $H:[0,1]^2\to[0,1]$ the following statements are equivalent:*

(a') *H induces pointwise a binary operation η on Δ, viz. for every F and G in Δ, $\eta(F,G)(t):=H(F(t),G(t))$ is a d.f.;*

(b') *H fulfils the conditions*

(b.1') *$H(0,0)=0$ and $H(1,1)=1$,*

(b.2') *H is increasing in each variable,*

(b.3') *H is left–continuous in each place.*

Because every P–increasing function satisfies (b.2') (see section 7.1), every function in $\mathcal{P}$ induces pointwise also a binary operation on Δ.

7.3 Some connected questions

Let A and B be bivariate d.f.'s defined for all $x, y \in \overline{\mathbb{R}}$ by

$$A(x,y) = C\left(F_1(x), G_1(y)\right) \qquad \text{and} \qquad B(x,y) = D\left(F_2(x), G_2(y)\right),$$

where F_i, G_i $(i = 1, 2)$ are their respective margins and C and D are their respective copulas (we adopt, if necessary, the method of bilinear interpolation in order to single out one copula, see [140]). In other words, A and B are, respectively, in the Fréchet classes $\Gamma(F_1, G_1)$ and $\Gamma(F_2, G_2)$. If H is in $\mathcal{P}$, we can obtain some information on the margins of the pointwise induced d.f. $\eta(A, B)$ defined as in (7.1).

Proposition 7.3.1. *Under the above assumptions, $\eta(A, B)$ belongs to the Fréchet class determined by the (unidimensional) d.f.'s*

$$x \mapsto H(F_1(x), F_2(x)) \quad \textit{and} \quad y \mapsto H(G_1(y), G_2(y)).$$

Proof. For all $x, y \in \overline{\mathbb{R}}$, we have

$$\eta(A, B)(x, +\infty) = H(A(x, +\infty), B(x, +\infty)) = H(F_1(x), F_2(x)),$$

and, analogously,

$$\eta(A, B)(+\infty, y) = H(A(+\infty, y), B(+\infty, y)) = H(G_1(y), G_2(y)),$$

as claimed. □

Moreover, if H satisfies the assumptions of Theorem 7.2.1 and, then, it induces pointwise a binary operation η on Δ^2, it is entirely natural to ask whether anything may be said about the copula $\widetilde{C}$ of $\eta(A, B)$ for all A and B in Δ^2.

Proposition 7.3.2. *Under the above assumptions, if $F_1 = F_2 = F$, $G_1 = G_2 = G$ and H is idempotent, then $\widetilde{C}(x, y) = H(C(x, y), D(x, y))$.*

Proof. For every H in the Fréchet class $\Gamma(F, G)$, $(x, y) \mapsto H(A(x, y), B(x, y))$ is a bivariate d.f. with marginal d.f.'s given by

$$H(F(x), F(x)) = F(x) \qquad \text{and} \qquad H(G(y), G(y)) = G(y).$$

It follows that there exists a copula $\widetilde{C}$ such that

$$\widetilde{C}\left(F(x), G(y)\right) = H\left(A(x, y), B(x, y)\right) = H\left[C(F(x), G(y)), D(F(x), G(y))\right],$$

from which an argument similar to that used in the proof of Sklar's theorem ([114]) yields $\widetilde{C}(s, t) = H\left(C(s, t), D(s, t)\right)$ for all $s, t \in [0, 1]$. □

In general, when $F_1 \neq F_2$ and $G_1 \neq G_2$, the above result is not true.

Example 7.3.1. Let $H(x, y) = \lambda x + (1 - \lambda)y$ be the weighted arithmetic mean and let $C = D = \Pi$ be the product copula, then, for $\lambda \in]0, 1[$, we have

$$\begin{aligned} H(A(x,y), B(x,y)) &= \lambda F_1(x)G_1(y) + (1-\lambda)F_2(x)G_2(y) \\ &\neq [\lambda F_1(x) + (1-\lambda)F_2(x)]\,[\lambda G_1(y) + (1-\lambda)G_2(y)] \\ &= \Pi(H(F_1(x), F_2(x)), H(G_1(y), G_2(y))). \end{aligned}$$

We conclude this section with a remark on the convergence in Δ^2. Assume that $\{A_n\}$ and $\{B_n\}$ are two sequences of d.f.'s in Δ^2 that converge weakly to the d.f.'s A and B, respectively; in other words, if $C(A)$ and $C(B)$ are the dense subsets of $\overline{\mathbb{R}}^2$ formed by the points of continuity of A and B, respectively, then

$$\forall (x,y) \in C(A) \qquad \lim_{n\to+\infty} A_n(x,y) = A(x,y),$$

and

$$\forall (x,y) \in C(B) \qquad \lim_{n\to+\infty} B_n(x,y) = B(x,y).$$

The question naturally arises of whether, for $H \in \mathcal{P}$ that induces the operation η on Δ^2, the sequence of bivariate d.f.'s $\{\eta(A_n, B_n)\}$ converges weakly to $\eta(A, B)$. While we do not know a general answer to this question, the following result provides a useful sufficient condition.

Theorem 7.3.1. *Under the conditions just specified, if H is continuous in each place, then the sequence $\{\eta(A_n, B_n)\}_{n\in\mathbb{N}}$ converges weakly to $\eta(A, B)$.*

Proof. The set $C(A) \cap C(B)$ is dense in $\overline{\mathbb{R}}^2$. For every point (x, y) in $C(A) \cap C(B)$

$$A_n(x,y) \xrightarrow[n\to+\infty]{} A(x,y) \qquad \text{and} \qquad B_n(x,y) \xrightarrow[n\to+\infty]{} B(x,y).$$

In view of Lemma (7.2.1), we have

$$\begin{aligned} &|\eta(A_n, B_n)(x,y) \;-\; \eta(A,B)(x,y)| \\ &= |H\left(A_n(x,y), B_n(x,y)\right) - H\left(A(x,y), B(x,y)\right)| \\ &\leq |H\left(A_n(x,y), 1\right) - H\left(A(x,y), 1\right)| + |H\left(1, B_n(x,y)\right) - H\left(1, B(x,y)\right)|\,. \end{aligned}$$

The assertion now follows directly from the continuity of H. □

7.4 Remarks on the composition of copulas

Since every copula is also the restriction of a bivariate d.f. to the unit square, it is natural to study also induced pointwise binary operations on $\mathcal{C}$. Note that the function $H(x, y) = \lambda x + (1 - \lambda)y$ induces pointwise a binary operation on $\mathcal{C}$, which is a convex set.

Proposition 7.4.1. *If $H : [0,1]^2 \to [0,1]$ induces pointwise a binary operation ρ on $\mathcal{C}$, then H is idempotent.*

Proof. Suppose that there exists a binary aggregation operator H that induces pointwise a binary operation ρ on $\mathcal{C}$, namely, for all A and B in $\mathcal{C}$,

$$\rho(A,B)(x,y) = H(A(x,y), B(x,y))$$

is a copula. It can be easily proved that $\rho(A,B)$ satisfies the boundary conditions (C1) if, and only if, $H(x,x) = x$ for all x in $[0,1]$. □

In particular, *no copula induces pointwise a binary operation on* $\mathcal{C}$: in fact, M is the only idempotent copula but the minimum of two copulas need not be a copula (see Example 2.3.2).

Because the P–increasing property preserves the 2–increasing property, we have that, if H is a P–increasing and idempotent agop, then H induces pointwise a binary operation on copulas. However, this procedure is not useful in view of the following result.

Proposition 7.4.2. *Let A be a binary aggregation operator such that $A(x,x) \geq x$ for every $x \in [0,1]$. Then A is P–increasing if, and only if, there exists $a \in [0,1]$ such that $A(x,y) = ax + (1-a)y$.*

Proof. Let A be a P–increasing agop such that $A(x,x) \geq x$ for every $x \in [0,1]$. In particular, on account of Theorem 7.1.2, A is 2–increasing and its horizontal and vertical sections are convex. Set $a := A(1,0)$ and $b := A(0,1)$ and notice that $a+b \leq 1$.

In view of the 2–increasing property, for every $y \in [0,1]$ we have

$$A(0,y) + A(y,1) \geq A(y,y) + A(0,1) \geq y + b, \tag{7.7}$$

and, from the convexity of $y \mapsto A(0,y)$,

$$A(0,y) \leq yA(0,1) + (1-y)A(0,0) = by.$$

Therefore, connecting the two inequalities above, we obtain $A(y,1) \geq y + (1-y)b$. On the other hand, from the convexity of $y \mapsto A(y,1)$,

$$A(y,1) \leq yA(1,1) + (1-y)A(0,1) = y + (1-y)b,$$

viz. $A(y,1) = y + (1-y)b$. Analogously $A(1,y) = (1-a)y + a$.

From (7.7), it follows also that

$$A(0,y) \geq y + b - (1-b)y - b = by$$

and, because $A(0,y) \leq yA(0,1) = by$, we have $A(0,y) = by$. In the same manner, $A(x,0) = ax$.

Now, because A is 2–increasing, for every $y \geq x$, we have

$$A(x,y) \geq A(x,1) + A(y,y) - A(y,1) \geq (1-b)x + by$$

and

$$A(x,y) \leq A(x,1) + A(0,y) - b = (1-b)x + by,$$

viz. $A(x,y) = (1-b)x + by$. In the same manner, for every $x \geq y$, we obtain $A(x,y) = ax + (1-a)y$.

Finally, notice that

$$A(x,1/2) = \begin{cases} (1-b)x + b/2, & \text{if } x \leq 1/2; \\ ax + (1-a)/2, & \text{if } x > 1/2; \end{cases}$$

and, from the convexity of $x \mapsto A(x,1/2)$, we have

$$A\left(\frac{1}{2},\frac{1}{2}\right) \leq \frac{1}{2}A\left(0,\frac{1}{2}\right) + \frac{1}{2}A\left(1,\frac{1}{2}\right),$$

which is equivalent to $a + b \geq 1$. Therefore $a + b = 1$ and, for every $(x,y) \in [0,1]^2$, $A(x,y) = ax + (1-a)y$. □

Corollary 7.4.1. *Let A be a P–increasing agop. The following statements are equivalent:*

(a) *A is idempotent;*

(b) *there exists $a \in [0,1]$ such that $A(x,y) = ax + (1-a)y$.*

Thus, in the class of copulas, the characterization of induced pointwise operation is still an open problem.

Chapter 8

Generalized composition of aggregation operators

Let $\mathcal{A}$ be the class of binary aggregation operators (=agops). In this section, we denote by Θ the class of all increasing functions $f : [0,1] \to [0,1]$. Given f_1, f_2, g_1 and g_2 in Θ and a binary operation H on $[0,1]$, let F be the mapping defined on $[0,1]^2$ by

$$F(x,y) := H\left(A(f_1(x), g_1(y)), B(f_2(x), g_2(y))\right), \tag{8.1}$$

for all A and B in $\mathcal{A}$. The function F is called *generalized composition* of (A,B) with respect to the 5–ple (f_1, g_1, f_2, g_2, H), which is called *generating system*. The prefix "generalized" is used here to distinguish the function F from the classical composition that is obtained when $f_1 = g_1 = f_2 = g_2 = \text{id}_{[0,1]}$, and already studied for agops (see, for instance, [10, 90]).

