Introduction

Functions of bounded variation, usually denoted by BV, have had and have an important role in several problems of calculus of variations. The main features that make BV functions suitable for dealing with specific variational problems are their compactness properties, in connection with integral functionals with linear growth on the gradient, and their property of allowing for discontinuities along hypersurfaces, which is important in several geometrical and physical problems. The prototype of integral functional with linear growth on the gradient is the area functional, whereas, among variational problems with discontinuities, maybe the first success of the theory has been the complete solution of the isoperimetric problem in $\mathbf{R}^n$, and more recently free discontinuity problems (a term introduced by E. De Giorgi in [17]) have been studied. These problems come from image segmentation and smoothing and fracture mechanics, motivated by biology and physics, where digital image processing and the study of elasticity properties of materials are of relevant importance. Notice that Sobolev functions do not either share compactness properties as general as BV, or allow for $(n-1)$-dimensional discontinuity sets (like boundaries).

BV functions have nowadays a satisfactory theory that regards their functional properties, including approximation, embedding theorems, smoothing, boundary trace theorems and fine properties. For a systematic and self-contained treatment of the theory of functions of bounded variation we consider as main reference the book of L. Ambrosio, N. Fusco and D. Pallara [5]. Other references are the monographs of E. Giusti [23], U. Massari and M. Miranda [32], L. C. Evans and R. F. Gariepy [20], and W. P. Ziemer [49].

Given Ω an open subset of $\mathbf{R}^n$, functions with bounded variation in Ω are defined as those $L^1(\Omega)$ functions whose distributional derivative is representable by a finite $\mathbf{R}^n$-valued Radon measure, denoted by Du, whose total variation defined as

$$|Df|(\Omega) = \sup\left\{ \int_{\mathbf{R}^n} f \operatorname{div}\phi \; dx \; : \phi \in C_c^1(\Omega, \mathbf{R}^n), \|\phi\|_{L^\infty(\Omega)} \leq 1 \right\} \tag{1}$$

is finite. A particular case of interest is when $f = \chi_E$, the characteristic function of $E \subset \mathbf{R}^n$. In this case, we set $\mathcal{P}(E, \Omega) = |D\chi_E|(\Omega)$, and E is said to be a set of finite perimeter in Ω if $\mathcal{P}(E, \Omega) < \infty$.

The theory of BV functions is closely related to that of sets with finite perimeter. The link is established by the coarea formula, that relates the variation measure of u and the perimeter of its level sets:

$$|Du|(\Omega) = \int_{\mathbf{R}} \mathcal{P}(E_t, \Omega)dt, \tag{2}$$

where $E_t = \{x \in \Omega : \ u(x) > t\}$.

One of the starting points of this thesis is the paper [15], where De Giorgi defines for the first time the perimeter of a set. At that time, it was more or less clear (see also [10]) that a class of sets enjoying good geometric and variational properties would come from an approximation procedure. De Giorgi's idea was to start from a convolution with real analytic kernels. With the aim of extending the isoperimetric inequality and the Gauss-Green formula, for a given function $f \in L^\infty(\mathbf{R}^n)$, he defines the approximating functions as

$$W(t)f(x) = (4\pi t)^{-n/2} \int_{\mathbf{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy.$$

This choice of convolution kernel $G_t(x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}$ has an advantage with respect to the compactly supported mollifiers, i.e., the function $W(t)f$ satisfies a semigroup law:

$$W(t+s)f(x) = W(t)W(s)f(x) \qquad t, s > 0.$$

In fact, the function $u(t,x) = W(t)f(x)$ is the solution of the parabolic problem

$$\begin{cases} \partial_t w(t,x) = \Delta w(t,x) & t \in (0,\infty),\ x \in \mathbf{R}^n \\ w(0,x) = f(x) & x \in \mathbf{R}^n \end{cases}. \tag{3}$$

The heat semigroup $(W(t))_{t\geq 0}$ is contractive on $L^1(\mathbf{R}^n)$ and commutes with the spatial derivatives, so that

$$\|DW(t+s)f\|_{L^1(\mathbf{R}^n)} = \|DW(t)W(s)f\|_{L^1(\mathbf{R}^n)} = \|W(t)DW(s)f\|_{L^1(\mathbf{R}^n)} \leq \|DW(s)f\|_{L^1(\mathbf{R}^n)}$$

hence the function

$$t \mapsto \int_{\mathbf{R}^n} |DW(t)f|\, dx$$

is non increasing and the existence of the limit as $t \to 0$ is guaranteed.