This chapter aims to establish which conditions on the generating system ensure that, for every choice of A and B in a given subset $\mathcal{B} \subseteq \mathcal{A}$ (for instance, $\mathcal{B}$ is the set of copulas, semicopulas, etc.), F is also an agop belonging to $\mathcal{B}$. Thus, in section 8.1 we analyse the case of agops and sections 8.2, 8.3 and 8.4 are devoted, respectively, to the study of generalized composition in the class of semicopulas, 1–Lipschitz and 2–increasing agops. The case of copulas is considered in section 8.5, where several examples are given together with an interesting application of this method.

The results of this chapter can be also found in [35, 38, 37].

8.1 Composition of agops

As above, given a generating system (f_1, g_1, f_2, g_2, H), for all agops A and B, let F be the mapping defined by (8.1). If H is an agop, then F is increasing, because it is a composition of increasing functions. Moreover, in order to ensure that

$$F(0,0) = H\left(A(f_1(0), g_1(0)), B(f_2(0), g_2(0))\right) = 0,$$

one among the following conditions is sufficient:

$$f_1(0) = g_1(0) = 0 \quad \text{and} \quad f_2(0) = g_2(0) = 0, \tag{8.2}$$

$$f_1(0) = g_1(0) = 0 \quad \text{and} \quad H(0,b) = 0, \text{ for every } b \in [0,1], \tag{8.3}$$

$$f_2(0) = g_2(0) = 0 \quad \text{and} \quad H(a,0) = 0, \text{ for every } a \in [0,1]. \tag{8.4}$$

Analogously, in order to obtain

$$F(1,1) = H\left(A(f_1(1), g_1(1)), B(f_2(1), g_2(1))\right) = 1,$$

one among the following conditions is sufficient:

$$f_1(1) = g_1(1) = 1 \quad \text{and} \quad f_2(1) = g_2(1) = 1, \tag{8.5}$$

$$f_1(1) = g_1(1) = 1 \quad \text{and} \quad H(1,b) = 1, \text{ for every } b \in [0,1], \tag{8.6}$$

$$f_2(1) = g_2(1) = 1 \quad \text{and} \quad H(a,1) = 1, \text{ for every } a \in [0,1]. \tag{8.7}$$

In the sequel, we suppose that a generating system *satisfies one condition among* (8.2)–(8.4) *and another one among* (8.5)–(8.7).

Proposition 8.1.1. *Let* $(A,B) \in \mathcal{A}\times\mathcal{A}$ *and* (f_1, g_1, f_2, g_2, H) *be a generating system. Then the function* F *given by* (8.1) *is an agop.*

The very general form of composition (8.1) allows a great flexibility in constructing new agops and, in particular, the new method includes well-known procedures (see [10] for more details about them), as the following examples show.

Example 8.1.1. Let (f_1, g_1, f_2, g_2, H) be a generating system such that, for every $(x,y) \in [0,1]^2$, $H(x,y) = x$. For every A and B in $\mathcal{A}$, the function F given in (8.1) is equal to the agop $A(f_1(x), g_1(y))$. In particular, if f_1 and g_1 are greater than $\mathrm{id}_{[0,1]}$, this transformation was used for augmenting the output given by A (see [92]).

Example 8.1.2. If $f_1 = g_1 = f_2 = g_2 = \mathrm{id}_{[0,1]}$ and H is an agop, then the generating system (f_1, g_1, f_2, g_2, H) generates, for all agops A and B, an agop F that is the classical composition in $\mathcal{A}$, which includes as special cases the weighted arithmetic mean of agops, by taking $H(x,y) = \lambda x + (1-\lambda)y$ ($\lambda \in [0,1]$), and the weighted geometric mean of agops, by taking $H(x,y) = x^{\lambda} \cdot y^{1-\lambda}$ ($\lambda \in [0,1]$). In particular, if A is a t–norm and B is a t–conorm, F is a *triangular norm–based compensatory operator*, a special agop introduced as a means for providing compensation between the small and the large degrees of memberships when we combine fuzzy sets (see [92] and the references therein).

Example 8.1.3. Let $A, B \in \mathcal{A}$ and let (f_1, g_1, f_2, g_2, H) be a generating system of the function F given by (8.1). If $f_1 = g_1$ is a bijection and $H(x, y) := f_1^{-1}(x)$, then $F(x, y) = f_1^{-1}(A(f_1(x), f_1(y)))$ is the *transformation of* A by f_1, considered in chapter 9.

Example 8.1.4. Let $A, B \in \mathcal{A}$ and let (f_1, g_1, f_2, g_2, H) be a generating system of the function F given by (8.1). If $f_1 = g_1 = f_2 = g_2 = \mathrm{id}_{[0,1]}$, take $H(x, y) = \min\{1, x + \beta y\}$ $(\beta \in [0, 1])$. Then F is the *augmentation* of A. Similarly, taking $H(x, y) = \max\{0, x - \beta(1 - y)\}$ $(\beta \in [0, 1])$, the corresponding F is the *reduction* of A. This is another method proposed in [31] for augmenting (reducing) the outputs.

Remark 8.1.1. Given two associative agops A and B and a generating system (f_1, g_1, f_2, g_2, H) with $f_1 = g_1$ and $f_2 = g_2$, the function F defined by (8.1) is called a *quasi–associative operator* (see [158]).

8.2 Composition of semicopulas

Now, we give some sufficient conditions for the generalized composition of semicopulas.

Proposition 8.2.1. *Let* (A, B) *be in* $\mathcal{S} \times \mathcal{S}$ *and let* (f_1, g_1, f_2, g_2, H) *be a generating system satisfying* (8.5) *and one condition among* (8.2)–(8.4)*. The function* F *given by* (8.1) *is a semicopula if, and only if,*

$$H(f_1(x), f_2(x)) = x \quad \text{and} \quad H(g_1(x), g_2(x)) = x \qquad \text{for every } x \in [0, 1]. \tag{8.8}$$

Proof. In view of Proposition 8.1.1 it suffices to show that F has neutral element equal to 1. Let x be in $[0, 1]$. We have

$$F(x, 1) = H\left(A(f_1(x), g_1(1)), B(f_2(x), g_2(1))\right) = H(f_1(x), f_2(x)) = x,$$

and, analogously, $F(1, x) = x$. □

Example 8.2.1. For every $a \geq 1$, we consider the following generating system:

$$f_1(x) = g_1(x) = \min\{ax, 1\}, \quad f_2(x) = g_2(x) = x, \quad H = \min\{x, y\}.$$

For all $A, B \in \mathcal{S}$, the semicopula F defined by (8.1) is given by

$$F(x, y) := \begin{cases} \min\{A(ax, ay), B(x, y)\}, & \text{if } (x, y) \in [0, 1/a[^2; \\ B(x, y), & \text{otherwise.} \end{cases}$$

Example 8.2.2. For every $a \geq 1$, we consider the following generating system:

$$f_1(x) = g_1(x) = \max\{ax + (1 - a), 0\}, \quad f_2(x) = g_2(x) = x, \quad H = \max\{x, y\}.$$

For all $A, B \in \mathcal{S}$, the semicopula F defined by (8.1) is given by

$$F(x,y) := \begin{cases} B(x,y), & \text{if } (x,y) \in \left[0, \dfrac{a-1}{a}\right]^2; \\ \max\left\{A(ax+(1-a), ay+(1-a)), B(x,y)\right\}, & \text{otherwise.} \end{cases}$$

Example 8.2.3. For all $\alpha, \beta > 0$, we consider the following generating system:

$$f_1(x) = x^{\alpha}, \quad g_1(x) = x^{\beta}, \quad f_2(x) = x^{1-\alpha}, \quad g_2(x) = x^{1-\beta}, \quad H = \Pi.$$

For all $A, B \in \mathcal{S}$, the semicopula F defined by (8.1) is given by

$$F(x,y) := A(x^{\alpha}, y^{\beta}) \cdot B(x^{1-\alpha}, y^{1-\beta}),$$

which is a non–symmetric agop for $\alpha \neq \beta$.

In the case of semicopulas, we can give a full characterization of the classical composition. To this end, first, we give a technical result.

Lemma 8.2.1. *Let s_1, s_2 and t be points in $[0,1[$ with $s_1 \le s_2$. Then there exist two semicopulas A and B and two points (x_1, y_1) and (x_2, y_2) in $[0,1]^2$, with $x_1 \le x_2$ and $y_1 \le y_2$ such that*

$$A(x_1,y_1) = s_1 \qquad \textit{and} \qquad A(x_2,y_2) = s_2, \qquad B(x_1,y_1) = t = B(x_2,y_2).$$

Proof. Three cases will be considered.
Case 1: $t \le s_1 \le s_2$. Let A be the ordinal sum given by

$$A = \left(\langle s_i, s_{i+1}, Z\rangle\right)_{i\in I},$$

with $I = \{0,1,2,3\}$ and $s_0 = 0$, $s_3 = 1$, so that

$$A(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0, s_1[^2; \\ s_1, & \text{if } (x,y) \in [s_1, s_2[^2; \\ s_2, & \text{if } (x,y) \in [s_2, 1[^2; \\ x \wedge y, & \text{otherwise;} \end{cases}$$

and let B be the ordinal sum given by $B = (\langle 0, t, Z\rangle, \langle t, 1, Z\rangle)$, so that

$$B(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0, t[^2; \\ t, & \text{if } (x,y) \in [t, 1[^2; \\ x \wedge y, & \text{otherwise.} \end{cases}$$

Then

$$A(s_1,s_1) = s_1, \qquad A(s_2,s_2) = s_2, \qquad B(s_1,s_1) = t = B(s_2,s_2).$$

Case 2: $s_1 \le t \le s_2$. Choose B as in the previous case and let A be the frame semicopula defined by

$$A(x,y) := \begin{cases} 0, & (x,y) \in [0,1[^2 \setminus [s_1,1[^2, \\ s_1, & (x,y) \in [s_1,1[^2 \setminus]t,1[^2, \\ t, & (x,y) \in]t,1[^2 \setminus [s_2,1[^2, \\ s_2, & (x,y) \in [s_2,1[^2, \\ x \wedge y, & x \vee y = 1. \end{cases}$$

Then

$$A(t,t) = s_1, \qquad A(s_2,s_2) = s_2 \qquad \text{and} \qquad B(t,t) = B(s_2,s_2) = t.$$

Case 3: $s_1 \le s_2 \le t$. Choose B as in the two previous cases and let A be the frame semicopula

$$A(x,y) := \begin{cases} 0, & (x,y) \in [0,1[^2 \setminus [t,1[^2, \\ s_1, & (x,y) \in [t,1[^2 \setminus [x_1,1[^2, \\ s_2, & (x,y) \in [x_1,1[^2, \\ x \wedge y, & x \vee y = 1, \end{cases}$$

where the point x_1 belongs to $]t,1]$. Then we have

$$A(t,t) = s_1, \qquad A(x_1,x_1) = s_2, \qquad B(x_1,x_1) = B(t,t) = t,$$

which proves the assertion. □

Theorem 8.2.1. *Let A and B be semicopulas and let H be a binary operation on $[0,1]$. Let $F(x,y) := H\left(A(x,y), B(x,y)\right)$. The following statements are equivalent:*

(a) *for all semicopulas A and B, F is a semicopula;*

(b) *H is an idempotent agop.*

Proof. (a) $\Longrightarrow$ (b): If F is a semicopula, then for every $x \in [0,1]$

$$x = F(x,1) = H\left(A(x,1), B(x,1)\right) = H(x,x).$$

Let s_1, s_2 and t be in $[0,1[$ with $s_1 \le s_2$. Hence, because of Lemma 8.2.1, there are two points (x_1,y_1) and (x_2,y_2) in $[0,1]^2$ with $x_1 \le x_2$ and $y_1 \le y_2$ such that $A(x_1,y_1) = s_1$, $A(x_2,y_2) = s_2$ and $B(x_1,y_1) = B(x_2,y_2) = t$. Therefore

$$\begin{aligned} H(s_1,t) &= H\left(A(x_1,y_1), B(x_1,y_1)\right) = F(x_1,y_1) \\ &\le F(x_2,y_2) = H\left(A(x_2,y_2), B(x_2,y_2)\right) = H(s_2,t). \end{aligned}$$

In an analogous manner, we prove that, for all $s \in [0,1[$, the function $t \mapsto H(s,t)$ is increasing. Thus H is an idempotent agop.

The converse implication, (b) $\Longrightarrow$ (a), is a consequence of Proposition 8.2.1. □

8.3 Composition of 1–Lipschitz agops

The following result gives a sufficient condition for the generalized composition of 1–Lipschitz agops, whose class is denoted by $\mathcal{A}_1$.

Theorem 8.3.1. *Let (A, B) be in $\mathcal{A}_1 \times \mathcal{A}_1$ and let (f_1, g_1, f_2, g_2, H) be a generating system. Let F be the function defined by (8.1). If H has the kernel property and f_i and g_i are 1–Lipschitz $(i = 1, 2)$, then F is in $\mathcal{A}_1$.*

Proof. Set $\widetilde{A}(x, y) := A(f_1(x), g_1(y))$ and $\widetilde{B}(x, y) := B(f_2(x), g_2(y))$. For every x, x', y, y' in $[0, 1]$ we have

$$\begin{aligned}
&|F(x,y) - F(x',y')| = |H\left(\widetilde{A}(x,y), \widetilde{B}(x,y)\right) - H\left(\widetilde{A}(x',y'), \widetilde{B}(x',y')\right)|\\
&\leq \max\{|\widetilde{A}(x,y) - \widetilde{A}(x',y')|, |\widetilde{B}(x,y) - \widetilde{B}(x',y')|\}\\
&\leq \max\{|f_1(x) - f_1(x')| + |g_1(y) - g_1(y')|, |f_2(x) - f_2(x')| + |g_2(y) - g_2(y')|\}\\
&\leq |x - x'| + |y - y'|,
\end{aligned}$$

which concludes the proof. □

Example 8.3.1. Let (A, B) be in $\mathcal{A}_1 \times \mathcal{A}_1$ and let $(\mathrm{id}_{[0,1]}, \mathrm{id}_{[0,1]}, \mathrm{id}_{[0,1]}, \mathrm{id}_{[0,1]}, H_a)$ be a generating system where, for every $a \in [0, 1]$, $H_a(x, y) = \mathrm{med}(x, y, a)$ is the median among x, y and a. Then the corresponding 1–Lipschitz agop F_a defined by (8.1) is

$$F_a(x, y) = \mathrm{med}\left(A(x, y), B(x, y), a\right).$$

In particular, if $a = 0$ (resp. $a = 1$), then we obtain that the minimum (resp. maximum) of two 1–Lipschitz agops is a 1–Lipschitz agop.

Example 8.3.2. Let (A, B) be in $\mathcal{A}_1 \times \mathcal{A}_1$ and let $(f_a, f_a, \mathrm{id}_{[0,1]}, \mathrm{id}_{[0,1]}, H)$ be a generating system where $a \in \,]0, 1[$ and

$$f_a(x) = \frac{ax}{a + (1 - a)x}, \quad H(x, y) = \min\{x, y\}.$$

Then the corresponding 1–Lipschitz agop F_a defined by (8.1) is

$$F_a(x, y) = \min\left\{A\left(\frac{ax}{a + (1 - a)x}, \frac{ay}{a + (1 - a)y}\right), B(x, y)\right\}.$$

Notice that the range of f_a is not the whole $[0, 1]$.

Corollary 8.3.1. *Let A and B be quasi–copulas and let (f_1, g_1, f_2, g_2, H) be a generating system satisfying the assumptions of Proposition 8.2.1. Let F be the function defined by (8.1). If one of the following statementes holds:*

(a) $f_i = g_i = id_{[0,1]}$ $(i = 1, 2)$ *and H is a kernel agop;*

(b) $f_1 = g_1 = id_{[0,1]}$, f_2 *and* g_2 *are* 1*–Lipschitz and* $H(x, y) = \min\{x, y\}$*;*

(c) f_1 *and* g_1 *are* 1*–Lipschitz,* $f_2 = g_2 = id_{[0,1]}$ *and* $H(x, y) = \min\{x, y\}$*;*

then F *is a quasi–copula.*

Proof. The assertion follows easily from both Theorem 8.3.1 and Proposition 8.2.1 because the function $\mathrm{id}_{[0,1]}$ is 1–Lipschitz and the function $H(x, y) = \min\{x, y\}$ is a kernel agop. □

The characterization of the classical composition of quasi–copulas was given in [90] and it is reproduced here.

Proposition 8.3.1. *Let* H *be a binary operation on* $[0, 1]$ *and denote by* Ω *the subset of the unit square defined by*

$$\Omega := \left\{(u, v) \in [0, 1]^2 : v \in \left[\max\{2u - 1, 0\}, \frac{u + 1}{2}\right]\right\}.$$

The following statements are equivalent:

(a) *for all quasi–copulas* A *and* B*,* $H(A(x, y), B(x, y))$ *is a quasi–copula;*

(b) H *is an agop which satisfies the kernel property on* Ω*.*

8.4 Composition of 2–increasing agops

We denote by $\mathcal{A}_2$ the class of 2–increasing agops.

Theorem 8.4.1. *Let* A *and* B *be* 2*–increasing agops and let* (f_1, g_1, f_2, g_2, H) *be a generating system. If* H *is* P*–increasing, then the function* F *defined by* (8.1) *is a* 2*–increasing agop.*

Proof. Set $\widetilde{A}(x, y) := A(f_1(x), g_1(y))$ and $\widetilde{B}(x, y) := B(f_2(x), g_2(y))$. The function F given by (8.1) satisfies the 2–increasing property if, and only if, for all x, x', y, y' in $[0, 1]$, $x \le x'$ and $y \le y'$,

$$\begin{aligned}
&F(x', y') - F(x', y) - F(x, y') + F(x, y) \\
&= H(\widetilde{A}(x', y'), \widetilde{B}(x', y')) - H(\widetilde{A}(x', y), \widetilde{B}(x', y)) \\
&\quad - H(\widetilde{A}(x, y'), \widetilde{B}(x, y')) + H(\widetilde{A}(x, y), \widetilde{B}(x, y)) \ge 0.
\end{aligned}$$

Now, take

$$\begin{aligned}
&s_1 = \widetilde{A}(x, y),\ s_2 = \widetilde{A}(x', y),\ s_3 = \widetilde{A}(x, y'),\ s_4 = \widetilde{A}(x', y') \\
&t_1 = \widetilde{B}(x, y),\ t_2 = \widetilde{B}(x', y),\ t_3 = \widetilde{B}(x, y'),\ t_4 = \widetilde{B}(x', y').
\end{aligned}$$

The functions $\widetilde{A}$ and $\widetilde{B}$ are increasing in each place and 2–increasing (in view of Proposition 3.2.1). Therefore the points s_i and t_i ($i \in \{1, 2, 3, 4\}$) satisfy (7.3) and (7.4) and, because H is P–increasing, it follows that F is 2-increasing. □

Notice that the assumptions of Theorem 8.4.1 are only sufficient: for particular agops A and B, in fact, they could be weakened, as the following example shows.

Example 8.4.1. Let A_S be the smallest agop. Consider $f_i = g_i = \mathrm{id}_{[0,1]}$ $(i = 1,2)$ and let B be an agop in $\mathcal{A}_2$. For every P–increasing agop H, the composition F of A_S and B given by (8.1) is equal to $H(0, B(x,y))$ for every $(x,y) \neq (1,1)$. Therefore, in order to ensure that F is 2–increasing, it is sufficient to give conditions only on the vertical section $y \mapsto H(0,y)$, and no other assumption on the values of H on $[0,1]^2$ is required.

The classical composition of 2–increasing agops is characterized here.

Theorem 8.4.2. *Let H be an agop. The following statements are equivalent:*

(a) *H is P–increasing;*

(b) *for every $(A,B) \in \mathcal{A}_2 \times \mathcal{A}_2$, $F(x,y) = H(A(x,y), B(x,y))$ is a 2–increasing agop.*

Proof. Part $(a) \Longrightarrow (b)$ is a particular case of Theorem 8.4.1. Conversely, let $s_i, t_i \in [0,1]$ $(i \in \{1,2,3,4\})$ such that (7.3) and (7.4) hold, namely

$$s_1 \le s_2 \wedge s_3 \le s_2 \vee s_3 \le s_4, \qquad t_1 \le t_2 \wedge t_3 \le t_2 \vee t_3 \le t_4, \tag{8.9}$$

$$s_1 + s_4 \ge s_2 + s_3, \qquad t_1 + t_4 \ge t_2 + t_3. \tag{8.10}$$

Define the following agops:

$$A(x,y) := \begin{cases} 0, & \text{if } \min\{x,y\} = 0; \\ s_1; & \text{if } (x,y) \in \,]0,1/2] \times]0,1/2]; \\ s_2; & \text{if } (x,y) \in \,]0,1/2] \times]1/2,1]; \\ s_3; & \text{if } (x,y) \in \,]1/2,1] \times]0,1/2]; \\ s_4; & \text{if } (x,y) \in \,]1/2,1[\times]1/2,1[; \\ 1; & \text{if } (x,y) = (1,1); \end{cases}$$

$$B(x,y) := \begin{cases} 0, & \text{if } \min\{x,y\} = 0; \\ t_1; & \text{if } (x,y) \in \,]0,1/2] \times]0,1/2]; \\ t_2; & \text{if } (x,y) \in \,]0,1/2] \times]1/2,1]; \\ t_3; & \text{if } (x,y) \in \,]1/2,1] \times]0,1/2]; \\ t_4; & \text{if } (x,y) \in \,]1/2,1[\times]1/2,1[; \\ 1; & \text{if } (x,y) = (1,1). \end{cases}$$

Let $F(x,y) = H(A(x,y), B(x,y))$ be the composition of A and B. Then

$$V_H\left([1/3,2/3]^2\right) = H(s_1,t_1) + H(s_4,t_4) - H(s_2,t_2) - H(s_3,t_3) \ge 0,$$

viz. H is P–increasing. □

Corollary 8.4.1. *Let $f : [0,1] \to [0,1]$ be such that $f(0) = 0$ and $f(1) = 1$. The following statements are equivalent:*

(a) *f is convex and increasing;*

(b) *for every $(A, B) \in \mathcal{A}_2 \times \mathcal{A}_2$, $F(x, y) = f(A(x, y))$ is a 2–increasing agop.*

Proof. It suffices to apply the above Theorem to the function $H(x, y) = f(x)$, which is P–increasing because of Theorem 7.1.2. □

8.5 Composition of copulas

The following result on the generalized composition of copulas is a direct consequence of Theorem 8.4.1 and Proposition 8.2.1.