In particular, given $E \subset \mathbf{R}^n$, De Giorgi defines the perimeter of E through the limit

$$P(E) := \lim_{t\to 0} \int_{\mathbf{R}^n} |DW(t)\chi_E|\, dx. \tag{4}$$

Now, since definition (4) makes sense also for functions in $L^1(\mathbf{R}^n)$, one could compute the limit in the right hand side of (4) (with a generic $f \in L^1(\mathbf{R}^n)$ in place of χ_E) and prove that

$$|Df|(\mathbf{R}^n) = \lim_{t\to 0} \int_{\mathbf{R}^n} |DW(t)f|\, dx, \tag{5}$$

i.e. that the limit in (5) coincides with the supremum in (1) for every $f \in L^1(\mathbf{R}^n)$.

The aim of this thesis is to investigate if the same result is true if $|Df|$ in (1) is replaced by a more general weighted variation of f, and the heat semigroup $(W(t))_{t\geq 0}$ in (5) is replaced by the semigroup generated by a general elliptic operator of second order in an open set $\Omega \subset \mathbf{R}^n$, with suitable boundary conditions.

Let us briefly describe the problem considered.
Let Ω be a (possibly unbounded) domain in $\mathbf{R}^n$ with uniformly C^2 boundary and let $\mathcal{A}$ be a uniformly elliptic second order operator in divergence form:

$$\mathcal{A}(x,D)=\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j)+\sum_{i=1}^{n} b_i(x)D_i+c(x). \tag{6}$$

If $\Omega \neq \mathbf{R}^n$ we consider the (conormal) operator $\mathcal{B}$ acting on the boundary $\partial\Omega$

$$\mathcal{B}(x,D)=\sum_{i,j=1}^{n} a_{ij}(x)\nu_i(x)D_j=\langle AD,\nu\rangle, \tag{7}$$

where ν is the outward unit normal to $\partial\Omega$ and $A=(a_{ij})$. We consider the following problem

$$\begin{cases} \partial_t w-\mathcal{A}w=0 & \text{in } (0,\infty)\times\Omega \\ w(0)=u_0 & \text{in } \Omega \\ \mathcal{B}w=0 & \text{in } (0,\infty)\times\partial\Omega. \end{cases}, \tag{8}$$

with initial datum $u_0 \in L^1(\Omega)$. Let us briefly comment on the homogeneous boundary condition $\langle ADw,\nu\rangle=0$. In the simplest case when $\mathcal{A}=\Delta$ and $u_0=\chi_E$ in (8), the natural boundary condition to obtain $\mathcal{P}(E,\Omega)$ as the limit as $t\to 0$ is the Neumann condition $\frac{\partial w}{\partial\nu}=0$, because in this way the function u_0 is not immediately modified near the boundary, and then for short times the contribution of the gradient of the solution is significant only in the interior of Ω, thus measuring only the relative boundary of E. The natural extension of $\frac{\partial w}{\partial\nu}=0$ in $(0,\infty)\times\partial\Omega$ when we consider a generic operator $\mathcal{A}$ is $\langle ADw,\nu\rangle=0$ in $(0,\infty)\times\partial\Omega$.
In order to study our problem, it has proved to be convenient to translate it in the language of semigroups, and exploit the relative techniques. This leads us to consider the realization $A_1: D(A_1)\subset L^1(\Omega)\to L^1(\Omega)$ of $\mathcal{A}$ in $L^1(\Omega)$, where the domain $D(A_1)$ takes into account the boundary conditions. We shall prove that $(A_1,D(A_1))$ is sectorial in $L^1(\Omega)$, hence it is the generator of an analytic semigroup $(T(t))_{t\geq 0}$.
In order to prove that a linear operator $A: D(A)\subset X\to X$ is sectorial it is needed to prove first of all that the resolvent set $\rho(A)$ contains a sector