Proposition 8.5.1. *Let (A, B) be in $\mathcal{C} \times \mathcal{C}$ and let (f_1, g_1, f_2, g_2, H) be a generating system satisfying the assumptions of Proposition 8.2.1. If H is P–increasing, then the function F defined by (8.1) is a copula.*

Example 8.5.1. Consider, for all $0 < \alpha < \beta < 1$,

$$f_1(x) = \frac{\beta x}{(\beta - \alpha)x + \alpha} \quad \text{and} \quad f_2(x) = \frac{(\beta - \alpha)x + \alpha}{\beta};$$

for every $\gamma \in [0, 1]$,

$$g_1(x) = x^\gamma \quad \text{and} \quad g_2(x) = x^{1-\gamma};$$

and $H = \Pi$. For all copulas A and B, in view of Proposition 8.5.1 we have the following family of copulas:

$$C_{\alpha,\beta,\gamma}(x, y) = A\left(\frac{\beta x}{(\beta - \alpha)x + \alpha}, y^\gamma\right) \cdot B\left(\frac{(\beta - \alpha)x + \alpha}{\beta}, y^{1-\gamma}\right).$$

Example 8.5.2. Consider, for all α and β in $]0, 1]$,

$$\begin{aligned} f_1(x) &= \alpha x + (1 - \alpha), & f_2(x) &= (1 - \alpha)x + \alpha, \\ g_1(x) &= \beta x + (1 - \beta), & f_2(x) &= (1 - \beta)x + \beta, \end{aligned}$$

and $H = W$. For all copulas A and B, in view of Proposition 8.5.1 we obtain the following family of copulas:

$$C_{\alpha,\beta}(x, y) = \max\left(A(\alpha x + \overline{\alpha}, \beta x + \overline{\beta}) + B(\overline{\alpha} x + \alpha, \overline{\beta} x + \beta) - 1, 0\right),$$

where $\overline{\alpha} := 1 - \alpha$ and $\overline{\beta} := 1 - \beta$.

Remark 8.5.1. For particular copulas A and B, the conditions of the previous proposition are only sufficient. In fact, let (A, B) be in $\mathcal{C} \times \mathcal{C}$ with $B(x, y) = \min\{x, y\}$ and let (f, g, H) be the generating triple defined, for every $\lambda \geq 1$, by

$$f(x) = \min\{\lambda x, 1\}, \quad g(x) = x, \quad H(x, y) = \min\{x/\lambda, y\}.$$

Thus H is not P–increasing (the horizontal section of H is concave), but, for every copula A, the function F given in (8.1) is

$$F(x, y) = \begin{cases} \dfrac{1}{\lambda} A(\lambda x, \lambda y), & \text{if } (x, y) \in \left[0, \frac{1}{\lambda}\right]^2; \\ \min\{x, y\}, & \text{otherwise;} \end{cases}$$

and F is the ordinal sum $(\langle 0, 1/\lambda, A\rangle)$ and, hence, it is a copula.

In Proposition 8.5.1, when either $f_1 \neq g_1$ or $f_2 \neq g_2$, we generate a family of non–symmetric copulas. In fact, the idea of this kind of composition arises from the paper [61] where the following mechanism is given.

Proposition 8.5.2 (Khoudraji, 1995)**.** *Let C be a symmetric copula, $C \neq \Pi$. A family of non–symmetric copulas $C_{\alpha,\beta}$ with parameters $0 < \alpha, \beta < 1$ $(\alpha \neq \beta)$ that includes C as a limiting case is defined by*

$$C_{\alpha,\beta}(x, y) := x^{1-\alpha} y^{1-\beta} C(x^\alpha, y^\beta).$$

Proof. It suffices to apply Proposition 8.5.1 with $H = \Pi$, $f_2(t) = t^\alpha$, $f_1 = f_2^{-1}$, $g_2(t) = t^\beta$, $g_1 = g_2^{-1}$. Then $C_{\alpha,\beta}$ is the generalized composition of (Π, C) with respect to the generating system $(f_1, f_2, g_1, g_2, \Pi)$. □

In the same manner, we prove:

Proposition 8.5.3. *Let A and B be symmetric copulas. A family of non–symmetric copulas $C_{\alpha,\beta}$ with parameter $0 < \alpha, \beta < 1$, $\alpha \neq 1/2$, is defined by*

$$C_{\alpha,\beta}(x, y) := A(x^\alpha, y^\beta) \cdot B(x^{1-\alpha}, y^{1-\beta}). \tag{8.11}$$

An interesting statistical interpretation can be given for this family. Let U_1, V_1, U_2 and V_2 be random variables uniformly distributed on $[0, 1]$. If A is the connecting copula of (U_1, V_1) and B is the connecting copula of (U_2, V_2) and the pairs (U_1, V_1) and (U_2, V_2) are independent, then $C_{\alpha,\beta}$ is the joint d.f. of

$$U = \max\{U_1^{1/\alpha}, U_2^{1/(1-\alpha)}\} \quad \text{and} \quad V = \max\{V_1^{1/\beta}, V_2^{1/(1-\beta)}\}.$$

Example 8.5.3. In the recent paper [96], a generalization of the bivariate survival d.f. of type Marshall–Olkin was considered. This function is given, for every $x, y \geq 0$, by

$$S^*(x, y) = S(x, y) \exp(-\lambda_{12} \max\{x, y\}),$$

where S is a bivariate survival d.f. with continuous survival marginal d.f.'s $F(x) = e^{-\lambda x}$, $\lambda > 0$ and $\lambda_{12} > 0$. If A is the copula of S, it is an easy computation to obtain that the copula of S^* is of the type (8.11), where $A = C$, $B = M$ and $\alpha = \beta = \lambda/(\lambda + \lambda_{12})$.

Example 8.5.4. Let A and B two Archimedean copulas generated, respectively, by φ and ϕ. In view of Proposition 8.5.3, for every α and β in $[0, 1]$ the following functions are copulas

$$C_{\alpha,\beta}(x, y) := \varphi^{[-1]}(\varphi(x^{\alpha}) + \varphi(y^{\beta})) \cdot \phi^{[-1]}(\phi(x^{1-\alpha}) + \phi(y^{1-\beta})). \tag{8.12}$$

In particular, if $\varphi(t) = \phi(t) = (-\ln t)^{\gamma}$ $(\gamma \geq 1)$, then A and B are the members of the so–called *Gumbel–Hougaard* family of copulas. By considering (8.12), we obtain a three–parameter family of non–symmetric copulas,

$$C_{\alpha,\beta,\gamma}(x, y) := \exp\left(-\left[(-\alpha \ln x)^{\gamma} + (-\beta \ln y)^{\gamma}\right]^{1/\gamma} - \left[(-\overline{\alpha} \ln x)^{\gamma} + (-\overline{\beta} \ln y)^{\gamma}\right]^{1/\gamma}\right),$$

where $\overline{\alpha} := 1 - \alpha$ and $\overline{\beta} := 1 - \beta$, which can be considered a non–symmetric generalization of the Gumbel–Hougaard family.

The importance of having at disposal families of asymmetric copulas is crucial in copula modelling. In applications, in fact, we have a (bivariate) data set and we have interested in the joint d.f. H that is the best–possible approximation to our data. Thanks to Sklar's theorem, this problem can be decomposed into two steps: the modelling of the marginal d.f.'s and the estimating of a copula that summarizes the dependence between the margins. In several practical cases, we select a large family of copulas C_{θ}, where $\theta = (\theta_1, \ldots, \theta_n)$ is a multiparameter belonging to a subset $J^n \subseteq \mathbb{R}^n$, and we choose $\widehat{\theta} \in J^n$ such that $C_{\widehat{\theta}}$ optimaly fits our data (see [60] for more details on the copula modelling). A suitable family C_{θ} could have a simple representation (like the Archimedean copulas), or a simple way to computing it by numerical procedure (like the normal copula), and a sufficiently large dependence structure. In particular, and this is often neglected, *no assumptions on the simmetry of the copulas should be made*, unless it is explicitly required by the problem at hand. In fact, if the copula C is symmetric and the marginals d.f.'s F_1 and F_2 are continuous and both equal to a d.f. F, then the joint d.f. $H = C(F, F)$ is *exchangeable* and, therefore, it is not suitable to describe situations in which the appropriateness of this symmetry condition is doubtful.

Chapter 9

Copula and semicopula transforms

In this chapter, a method will be studied for transforming a copula into another one via a continuous and strictly increasing function. For the first time, this method appeared in the theory of semigroups and it was already used for triangular norms ([141, 83]). Recently, it has been studied in the theory of copulas in [49], where strong conditions on the transformating function are given, and in [87], where the authors are interested, in particular, in the study of the invariance of copulas under such transformations. However, the approach presented here takes into account the ideas presented in [7], where transformations of copulas and semicopulas are a useful tool to investigate bivariate notions of aging.

Therefore, in section 9.1 we study first the transformation of semicopulas; then sections 9.2 and 9.3 are devoted to a characterization of this transformation in the class of copulas and to the study of its properties.

For the results here presented, we can also see [46].

9.1 Transformation of semicopulas

We denote by Θ the set of continuous and strictly increasing functions $h : [0,1] \to [0,1]$ with $h(1) = 1$ and we denote by Θ_i the subset of Θ defined by those $h \in \Theta$ that are invertible. The following theorem is basic for what follows.

Theorem 9.1.1. *For all $h \in \Theta$ and $S \in \mathcal{S}$, the function $S_h : [0,1]^2 \to [0,1]$, defined, for all x and y in $[0,1]$, by*

$$S_h(x,y) := h^{[-1]}\left(S(h(x), h(y))\right) \tag{9.1}$$

is a semicopula. Moreover, if S is continuous, then also S_h is continuous.

Proof. If t is in $[0,1]$, then

$$S_h(t,1) = h^{[-1]}\left(S(h(t),h(1))\right) = h^{[-1]}(h(t)) = t = S_h(1,t).$$

Let x, x', y be in $[0,1]$ with $x \le x'$. Then

$$\begin{aligned} h(x) \le h(x') &\Longrightarrow S(h(x),h(y)) \le S(h(x'),h(y)) \\ &\Longrightarrow h^{[-1]}\left(S(h(x),h(y))\right) \le h^{[-1]}\left(S(h(x'),h(y))\right), \end{aligned}$$

namely $x \mapsto S_h(x,y)$ is increasing; similarly, $y \mapsto S_h(x,y)$ is increasing. □

The function S_h given by (9.1) is said to be the *transformation* of S via h, or the *h–transformation* of S.

Theorem 9.1.1 introduces a mapping $\Psi : \mathcal{S} \times \Theta \to \mathcal{S}$ defined, for all x and y in $[0,1]$, by

$$\Psi(S,h)(x,y) := h^{[-1]}\left(S(h(x),h(y))\right).$$

We shall often set $\Psi_h\, S := \Psi(S,h)$.

The set $\{\Psi_h, h \in \Theta\}$ is closed with respect to the composition $\circ$. Moreover, given $h, g \in \Theta$, for all $S \in \mathcal{S}$ we have

$$\begin{aligned} (\Psi_g \circ \Psi_h)\left(S(x,y)\right) &= \Psi\left(\Psi(S,h),g\right)(x,y) = g^{[-1]}\left(\Psi_h\, S\left(g(x),g(y)\right)\right) \\ &= g^{[-1]}\Big(h^{[-1]}\, S\left((h\circ g)(x),(h\circ g)(y)\right)\Big) \\ &= (h\circ g)^{[-1]}\left(S\left((h\circ g)(x),(h\circ g)(y)\right)\right) = \Psi_{h\circ g} S(x,y). \end{aligned}$$

The identity mapping in $\mathcal{S}$, which coincides with $\Psi_{\mathrm{id}_{[0,1]}}$, is, obviously, the neutral element of the composition operator $\circ$ in $\{\Psi_h, h \in \Theta\}$. Moreover, if $h \in \Theta_i$, then Ψ_h admits an inverse function given by $\Psi_h^{-1} = \Psi_{h^{-1}}$ and the mapping $\Psi : \mathcal{S} \times \Theta_i \to \mathcal{S}$ is the so–called *action* of the group Θ_i on $\mathcal{S}$.

Notice that, given the copula Π, for all $h \in \Theta$ $\Psi_h\Pi$ is an Archimedean and continuous t-norm with additive generator $\varphi(t) = -\ln(h(t))$ (see Theorem 1.4.2). Moreover, for all $h \in \Theta$, we have $\Psi_h M = M$ and $\Psi_h Z = Z$.

Definition 9.1.1. A subset $\mathcal{B}$ of $\mathcal{S}$ is said to be *stable* (or *closed*) with respect to (or under) Ψ if the image of $\mathcal{B} \times \Theta$ under Ψ is contained in $\mathcal{B}$, $\Psi_h\mathcal{B} \subseteq \mathcal{B}$ for every $h \in \Theta$.

It is easily proved that the subsets of commutative and continuous semicopulas are closed under Ψ. Moreover, the following result can be proved (see also [141, 83]).

Proposition 9.1.1. *The class $\mathcal{T}$ of all t–norms is closed under Ψ.*

Proof. For each $h \in \Theta$ and $T \in \mathcal{T}$, it suffices to show that the function $T_h := \Psi_h T$, defined by

$$T_h(x,y) := h^{[-1]}\left(T(h(x),h(y))\right) \qquad \text{for all } x, y \in [0,1],$$

is associative. Set $\delta := h(0) \geq 0$. For all s, t and u all belonging to $[0,1]$, simple calculations lead to the two expressions

$$T_h\left[T_h(s,t),u\right] = h^{[-1]}\left\{T\left[T(h(s),h(t)) \vee \delta, h(u)\right]\right\}$$
$$T_h\left[s,T_h(t,u)\right] = h^{[-1]}\left\{T\left[h(s),T(h(t),h(u)) \vee \delta\right]\right\}.$$

If $T(h(s),h(t)) \leq \delta$, then

$$T_h\left[T_h(s,t),u\right] = h^{[-1]}\left(T(\delta,h(u))\right) \leq h^{[-1]}(\delta) = 0,$$

and either

$$\begin{aligned} T_h\left[s,T_h(t,u)\right] &= h^{[-1]}\left(T\left(h(s),T(h(t),h(u))\right)\right) \\ &= h^{[-1]}\left(T\left(T(h(s),h(t)),h(u)\right)\right) \leq h^{[-1]}\left(T(\delta,h(u)\right) \leq h^{[-1]}(\delta) = 0, \end{aligned}$$

or

$$T_h\left[s,T_h(t,u)\right] = h^{[-1]}\left(T(h(s),\delta)\right) \leq h^{[-1]}(\delta) = 0.$$

Therefore T_h is associative.

If $T\left(h(s),h(t)\right) > \delta$, then

$$T_h\left[T_h(s,t),u\right] = h^{[-1]}\left\{T\left[T(h(s),h(t)),h(u)\right]\right\}$$

and either

$$\begin{aligned} T_h\left[s,T_h(t,u)\right] &= h^{[-1]}\left(T\left(h(s),T(h(t),h(u))\right)\right) \\ &= h^{[-1]}\left(T\left(T(h(s),h(t)),h(u)\right)\right) = T_h\left[T_h(s,t),u\right], \end{aligned}$$

or

$$T_h\left[s,T_h(t,u)\right] = h^{[-1]}\left(T(h(s),\delta)\right) \leq h^{[-1]}(\delta) = 0,$$

but, in this case, we have also

$$\begin{aligned} T_h\left[T_h(s,t),u\right] &= h^{[-1]}\left\{T\left[T(h(s),h(t)),h(u)\right]\right\} \\ &= h^{[-1]}\left(T\left(h(s),T(h(t),h(u))\right)\right) \leq h^{[-1]}\left(T\left(h(s),\delta\right)\right) \leq h^{[-1]}(\delta) = 0; \end{aligned}$$

which is the desired assertion. □

A t–norm T is said to be *isomorphic* to a t–norm T' if, and only if, there exists $h \in \Theta_i$ such that $T' = T_h$, viz. T' is the h–transformation of T. The following result characterizes in terms of transformations two important subsets of t–norms (see [83]).

Theorem 9.1.2. *Let T be a function from $[0,1]^2$ to $[0,1]$.*

(i) *T is a strict t–norm if, and only if, T is isomorphic to Π.*

(ii) *T is a nilpotent t–norm if, and only if, T is isomorphic to W.*

9.2 Transformation of copulas

Given a copula C and a function $h \in \Theta$, let C_h be the h–transformation of C,

$$C_h(x, y) := h^{[-1]}\left(C(h(x), h(y))\right). \tag{9.2}$$

From Theorem 9.1.1, it follows that C_h is a semicopula for all $h \in \Theta$ and for every copula $C \in \mathcal{C}$. However, it is easily checked that C_h need not be a copula, as the following example shows.

Example 9.2.1. Let h be in Θ defined by $h(t) := t^2$. Then

$$W_h(x, y) = h^{-1}\left(W(h(x), h(y))\right) = \sqrt{\max\{x^2 + y^2 - 1, 0\}},$$

namely

$$W_h(x, y) = \begin{cases} 0, & \text{if } x^2 + y^2 \le 1, \\ \sqrt{x^2 + y^2 - 1}, & \text{otherwise.} \end{cases}$$

And we have

$$W_h\left(1, \frac{6}{10}\right) - W_h\left(\frac{6}{10}, \frac{6}{10}\right) = \frac{6}{10} > \frac{4}{10}.$$

Thus W_h is not 1–Lipschitz, therefore neither the class of copulas nor the class of quasi–copulas are stable under Ψ.

In the following result, we characterize the transformations of copulas.

Theorem 9.2.1. *For each $h \in \Theta$, the following statements are equivalent:*

(a) *h is concave;*

(b) *for every copula C, the transform* (9.2) *is a copula.*

Proof. (a) $\Longrightarrow$ (b) In view of Theorem 9.1.1, it suffices to show that C_h satisfies the rectangular inequality (C2). To this end, let x_1, y_1, x_2, y_2 be points of $[0, 1]$ such that $x_1 \le x_2$ and $y_1 \le y_2$. Then the points s_i $(i = 1, 2, 3, 4)$, defined by

$$s_1 = C(h(x_1), h(y_1)), \quad s_2 = C(h(x_1), h(y_2)),$$
$$s_3 = C(h(x_2), h(y_1)), \quad s_4 = C(h(x_2), h(y_2)),$$

satisfy

$$s_1 \le s_2 \wedge s_3 \le s_2 \vee s_3 \le s_4 \quad \text{and} \quad s_1 + s_4 \ge s_2 + s_3, \tag{9.3}$$

viz. $(s_3, s_2) \prec_w (s_4, s_1)$. Because $h^{[-1]}$ is convex, continuous and increasing, it follows from Tomic's theorem 1.2.3 that

$$h^{[-1]}(s_3) + h^{[-1]}(s_2) \le h^{[-1]}(s_4) + h^{[-1]}(s_1).$$

Therefore we have

$$\begin{aligned}&h^{[-1]}(C(h(x_2),h(y_1))) + h^{[-1]}(C(h(x_1),h(y_2)))\\&\qquad\qquad \le h^{[-1]}(C(h(x_2),h(y_2))) + h^{[-1]}(C(h(x_1),h(y_1))),\end{aligned}$$

namely C_h satisfies (C2).

(b) $\Longrightarrow$ (a) It suffices to show that $h^{[-1]}$ is mid–convex, that is

$$\forall s,t \in [0,1] \qquad h^{[-1]}\left(\frac{s+t}{2}\right) \le \frac{h^{[-1]}(s) + h^{[-1]}(t)}{2}, \tag{9.4}$$

because, then, $h^{[-1]}$ is convex and, hence, h is concave.

Without loss of generality consider the copula W and s and t in $[0,1]$ with $s \le t$. If $(s+t)/2$ is in $[0, h(0)]$, then (9.4) is immediate. If $(s+t)/2$ is in $]h(0), 1]$, then we have

$$\begin{aligned}&W\left(\frac{s+1}{2}, \frac{s+1}{2}\right) = s, \quad W\left(\frac{t+1}{2}, \frac{t+1}{2}\right) = t\\&W\left(\frac{s+1}{2}, \frac{t+1}{2}\right) = \frac{s+t}{2} = W\left(\frac{t+1}{2}, \frac{s+1}{2}\right).\end{aligned}$$

There are points x_1 and x_2 in $[0,1]$ such that

$$h(x_1) = \frac{1+s}{2} \qquad \text{and} \qquad h(x_2) = \frac{1+t}{2}.$$

Since W_h is a copula, we have

$$W_h(x_1,x_1) - W_h(x_1,x_2) - W_h(x_2,x_1) + W_h(x_2,x_2) \ge 0;$$

and, as a consequence

$$h^{[-1]}(s) - h^{[-1]}\left(\frac{s+t}{2}\right) - h^{[-1]}\left(\frac{s+t}{2}\right) + h^{[-1]}(t) \ge 0,$$

which is the desired conclusion. □

Remark 9.2.1. In a special case, an interesting probabilistic interpretation of formula (9.2) is presented in [59, Theorem 5.2.3]: if $h(t) = t^{1/n}$ for some $n \ge 1$, then C_h is the copula associated with componentwise maxima, $X = \max\{X_1, \dots, X_n\}$ and $Y = \max\{Y_1, \dots, Y_n\}$, of a random sample (X_1, Y_1), (X_2, Y_2), …, (X_n, Y_n) of i.i.d. random vectors with the same copula C. Power transformations of copulas are useful in the theory of extreme value distributions ([104, 14, 20, 87]).