$$\Sigma_\theta=\{\lambda\in\mathbf{C}:\ \lambda\neq\omega, |\arg(\lambda-\omega)|<\theta\},$$

with $\omega\in\mathbf{R}$ and $\theta>\frac{\pi}{2}$; then, that there is $M>0$ such that the resolvent operator of A, $R(\lambda,A)=(\lambda-A)^{-1}$ verifies

$$\|R(\lambda,A)\|\leq M/|\lambda-\omega| \quad \text{for } \lambda\in\Sigma_\theta. \tag{9}$$

For the first requirement one has to prove existence and uniqueness of the solution of elliptic boundary value problems in $L^1(\Omega)$.
Basically, two ways are known to show the sectoriality of $(A_1,D(A_1))$. One is based on the integral representation

$$(T(t)f)(x):=\int_\Omega p(t,x,y)f(y)\,dy, \tag{10}$$

and consists in proving the existence of the kernel p, and subsequently in deriving suitable estimates on p and its derivative. Relying on earlier ideas of R. Beals and L. Hörmander, this point of view is deeply pursued by H. Tanabe in his book [45].
The other way is based on a duality argument. There is a serious obstruction in extending to $L^1(\Omega)$ the L^p-theory ($1 < p < \infty$), because the classical Calderón-Zygmund and Agmon-Douglis-Nirenberg estimates are known to fail for $p = 1, \infty$. A way to circumvent this difficulty for $p = \infty$ has been devised by K. Masuda and H. B. Stewart (see [42], [43] and also [31]) and consists in a clever passage to the limit as $p \to \infty$ in the L^p estimates. Then, a duality argument can be used to pass from L^∞ estimates to L^1 estimates and the sectoriality in $L^1(\Omega)$. This has been done in the case Ω bounded and Dirichlet boundary conditions by G. Di Blasio [18], H. Amann [4], A. Pazy [35], and D. Guidetti [24] for the case of elliptic systems in L^1. In the same vein, we have proved sectoriality of $(A_1, D(A_1))$ in $L^1(\Omega)$ for Ω (possibly) unbounded and homogeneous co-normal boundary conditions. After proving the existence and analyticity of the semigroup $(T(t))_{t\geq 0}$, we need precise estimates on the first and second order derivatives, in order to prove that the limit in (5) exists, and to evaluate it.

Let us come to our standing hypotheses.
We suppose that the operator $\mathcal{A}$ has real valued coefficients satisfying the following assumptions
$$a_{ij} = a_{ji} \in W^{2,\infty}(\Omega) \quad \text{and} \quad b_i, c \in L^\infty(\Omega).$$
and that the uniform ellipticity condition holds, namely there exists a positive constant $\mu \geq 1$ such that for any $x \in \overline{\Omega}$ and $\xi \in \mathbf{R}^n$
$$\mu^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \mu|\xi|^2.$$
With these assumptions on the coefficients it turns out that $(A_1, D(A_1))$, where $D(A_1)$ is the closure in the graph norm $\|\cdot\|_{L^1(\Omega)} + \|A\cdot\|_{L^1(\Omega)}$ of the space
$$\{u \in L^1(\Omega) \cap C^2(\bar{\Omega});\ \mathcal{A}u \in L^1(\Omega), \mathcal{B}u = 0 \text{ in } \partial\Omega\},$$
is a sectorial operator so it generates a bounded analytic semigroup $T(t)$ in L^1, and $T(t)u_0$ is the solution of
$$\begin{cases} \partial_t w(t,x) = \mathcal{A}w(t,x) & t \in (0,\infty),\ x \in \Omega \\ w(0,x) = u_0(x) & x \in \Omega \\ \mathcal{B}w(t,x) = 0 & t \in (0,\infty),\ x \in \partial\Omega \end{cases} \tag{11}$$
By the density of $D(A_1)$ in L^1 and the fact that $D(A_1) \hookrightarrow W^{1,1}(\Omega)$ (see Remark 3.0.6) we can also deduce that $T(t)$ is strongly continuous in $D(A_1)$ with respect to the $W^{1,1}$ norm, and that
$$\lim_{t\to 0} \|T(t)u_0 - u_0\|_{W^{1,1}(\Omega)} = 0 \tag{12}$$
for every $u_0 \in D(A_1)$. Formula (12) implies the convergence of $\|DT(t)u_0\|_{L^1(\Omega)}$ to the total variation of Du_0 as $t \to 0$.