Remark 9.2.2. Let H be a bivariate distribution function with marginals F and G and let h be a concave and strictly increasing function. From the proof of Theorem 9.2.1, it is easily proved that the function $\widetilde{H}$ given, for every $(x,y) \in \overline{\mathbb{R}}^2$, by

$$\widetilde{H}(x,y) = h(H(x,y)) \tag{9.5}$$

is a bivariate distribution function with margins $h(F)$ and $h(G)$. Moreover, if the margins are continuous, the copula of $\widetilde{H}$ is $C_{h^{-1}}$. Transformations of type (9.5) were used in the field of insurance pricing ([58, 156]) and they are also called *distorted probability measure* in the context of non–additive probabilities ([30]).

9.3 Properties of the transformed copula

We denote by Θ_C the set of concave functions in Θ. These properties can be easily proved:

Proposition 9.3.1. *Let h and g be two functions in Θ_C. Then*

(a) *$\lambda h + (1-\lambda)g$ is in Θ_C for every $\alpha \in [0,1]$;*

(b) *$h \circ g$ is in Θ_C;*

(c) *$h(t^\alpha)$ and $(h(t))^\alpha$ are in Θ_C for all $\alpha \in \,]0,1[$.*

$h(x)$	$h^{[-1]}(x)$	Parameter
$x^{1/\alpha}$	x^α	$\alpha \geq 1$
$\frac{1-e^{-\alpha x}}{1-e^{-\alpha}}$	$-\frac{1}{\alpha}\log\left(1-x(1-e^{-\alpha})\right)$	$\alpha > 0$
$\frac{bx}{bx+a(1-x)}$	$\frac{ax}{ax-bx+b}$	$0 < a < b$
$\sin(\pi x/2)$	$(2/\pi)\arcsin x$	
$(4/\pi)\arctan x$	$\tan(\pi x/4)$	

Table 9.1: Examples of functions in Θ_C

Example 9.3.1. Let C be a copula and let r be a function defined on $[0,1]$ by $r(t) = at + b$, with $a, b \in \,]0,1[$, $a + b = 1$. Then $r^{[-1]}(t) = \max\{0, (t-b)/a\}$ and we have

$$C_r(x,y) = \begin{cases} \frac{1}{a}\left[C(ax+b, ay+b) - b\right], & \text{if } C(ax+b, ay+b) \geq b; \\ 0, & \text{otherwise.} \end{cases}$$

The copula C_r is said to be *linear transformation of C*.

In particular, given $r(t) = (t+1)/2$, let C' be an ordinal sum of type $(\langle 0, 1/2, C \rangle)$. Then $C_r = M$.

Remark 9.3.1. Let h and g be in Θ_C. Given a copula C, the transformations C_h and C_g may be equal, $C_h = C_g$, even though the functions h and g are not equal,

$h \neq g$. For instance, we consider the copula W and let h be the function defined on $[0,1]$ by $h(t) = (t+1)/2$. Then $W_h = W$ and $W_{\text{id}} = W$, but id $\neq h$.

Conversely, Let C and D be copulas. Given $h \in \Theta_C$, we may have $C_h = D_h$ even though $C \neq D$. In fact, $C_h(x,y) = D_h(x,y)$ if, and only if,

$$\max\{h(0), C(h(x), h(y))\} = \max\{h(0), D(h(x), h(y))\},$$

viz. it suffices $C = D$ on $[h(0), 1]^2$.

Theorem 9.2.1 introduces, for all $h \in \Theta_C$, a mapping

$$\Psi_h : \mathcal{C} \to \mathcal{C}, \qquad C \mapsto \Psi_h C := C_h,$$

which verifies the properties given in the proposition below.

Proposition 9.3.2. *For every h and g in Θ_C, we have*

(a) $\Psi_h \circ \Psi_g = \Psi_{g \circ h}$;

(b) *if $\{C^n\}$ is a sequence of copulas that converges pointwise to the copula C, then $\{\Psi_h C^n\}$ converges pointwise to $\Psi_h C$;*

(c) *Ψ_h is continuous, in the sense that, for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\forall A, B \in \mathcal{C} \quad \|A - B\|_\infty < \delta \quad \Longrightarrow \quad \|\Psi_h A - \Psi_h B\|_\infty < \epsilon.$$

(d) *Ψ_h is convex, in the sense that, for every $A, B \in \mathcal{C}$ and $\lambda \in [0,1]$*

$$\Psi_h(\lambda A + (1-\lambda)B) \leq \lambda \Psi_h A + (1-\lambda)\Psi_h B.$$

Proof. Let h and g be in Θ_C.
(a) For every copula C, we have

$$\begin{aligned}\Psi_h \circ \Psi_g(C) &= \Psi_h \left(g^{[-1]}\left(C(g(x), g(y))\right)\right)\\ &= h^{[-1]}\left(g^{[-1]}\left(C(g(h(x)), g(h(y)))\right)\right) = \Psi_{g\circ h} C,\end{aligned}$$

and, from Proposition 9.3.1, $g \circ h$ is in Θ_C.

(b) For every (x,y) in $[0,1]^2$, we have

$$C_n(x,y) \xrightarrow{n\to+\infty} C(x,y);$$

and, in particular,

$$C_n(h(x), h(y)) \xrightarrow{n\to+\infty} C(h(x), h(y)).$$

Now, the assertion follows from the continuity of $h^{[-1]}$.

(c) Given two copulas A and B, since $h^{[-1]}$ is convex, we obtain

$$\begin{aligned}
&\Psi_h\left(\lambda A(x,y)+(1-\lambda)B(x,y)\right)\\
&\quad= h^{[-1]}\left(\lambda A(h(x),h(y))+(1-\lambda)B(h(x),h(y))\right)\\
&\quad\leq \lambda h^{[-1]}\left(A(h(x),h(y))\right)+(1-\lambda)h^{[-1]}\left(B(h(x),h(y))\right)\\
&\quad= \lambda\Psi_h A(x,y)+(1-\lambda)\Psi_h B(x,y),
\end{aligned}$$

which concludes the proof. □

As in section 9.1, a subset $\mathcal{B}$ of $\mathcal{C}$ is said to be *stable* with respect to Ψ if the image of $\mathcal{B}\times\Theta_C$ under Ψ is contained in $\mathcal{B}$, $\Psi(\mathcal{B}\times\Theta_C)\subseteq\mathcal{B}$.

Proposition 9.3.3. *The following class of copulas are stable with respect to* Ψ*:*

(a) *the Archimedean family;*

(b) *the class of associative copulas;*

(c) *the Archimax family.*

Proof. (a) Let C be an Archimedean copula additively generated by φ. For every $h\in\Theta_C$, the h–transformation of C is given by

$$C_h(x,y)=h^{[-1]}\left(\varphi^{[-1]}\left(\varphi(h(x))+\varphi(h(y))\right)\right),$$

viz. C_h is the Archimedean copula generated by $\varphi\circ h$.
Part (b) is a direct consequence of Proposition 9.1.1.
(c) Let C be an Archimax copula defined by the dependence function A and the Archimedean generator φ (see Example 1.6.9). As in part (a), we can prove that the h–transformation of C, C_h, is also an Archimax copula defined by the dependence function A and the Archimedean generator $\varphi\circ h$. □

In [7] some results are presented about the preservation of some dependence properties of a copula C that is transformed via a concave bijection (see Propositions 6.6 and 6.7). Here, we present only a result about the concordance order.

Proposition 9.3.4. *Given* C *and* C' *in* $\mathcal{C}$*, and* h *in* Θ_C*, we have*

(a) *the operation* Ψ_h *is order–preserving in the first place, i.e.,* $C\leq C'$ *implies* $\Psi_h C\leq\Psi_h C'$*;*

(b) *if* $\Psi_h C\leq\Psi_h C'$*, then* $C(x,y)\leq C'(x,y)$ *for all* $(x,y)\in[h(0),1]^2$.

Proof. Part (a) is a consequence of the fact that h and $h^{[-1]}$ are both increasing. Part (b) follows by considering that the restriction of h on $[h(0),1]$ is a bijection. □

Notice that, in general, C and its transformation C_h are not ordered in concordance order. It suffices to take, for $\alpha \in]0,1[$, the copula

$$C_\alpha(x,y) := \frac{xy}{[1+(1-x^\alpha)(1-y^\alpha)]^{1/\alpha}},$$

and $h(t) = t^{1/2}$ a function in Θ_C. Then $\Psi_h C_\alpha = C_{\alpha/2}$ and $C_{\alpha/2} \le C_\alpha$ if, and only if, $x^{\alpha/2} + y^{\alpha/2} \le 1$ (see also [114, Example 4.15]).

Chapter 10

Copulas and Schur–concavity

The notion of Schur–concavity (and the closely related concept of Schur–convexity) has a great importance in the recent applications of statistics; witness of this is the recent monograph of Spizzichino [152] where Schur–concavity is one of the central themes in the Bayesian models of aging. However, the study of Schur–concavity of copulas does not seem to have yet received any attention in the literature, although twenty years ago Alsina studied the same question for t–norms (see [1]). To this topic this chapter is devoted.

In section 10.1 we present some results about the class of Schur–concave copulas and several examples are given in section 10.2. The concept of Schur–concavity, moreover, allows us to discuss an open problem on the classes of copulas and triangular norms (section 10.3).

The presented resulted are also contained in [44, 33].

10.1 The class of Schur–concave copulas

At the beginning of the study on Schur–concavity of copulas, we recall some properties that can be directly derived from section 1.2.

Proposition 10.1.1. *Let $C\colon [0,1]^2 \to [0,1]$ be a semicopula.*

(a) *If C is Schur–concave (or Schur-convex), then it is symmetric.*

(b) *If C is Schur–concave (or Schur–convex) on $\Delta_+ := \{(x,y) \in [0,1]^2 : x \geq y\}$, then C is Schur–concave (or Schur–convex) on $[0,1]^2$.*

Proposition 10.1.2. *A semicopula $C\colon [0,1]^2 \to [0,1]$ is Schur–concave if, and only if, for all x, y and λ in $[0,1]$*

$$C(x,y) \leq C(\lambda x + (1-\lambda)y, (1-\lambda)x + \lambda y).$$

Proof. It suffices to consider the definition of Schur–concavity and Corollary 1.2.1. □

Example 10.1.1. Consider the copula M. For every $x \geq y$, we have $y \leq \lambda x + (1-\lambda)y$ and $y \leq (1-\lambda)x + \lambda y$, so that

$$M(x,y) \leq M(\lambda x + (1-\lambda)y, (1-\lambda)x + \lambda y);$$

and, analogously, we have the same result for $x < y$. Therefore M is Schur–concave.

Proposition 10.1.3. *Let C be a continuously differentiable semicopula. Then C is Schur–concave on $[0,1]^2$ if, and only if,*

(i) *C is symmetric;*

(ii) *for all $(x,y) \in \Delta_+$, $\partial_1 C(x,y) \leq \partial_2 C(x,y)$.*

As a consequence, it is easily proved that the copula Π is Schur–concave. Note that not every symmetric copula is Schur–concave, as the following example shows.