But for general $f \in L^1(\Omega)$ the existence of the limit in the right hand side of (5), with $T(t)$ in place of $W(t)$, relies on precise estimates on the first and second order derivatives of $T(t)f$. We prove that, for every $t > 0$, the inequalities

$$\begin{aligned} \|DT(t)u\|_{L^1(\Omega)} &\leq \frac{C}{\sqrt{t}}\|u\|_{L^1(\Omega)} \\ \|D^2T(t)u\|_{L^1(\Omega)} &\leq \frac{C}{t}\|u\|_{L^1(\Omega)} \end{aligned} \tag{13}$$

hold for every $u \in L^1(\Omega)$ and some constant $C > 0$ independent of u. Estimate (13) has to be improved to go ahead, and the improvement is obtained via a characterization of the interpolation space between the domain $D(A_1)$ and $L^1(\Omega)$. As a consequence, we prove that there exists $\delta \in (1/2, 1)$ such that

$$t^\delta \|D^2T(t)u\|_{L^1(\Omega)} \leq C\|u\|_{W^{1,1}(\Omega)} \quad t \in (0,1) \tag{14}$$

holds for every $u \in D(A_1)$ and for some constant $C > 0$. Estimate (14) will be very useful to estimate the "defect of monotonicity" of the function

$$F(t) = \int_\Omega |DT(t)u_0|\, dx. \tag{15}$$

Actually, we prove that for $\delta \in (1/2, 1)$ as in (14) the inequality

$$\int_\Omega \eta |DT(t)v|_A \, dx \leq \int_\Omega \eta |Dv|_A \, dx + Ct^{1-\delta}\|v\|_{W^{1,1}(\Omega)} \qquad t \in (0,1) \tag{16}$$

holds for $v \in D(A_1)$ and for any nonnegative function $\eta \in C^1_b(\overline{\Omega})$. In (16), $|Dv|_A$ denotes the A-variation of Dv, namely the total variation weighted by the matrix of the coefficients $A = (a_{ij})_{ij}$ defined as follows

$$|Du|_A(\Omega) = \sup\left\{\int_\Omega u \mathrm{div}\psi dx : \psi \in C^1_c(\Omega, \mathbf{R}^n), \|A^{-1/2}\psi\|_\infty \leq 1\right\}.$$

Finally, using (16) and a result of approximation in variation for BV functions via functions belonging to $D(A_1)$, we get that the total variation of u_0 is the limit as $t \to 0$ of the L^1 norm of the gradient of $T(t)u_0$, that is the following equality

$$|Du_0|(\Omega) = \lim_{t\to 0} \int_\Omega |D(T(t)u_0)|\, dx \tag{17}$$

holds for every $u_0 \in L^1(\Omega)$. As a consequence we get that $u_0 \in BV(\Omega)$ if and only if the above limit is finite. Let us point out that the previous characterization holds not only for classical BV functions, but also for weighted BV functions (see Theorem 4.3.4).
The proof of estimate (14) for the derivatives is a quite long tour. Following ideas introduced by V. Vespri in [47] and [48] for Dirichlet boundary conditions, we study the semigroup $(T(t))_{t\geq 0}$ in Sobolev spaces of negative order and use a complex interpolation result. We remark that in some intermediate steps (mainly, when we deal with the adjoint operator of $\mathcal{A}$) we need to assume higher regularity on the coefficients. However, a perturbation result will allow us to come back to the initial assumptions.