Example 10.1.2. Let C be the absolutely continuous copula defined by

$$C(x,y) := \begin{cases} xy/2, & \text{if } (x,y) \in [0,1/2] \times [0,1/2]; \\ x\,(3y-1)/2, & \text{if } (x,y) \in [0,1/2] \times [1/2,1]; \\ y\,(3x-1)/2, & \text{if } (x,y) \in [1/2,1] \times [0,1/2]; \\ (xy + x + y - 1)/2, & \text{if } (x,y) \in [1/2,1] \times [1/2,1]. \end{cases}$$

This copula is symmetric and has a density c given by

$$c(x,y) := \begin{cases} 1/2, & \text{if } (x,y) \in [0,1/2]^2 \cup [1/2,1]^2; \\ 3/2, & \text{otherwise.} \end{cases}$$

The three points $\mathbf{x} = (6/10, 4/10)$, $\mathbf{y} = (7/10, 3/10)$ and $\mathbf{z} = (8/10, 2/10)$ are such that $\mathbf{x} \prec \mathbf{y} \prec \mathbf{z}$, but

$$C\left(\frac{6}{10}, \frac{4}{10}\right) = \frac{32}{200} < \frac{33}{200} = C\left(\frac{7}{10}, \frac{3}{10}\right),$$
$$C\left(\frac{7}{10}, \frac{3}{10}\right) = \frac{33}{200} > \frac{28}{200} = C\left(\frac{8}{10}, \frac{2}{10}\right).$$

Therefore C is not Schur–concave.

The following result allows us to investigate only on the class of Schur–concave copulas.

Proposition 10.1.4. *The copula W is the only Schur–convex (quasi–)copula.*

Proof. Let C be a Schur–convex copula. Given $x, y \in [0,1]$ such that $x + y \leq 1$, we have $(x, y) \prec (x + y, 0)$, from which

$$C(x, y) \leq C(x + y, 0) = 0.$$

Furthermore, given $x, y \in [0,1]$ such that $x + y > 1$, we have $(x, y) \prec (1, x + y - 1)$, from which

$$C(x, y) \leq C(1, x + y - 1) = x + y - 1.$$

Then, for all $x, y \in [0,1]$

$$C(x, y) \leq \max(x + y - 1, 0) = W(x, y),$$

but, from the Fréchet–Hoeffding bounds inequalities (1.13) it follows that $C = W$. □

Notice that W is also the only Schur–constant (semi–)copula, as showed in Proposition 2.2.2.

Now, we give some results about the class $\mathcal{C}_{SC}$ of Schur–concave copulas.

Proposition 10.1.5. *The class $\mathcal{C}_{SC}$ is a compact subset of $\mathcal{C}$ with respect to the topology of uniform convergence.*

Proof. It is known that $\mathcal{C}$ is compact space with respect to the topology of uniform convergence. But, if $(C_n)_{n \in \mathbf{N}}$ is a sequence in $\mathcal{C}_{SC}$, then the pointwise limit

$$C(x, y) = \lim_{n \to +\infty} C_n(x, y)$$

is Schur–concave. It follows that the set $\mathcal{C}_{SC}$ is a closed subset of $\mathcal{C}$, and therefore it is also compact. □

Proposition 10.1.6. *The class $\mathcal{C}_{SC}$ is a convex subset of $\mathcal{C}$.*

Proof. Let (x_1, x_2) and (y_1, y_2) be two points in $[0,1]^2$ such that $(x_1, x_2) \prec (y_1, y_2)$ and suppose that C_1 and C_2 are Schur–concave copulas. Then, for every $\lambda \in [0,1]$

$$\begin{aligned} C(x_1, x_2) &= \lambda C_1(x_1, x_2) + (1 - \lambda) C_2(x_2, y_2) \\ &\geq \lambda C_1(y_1, y_2) + (1 - \lambda) C_2(y_1, y_2) = C(y_1, y_2), \end{aligned}$$

which concludes the proof. □

Proposition 10.1.7. *A copula C is Schur–concave if, and only if, the survival copula $\hat{C}$ associated with C is Schur–concave.*

Proof. If C is Schur–concave, then, given (x_1, x_2), (y_1, y_2) two points in Δ_+ such that $(x_1, x_2) \prec (y_1, y_2)$, we have

$$(1 - x_1, 1 - x_2) \prec (1 - y_1, 1 - y_2),$$

from which

$$C(1-x_1,1-x_2) \geq C(1-y_1,1-y_2),$$

and

$$x_1+x_2-1+C(1-x_1,1-x_2) \geq y_1+y_2-1+C(1-y_1,1-y_2),$$

Then $\hat{C}$ is Schur-concave. The same argument applies if $\hat{C}$ is assumed to be Schur–concave □

In view of Sklar's Theorem, given a copula C and two univariate d.f.'s F and G, it is possible to construct a bivariate d.f. $H(x,y) := C(F(x),G(y))$ for every $(x,y) \in \mathbb{R}^2$. Now, it is useful to stress the fact that, with suitable marginal d.f.'s, Schur–concave copulas may yield Schur–concave, –convex or constant bivariate d.f.'s (see [115]).

10.2 Families of Schur–concave copulas

Theorem 10.2.1. *Every associative copula is Schur–concave.*

In order to prove this result, first we establish the following two lemmas.

Lemma 10.2.1. *An ordinal sum of Schur–concave copulas is a Schur–concave copula.*

Proof. Let $\{J_i = [a_i, b_i]\}_{i\in\mathcal{I}}$ be a partition of the unit square and let $\{C_i\}_{i\in\mathcal{I}}$ be a family of Schur–concave copulas. Let C be the ordinal sum of $\{C_i\}_{i\in\mathcal{I}}$ with respect to $\{J_i\}_{i\in\mathcal{I}}$, viz.

$$C(x,y) := \begin{cases} a_i + (b_i - a_i)\ C_i\left(\dfrac{x-a_i}{b_i-a_i}, \dfrac{y-a_i}{b_i-a_i}\right), & \text{if } (x,y) \in J_i^2; \\ M(x,y), & \text{otherwise.} \end{cases}$$

Notice that C is symmetric and we shall show that, if every C_i is Schur–concave, then C is Schur–concave. Let $(x_1,x_2),(y_1,y_2)$ be two points in Δ_+ such that $(x_1,x_2) \prec (y_1,y_2)$. Suppose that there exists an index $i_0 \in \mathcal{I}$ such that $(x_1,x_2),(y_1,y_2) \in J_{i_0}^2$. We observe that

$$\left(\frac{x_1-a_{i_0}}{b_{i_0}-a_{i_0}}, \frac{x_2-a_{i_0}}{b_{i_0}-a_{i_0}}\right) \prec \left(\frac{y_1-a_{i_0}}{b_{i_0}-a_{i_0}}, \frac{y_2-a_{i_0}}{b_{i_0}-a_{i_0}}\right),$$

that implies

$$C_{i_0}\left(\frac{x_1-a_{i_0}}{b_{i_0}-a_{i_0}}, \frac{x_2-a_{i_0}}{b_{i_0}-a_{i_0}}\right) \geq C_{i_0}\left(\frac{y_1-a_{i_0}}{b_{i_0}-a_{i_0}}, \frac{y_2-a_{i_0}}{b_{i_0}-a_{i_0}}\right),$$

since C_{i_0} is Schur–concave, and it follows $C(x_1,x_2) \geq C(y_1,y_2)$. Similarly, if (x_1,x_2) and (y_1,y_2) does not belong to J_i^2 for all $i \in \mathcal{I}$, since M is also Schur–concave, it follows $C(x_1,x_2) \geq C(y_1,y_2)$. Finally, suppose that exists an index i_0 such that $(x_1,x_2) \in J_{i_0}^2$ and $(y_1,y_2) \notin J_i^2$ for all $i \in \mathcal{I}$. We set $k := x_1+x_2 = y_1+y_2$ and we distinguish two cases.

Case 1. If $2a_{i_0} \leq k \leq a_{i_0} + b_{i_0}$, then $(x_1, x_2) \prec (k - a_{i_0}, a_{i_0})$ and

$$C(x_1, x_2) \geq C(k - a_{i_0}, a_{i_0}) = a_{i_0} \geq M(y_1, y_2) = C(y_1, y_2);$$

hence C is Schur–concave.
Case 2. If $a_{i_0} + b_{i_0} < k < 2b_{i_0}$, then $(x_1, x_2) \prec (b_{i_0}, k - b_{i_0})$ and

$$C(x_1, x_2) \geq C(b_{i_0}, k - b_{i_0}) = k - b_{i_0} \geq M(y_1, y_2) = C(y_1, y_2),$$

from which it follows that C is Schur–concave. □

Lemma 10.2.2. *Every Archimedean copula is Schur–concave.*

Proof. Let (x_1, x_2) and (y_1, y_2) two points in $[0,1]^2$ such that $(x_1, x_2) \prec (y_1, y_2)$. It follows from Corollary 1.2.1 that there exists $\alpha \in [0,1]$ such that, if $\overline{\alpha} := 1 - \alpha$, then

$$x_1 = \alpha\, y_1 + \overline{\alpha}\, y_2, \qquad x_2 = \overline{\alpha}\, y_1 + \alpha\, y_2.$$

Let C_φ be an Archimedean copula with additive generator φ. Since φ is convex and strictly decreasing

$$\begin{aligned}
C(x_1, x_2) &= C(\alpha\, y_1 + \overline{\alpha}\, y_2, \overline{\alpha} y_1 + \alpha\, y_2) \\
&= \varphi^{[-1]}\left(\varphi(\alpha\, y_1 + \overline{\alpha}\, y_2) + \varphi(\overline{\alpha}\, y_1 + \alpha\, y_2)\right) \\
&\geq \varphi^{[-1]}\left(\alpha\, \varphi(y_1) + \overline{\alpha}\, \varphi(y_2) + \overline{\alpha}\, \varphi(y_1) + \alpha\, \varphi(y_2)\right) \\
&= \varphi^{[-1]}\left(\varphi(y_1) + \varphi(y_2)\right) = C(y_1, y_2),
\end{aligned}$$

which concludes the proof. □

Proof. (Theorem 10.2.1) It was shown that M and every Archimedean copula are Schur–concave, moreover the ordinal sum of two Schur–concave copulas is Schur–concave too. In view of Representation Theorem for associative copulas (Theorem 1.6.9), the assertion follows. □

Here we give some other examples of Schur–concave copulas.

Example 10.2.1 (The Fréchet family). Every copula $C_{\alpha,\beta}$ belonging to the Fréchet family (see Example 1.6.2), defined by

$$C_{\alpha,\beta}(x, y) = \alpha\, M(x, y) + (1 - \alpha - \beta)\, \Pi(x, y) + \beta\, W(x, y)$$

is Schur–concave, because it is a convex sum of Schur–concave copulas.

Example 10.2.2 (The FGM family). For all $x, y \in [0,1]$ and $\theta \in [-1,1]$

$$C_\theta(x, y) = xy + \theta xy\,(1 - x)\,(1 - y)$$

is a member of the FGM family (see Example 1.6.3). For every $x, y \in [0,1]$ we have

$$\partial_1 C_\theta(x,y) = y + \theta y\,(1-x)\,(1-y) - \theta xy\,(1-y),$$
$$\partial_2 C_\theta(x,y) = x + \theta x\,(1-x)\,(1-y) - \theta xy\,(1-x).$$

As a consequence of the inequality $|1-x-y+2xy| \leq 1$, which holds for all x and y in $[0,1]$, if $x \geq y$ we have

$$\partial_2 C_\theta(x,y) - \partial_1 C_\theta(x,y) = (x-y)[1+\theta(1-x-y+2xy)] \geq 0.$$

Thus, it follows from Proposition 10.1.3 that C_θ is Schur–concave.