We study also another connection between the short-time behavior of the semigroup $(T(t))_{t\geq 0}$ in $L^1(\Omega)$ and $BV(\Omega)$. In fact, this leads to a second characterization for BV functions. In this part, we use the integral representation (10) of the semigroup and the relative estimates quoted at the beginning of this Introduction.
More precisely we extend the results in [33], where the authors prove that a given function $u \in L^1(\mathbf{R}^n)$ is a function with bounded variation if and only if

$$\liminf_{t\to 0} \frac{1}{\sqrt{t}} \int_{\mathbf{R}^n\times\mathbf{R}^n} |u(x)-u(y)|G_t(x-y)\,dx\,dy < \infty$$

and in that case its total variation can be written as

$$|Du|(\mathbf{R}^n) = \lim_{t\to 0} \frac{\pi}{2\sqrt{t}} \int_{\mathbf{R}^n\times\mathbf{R}^n} |u(x)-u(y)|G_t(x-y)\,dx\,dy. \tag{18}$$

In order to extend (18) to functions with bounded variation in the domain Ω, we first consider the special case of the characteristic functions and we characterize sets with finite perimeter in Ω. We prove that if $E \subset \mathbf{R}^n$ is such that either E or E^c has finite measure in Ω, then E has finite perimeter in Ω if and only if

$$\liminf_{t\to 0} \frac{1}{\sqrt{t}} \int_{E^c\cap\Omega} T(t)\chi_E(x)dx < +\infty,$$

and in this case the following equality holds

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_{\Omega\cap E^c} T(t)\chi_E dx = \int_{\Omega\cap\mathcal{F}E} |A^{1/2}(x)\nu_E(x)|d\mathcal{H}^{n-1}(x), \tag{19}$$

where $\mathcal{F}E$ is the reduced boundary of E (see Definition 4.5). We remark that the right hand side of (19) reduces to the classical perimeter when $A = I$, since $\mathcal{P}(E,\Omega) = \mathcal{H}^{n-1}(\mathcal{F}E\cap\Omega)$. Then, using (19) in connection with the coarea formula (2), we prove that a given function $u \in L^1(\Omega)$ is of bounded variation if and only if

$$\liminf_{t\to 0} \frac{1}{\sqrt{t}} \int_\Omega\int_\Omega p(t,x,y)|u(x)-u(y)|\,dydx < \infty$$

and its A-variation can be written as follows

$$|Du|_A(\Omega) = \lim_{t\to 0} \frac{\sqrt{\pi}}{2\sqrt{t}} \int_{\Omega\times\Omega} p(t,x,y)|u(x)-u(y)|\,dydx. \tag{20}$$

Here, p is the kernel in (10).
Important tools for this second characterization are also the results of geometric measure theory concerning the structure of sets of finite perimeter and in particular their blow-up properties. We remark that this characterization is also in the spirit of [8], [14] and [27], where only kernels depending on $|x-y|$ are considered.

The two characterization of BV functions in terms of the short-time behavior of semigroups, described below, have been published in [6]. However we point out that the proofs in [6] rely on the kernel estimates recalled in Theorem B.1.1, whereas here we use such estimates only in Chapter 5. In fact, in this thesis the construction and the analysis of the semigroup $(T(t))_{t\geq 0}$, as well as the characterization of BV in Chapter 4, are independent of the kernel estimates and are rather based on the study of the resolvent equation. In this respect, the estimates we get are self-contained, and, even though the methods are based on previous works mainly confined to the Dirichlet problem, our presentation as a whole is original.