Example 10.2.3 (The Plackett family). For all $u, v \in [0,1]$ and $\theta > 0$, $\theta \neq 1$,

$$C_\theta(u,v) = \frac{[1+(\theta-1)\,(u+v)] - \sqrt{[1+(\theta-1)\,(u+v)]^2 - 4\theta uv\,(\theta-1)}}{2\,(\theta-1)}$$

is a family of copulas, known as *Plackett family* (see [130]). For all $x, y \in [0,1]$, we have

$$\partial_1 C_\theta(u,v) = \frac{1}{2} - \frac{1\,+(\theta-1)\;(u+v)\;-2\theta v}{2\,\sqrt{[1+(\theta-1)\,(u+v)]^2 - 4\theta uv\,(\theta-1)}},$$
$$\partial_2 C_\theta(u,v) = \frac{1}{2} - \frac{1\,+(\theta-1)\;(u+v)\;-2\theta u}{2\,\sqrt{[1+(\theta-1)\,(u+v)]^2 - 4\theta uv\,(\theta-1)}}.$$

Moreover, for $u \geq v$, it follows that

$$\partial_2 C_\theta(u,v) - \partial_1 C_\theta(u,v) = \frac{\theta\,(u-v)}{2\,\sqrt{[1+(\theta-1)\,(u+v)]^2 - 4\theta uv\,(\theta-1)}} \geq 0.$$

Thus C_θ is Schur–concave.

10.3 Solution of an open problem for associative copulas

Recently, E.P. Klement, R. Mesiar and E. Pap ([85]) posed some open problems concerning triangular norms and related operators. In particular, the following problem was formulated:

Problem 10.3.1. *Let T be a continuous Archimedean t–norm. Prove or disprove that:*

$$T\,(\max\{x-a,0\}, \min\{x+a,1\}) \leq T(x,x) \tag{10.1}$$

holds for all $x \in [0,1]$ and for all $a \in \,]0,1/2[$.

In particular, the authors added that "a positive solution of this problem would induce a new characterization of associative copulas". This comment spurs us to investigate inequality (10.1) in the class of copulas: to this end, the notion of Schur–concavity will be useful.

First, notice that inequality (10.1) is not true for every copula.

Example 10.3.1. Let C be the copula given in [114, Example 3.3],

$$C(x,y) := \begin{cases} x, & \text{if } 0 \le x \le \frac{y}{2} \le \frac{1}{2}; \\ \frac{y}{2}, & \text{if } 0 \le \frac{y}{2} < x < 1 - \frac{y}{2}; \\ x + y - 1, & \text{if } \frac{1}{2} \le 1 - \frac{y}{2} \le x \le 1. \end{cases}$$

Then

$$C\left(\frac{4}{10}, \frac{6}{10}\right) = \frac{3}{10} > C\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4}.$$

Note that C is not associative:

$$C\left(C\left(\frac{1}{2}, \frac{1}{2}\right), \frac{1}{2}\right) = C\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4} \neq \frac{1}{8} = C\left(\frac{1}{2}, C\left(\frac{1}{2}, \frac{1}{2}\right)\right).$$

But, in general, we have

Lemma 10.3.1. *Let A be a semicopula. If A is Schur–concave, then A satisfies* (10.1).

Proof. Let a be in $]0, 1/2[$. We distinguish three cases. If $x \le a$, then (10.1) follows since A is positive. If $a < x \le 1 - a$, then (10.1) is equivalent to $A(x - a, x + a) \le A(x, x)$, which is a direct consequence of the Schur–concavity. If $x > 1 - a$, then (10.1) is equivalent to $x - a \le A(x, x)$ and this last inequality follows from the fact that

$$A(x, x) \ge A(2x - 1, 1) = 2x - 1 > x - a.$$

□

Lemma 10.3.1 and Theorem 10.2.1 together yield:

Theorem 10.3.1. *If C is an associative copula, then C satisfies* (10.1).

Notice that, if a copula C satisfies (10.1), then it need not be associative.

Example 10.3.2. We consider the FGM family of copulas given, for all $x, y \in [0, 1]$ and $\theta \in [-1, 1]$, by $C_\theta(x, y) = xy + \theta xy\,(1 - x)\,(1 - y)$. From Example 10.2.2, C_θ is Schur–concave, and thus satisfies (10.1), but, if $\theta \neq 0$, C_θ is not associative.

Notice also that, if a copula C satisfies (10.1), then it need not be Schur–concave.

Example 10.3.3. Let C be the copula defined by

$$C(x,y) := \begin{cases} \frac{1}{3}M(3x, 3y-2), & \text{if } (x,y) \in \left[0, \frac{1}{3}\right] \times \left[\frac{2}{3}, 1\right]; \\ \frac{1}{3}M(3x-1, 3y-1), & \text{if } (x,y) \in \left[\frac{1}{3}, \frac{2}{3}\right] \times \left[\frac{1}{3}, \frac{2}{3}\right]; \\ \frac{1}{3}M(3x-2, 3y), & \text{if } (x,y) \in \left[\frac{2}{3}, 1\right] \times \left[0, \frac{1}{3}\right]; \\ W(x,y), & \text{otherwise.} \end{cases}$$

This copula is obtained by using the block–based construction method introduced in [28]. Simple, but tedious, calculations show that C satisfies (10.1), but C is not Schur–concave. In fact, given the points $(2/10, 7/10)$ and $(3/10, 6/10)$, we have

$$C\left(\frac{3}{10}, \frac{6}{10}\right) = 0 < \frac{1}{30} = C\left(\frac{2}{10}, \frac{7}{10}\right),$$

which implies that C is not Schur–concave.

Remark 10.3.1. A geometrical interpretation can be given of the difference between inequality (10.1) and Schur–concavity. If $z = C(s,t)$ is the surface associated with a copula C that satisfies (10.1), the intersections of the surface with all the vertical planes of the form $s + t = 2x$, for all $x \in [0,1]$ and $s \in [0,x]$, are curves that take the maximum value in the point (x,x). But, if C is Schur–concave, we have the stronger condition that such curves are also decreasing from (x,x) to $(2x,0)$ (resp. $(2x-1,1)$).

10.3.1 Discussion in the class of triangular norms

In the class of continuous Archimedean t–norms, inequality (10.1) was characterized in [67] (see also [98, 127]).

Theorem 10.3.2. *Let T be a continuous Archimedean t–norm with additive generator t. Let ξ be defined by $\xi := t^{-1}(t(0)/2)$. Then T satisfies* (10.1)*, for all $a \in]0,1/2[$ and $x \in [0,1]$, if, and only if, the two following statements hold:*

(a) *for all $z \in]0, \min\{\xi, 1-\xi\}[$, $\quad t(\xi - z) + t(\xi + z) \geq 1$;*

(b) *t is convex on $[\xi, 1]$.*

In particular, if T is strict (viz. $t(0) = +\infty$), then the following statements are equivalent:

(a') *T satisfies* (10.1)*, for all $a \in]0,1/2[$ and $x \in [0,1]$;*

(b') *t is convex on $[0,1]$.*

On the other hand, we have also the characterization of continuous Archimedean t–norms that are Schur–concave (see [1]).

Theorem 10.3.3. *Let T be a continuous Archimedean t–norm with additive generator t. Then we have:*

(a) *if T is strict, then T is Schur–concave if, and only if, t is convex;*

(b) *if T is nilpotent, then T is Schur–concave if, and only if, t satisfies the following inequality:*

$$t\left(\alpha x + (1-\alpha)y\right) + t\left((1-\alpha)x + \alpha y\right) \leq t(x) + t(y)$$

for every α in $[0,1]$ and for all x, y in $[0,1]$ such that $t(x) + t(y) \leq 1$.

From the two previous results, we derive

Theorem 10.3.4. *Let T be a strict Archimedean t–norm with additive generator t. The following statements are equivalent:*

(i) *T is a copula;*

(ii) *T is Schur–concave;*

(iii) *T satisfies* (10.1).

Proof. From Theorem 1.6.6, T is a copula if, and only if, the additive generator t is convex and, then, T is Schur–concave (Theorem 10.3.3). Moreover, from Lemma 10.3.1, (ii) implies (iii), which, in its turn, is equivalent to the convexity of t (Theorem 10.3.2), which concludes the proof. □

Remark 10.3.2. The previous result also holds in the case of a continuous t–norm T which is jointly strictly monotone, i.e. $T(x,y) < T(x,z)$ whenever $x > 0$ and $y < z$ (see [88]).

Looking at Theorem 10.3.4 in the class of nilpotent t–norm, we have $(i) \implies (ii) \implies (iii)$. But, there exists a Schur–concave nilpotent t–norm T, which is not a copula: consider, for example, a t–norm additively generated by $t(x) := \frac{1+\cos(\pi x)}{2}$ (see [1, Example 2.1]). Moreover, in the class of nilpotent t–norms, inequality (10.1) does not imply Schur–concavity as the following example shows.

Example 10.3.4. Consider a t–norm T with additive generator t given by

$$t(x) := \begin{cases} 1 - \frac{x}{10}, & \text{if } x \in \left[0, \frac{1}{10}\right]; \\ -\frac{49\sqrt{2}}{10\left(9\sqrt{2}-10\right)}\left(x - \frac{1}{10}\right) + \frac{99}{100} & \text{if } x \in \left]\frac{1}{10}, 1 - \frac{1}{\sqrt{2}}\right]; \\ (1-x)^2, & \text{otherwise.} \end{cases}$$

Then T satisfies the assumptions of Theorem 10.3.2, and thus the inequality (10.1), but

$$T(5/100, 95/100) = 25/1000 > 0 = T(1/10, 9/10),$$

which implies that T is not Schur–concave.

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Acknowledgments

Chi cerca trova.
(Mt 7,7)

At the end of this dissertation, I have the pleasure to express my gratitude to all the people that I met during my Ph.D. studies and who helped me by means of their ideas, suggestions and fruitful discussions.

Thanks to B. Bassan, B. De Baets, R. Ghiselli, R.B. Nelsen, E.P. Klement, A. Kolesárová, E. Pap, P.L. Papini, J.A. Rodríguez Lallena, M. Scarsini, F. Spizzichino and M. Úbeda Flores.

Thanks to Professor José Juan Quesada Molina for giving me the opportunity to collaborate with him and for his friendly hospitality when I was visiting the University of Granada.

Thanks to Professor Radko Mesiar, who introduced me to the interesting field of fuzzy theory and aggregation operators and who has given a continuous support on my investigations.

Thanks to Professors Berthold Schweizer and Abe Sklar: without them, there wouldn't have been much to write about copulas.

Most of all, I would like to thank my supervisor Prof. Carlo Sempi, who introduced me in the world of the probability and the statistics and suggested me to investigate on the subject of this thesis. His support and his contagious optimism have been absolutely essential for me. I have to thank him also for teaching me how to write mathematical papers in English: all the mistakes of this dissertation depend, of course, only on my incompetence.

Finally, I would like to thank for their support and the patience all the people who have a particular place in my life, especially my family. No word can be enough to express them all my gratitude.

Fabrizio Durante
May 2006

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