Let us describe the contents of the thesis. We tried to be as self-contained as possible, so we start in Chapter 1 by recalling some basic definitions and the most important properties of semigroups and a few relevant notions of measure theory. Mainly following [19] for the first part and [5] for the second one, we state (often without proof) some classical theorems that will be used throughout the thesis and fix our notation. We recall the main properties of sectorial operators and some perturbation results. Moreover analytic semigroup and intermediate spaces are mentioned in the first part. The second part consists in definitions and useful results of measure theory. Finally, Section 1.5 contains a collection of analytical tools helpful in the sequel.
Chapter 2 is devoted to results of generation of analytic semigroups in suitable Banach spaces. Since we get generation in $L^1(\Omega)$ from analogous results in L^∞ by duality and since the L^∞ theory makes use of that in L^p, $1 < p < \infty$, we start by recalling some classical result of generation in L^p spaces. Then, following [42] and [43], we deduce generation for elliptic operator with non tangential boundary conditions in the space of essentially bounded functions. Thus, using the adjoint boundary value problems in L^∞, we get existence and the estimate (9) for the solution of the elliptic boundary value problem associated with $\mathcal{A}$ and $\mathcal{B}$ in L^1. We also study elliptic boundary value problems in the dual space of some Sobolev spaces to deduce by duality estimates for the gradient of the resolvent operator $R(\lambda, A_1)$.
In Chapter 3 we derive estimates for the L^1 norm of the semigroup $T(t)$ generated by $(A_1, D(A_1))$. Other useful estimates are established for the first and the second order spatial derivatives of $T(t)$ also by mean of the characterization of some new real interpolation spaces.
After a brief introduction on the possibly weighted BV functions and sets of weighted finite perimeter we collect in Chapter 4 their main properties. In particular, a version of the classical Anzellotti-Giaquinta approximation theorem is derived, and a weighted version of the coarea formula is also shown. In the simplest case of the Laplacian defined in a convex domain with homogeneous Neumann boundary condition on $\partial\Omega$, the function F in (15) can be easily proved to be non increasing by differentiating under the integral sign. We remark that in such framework the convexity of the domain is essential: in fact a counterexample to the monotonicity is provided in [22]. In general, when we consider a generic operator like $\mathcal{A}$, the same procedure does not work as well as in the previous case as we do not get monotonicity. However estimate (16) and the approximation results allow us to conclude, without convexity assumption on Ω. The first part of Chapter 5 is devoted to collect known results concerning some connections between semigroups and perimeter. In particular we refer to [27], where Ledoux connects the L^2 norm of the heat semigroup in $\mathbf{R}^n$ with the isoperimetric inequality, and to [33] for the characterization

of the perimeter of a set $E \subset \mathbf{R}^n$ in terms of the behavior of

$$\int_{\mathbf{R}^n \setminus E} W(t)\chi_E \, dx$$

as $t \to 0$. Then we extend this latter result and we provide a second characterization for sets of finite perimeter and functions with bounded variation in Ω.
At the end of the thesis there are two appendices. The first one consists in an elementary treatment of the real and complex interpolation theory. Moreover a new characterization of a real interpolation space is given. More precisely, we prove that if $\theta \in (0, 1/2)$ the real interpolation space

$$(L^1(\Omega), W^{2,1}(\Omega) \cap W^{1,1}_{A,\nu}(\Omega))_{\theta,1},$$

where $W^{1,1}_{A,\nu}(\Omega)$ is the closure of $\{u \in C^1(\overline{\Omega}) \,|\, \langle A(x) \cdot Du, \nu(x) \rangle = 0 \text{ for } x \in \partial\Omega\}$ with respect to the topology of $W^{1,1}(\Omega)$, consists of functions that are in the fractional Sobolev space $W^{2\theta,1}(\Omega)$. This fact will be used in Chapter 3 to characterize the intermediate space $D_{A_1}(\theta, 1)$. Finally a brief recall on the complex interpolation spaces is provided in Section A.3. We present this argument in a quite general context, which still is not the most general possible, but is close to our applications.
In Appendix B we gather up some Gaussian upper and lower bounds for the integral kernel p in (10), (20). For the Gaussian lower bounds we study first the symmetric case then, the estimates are extended to the non-symmetric one. More details about this matter can be found in [34].

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