Appendix B

Heat kernel estimates on domains

In this section we collect some upper and lower estimates for the integral kernel of the semigroup associated with the parabolic problem

$$\begin{cases} \partial_t w - \mathcal{A}w = 0 & \text{in } (0,\infty)\times\Omega \\ w(0) = u_0 & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega. \end{cases} \tag{B.1}$$

under the hypotheses summarized at the beginning of Chapter 5. Since we shall deal with several semigroups, the exponential notation seems to us to be clearer, as it emphasizes the relevant elliptic generator. In fact we consider

$$\mathcal{A}_0 = \mathrm{div}(A\cdot D), \quad \mathcal{A}' = \mathrm{div}(A\cdot D) + B\cdot D \quad \text{and} \quad \mathcal{A} = \mathrm{div}(A\cdot D) + B\cdot D + c$$

and the related semigroups $e^{-t\mathcal{A}_0}, e^{-t\mathcal{A}'}$ and $e^{-t\mathcal{A}}$ whose kernels p_0, p' and p are such that, e.g.,

$$e^{-t\mathcal{A}}f(x) = \int_\Omega p(t,x,y)f(y)\,dy$$

and the analogous expressions for $e^{-t\mathcal{A}_0}$ and $e^{-t\mathcal{A}'}$ hold.
We first recall upper estimates directly for p, that are well-known. On the contrary, lower estimates are known in the symmetric case, i.e., for p_0. After observing that there is no difficulty in passing from p' to p, we shall deduce lower estimates for p', deducing them from those on p_0 via a perturbation argument. The proofs in Section B.2.2 are due to G. Metafune, E.M. Ouhabaz and D. Pallara whom we thank for communicating the above results and allowing us to reproduce them here.

B.1 Gaussian upper bounds for heat kernels

We collect the known Gaussian upper bound results in the following statement and we refer to [45, Theorem 5.7] for the proof.

Theorem B.1.1. *(Kernel estimates)*
Let Ω be an open set of $\mathbf{R}^n$ uniformly regular of class C^2. Let $\mathcal{A}$, $\mathcal{B}$ be as in (2.3)–(2.7) *and let $(T(t))_{t\geq 0}$ be the analytic semigroup generated by the realization of $\mathcal{A}$ in $L^1(\Omega)$ with homogeneous boundary conditions $\mathcal{B}u = 0$; for the kernel $p : (0, +\infty)\times\Omega\times\Omega \to \mathbf{R}$ of the semigroup $(T(t))_{t\geq 0}$ the following estimates hold: there exist $b, c_1 > 0$, a real number ω such that for $|\alpha|, |\beta| < 2$, $x, y \in \Omega$ and $t > 0$*

$$|D_x^\alpha D_y^\beta p(t,x,y)| \leq \frac{c_1}{t^{\frac{n+|\alpha|+|\beta|}{2}}} e^{-b\frac{|x-y|^2}{t}} e^{\omega t}. \tag{B.2}$$

B.1.1 Some norm estimates

Immediate consequences of the Gaussian upper bound are the following $L^1 - L^p$ and $L^p - L^\infty$ estimates.

Proposition B.1.2. *Let $p \geq 1$ and let $e^{-t\mathcal{A}}$ be the semigroup generated by $\mathcal{A}$. Then there exist $c_2, c_3 > 0$ such that*

$$\|e^{-t\mathcal{A}}\|_{\mathcal{L}(L^1,L^p)} \leq c_2 t^{-\frac{n}{2}(1-\frac{1}{p})} \quad 0 < t < 1, \tag{B.3}$$

and

$$\|e^{-t\mathcal{A}}\|_{\mathcal{L}(L^p,L^\infty)} \leq c_3 t^{-\frac{n}{2p}} \quad 0 < t < 1. \tag{B.4}$$

Proof. Let $f \in L^1(\Omega)$; then, using (B.2) we get

$$\begin{aligned}
\|e^{-t\mathcal{A}} f\|_{L^p(\Omega)}^p &= \int_\Omega \Big| \int_\Omega p(t,x,y) f(y)\, dy\Big|^p dx \\
&\leq \|f\|_{L^1(\Omega)}^p \int_\Omega \|p(t,x,\cdot)\|_{L^\infty(\Omega)}^p dx \\
&\leq c_1 t^{-np/2} \|f\|_{L^1(\Omega)}^p \int_\Omega e^{-b\frac{|x-y|^2}{t}} dx \\
&= c_1' t^{-\frac{n}{2}(p-1)} \|f\|_{L^1(\Omega)}^p
\end{aligned}$$

Thus

$$\|e^{-t\mathcal{A}} f\|_{L^p(\Omega)} \leq c_2 t^{-\frac{n}{2}(1-\frac{1}{p})} \|f\|_{L^1(\Omega)}$$

and (B.3) is proved. Similarly, let $f \in L^p$, and $p' = p/(p-1)$, then again by (B.2)

$$\begin{aligned}\|e^{-t\mathcal{A}}f\|_{L^\infty(\Omega)} &= \sup_{x\in\Omega}\Big|\int_\Omega p(t,x,y)f(y)\,dy\Big| \\ &\le \|f\|_{L^p(\Omega)}\sup_{x\in\Omega}\|p(t,x,\cdot)\|_{L^{p'}(\Omega)} \\ &\le c_1\|f\|_{L^p(\Omega)}\sup_{x\in\Omega}\Big(t^{-\frac{n}{2}p'}\int_\Omega e^{-bp'\frac{|x-y|^2}{t}}\,dy\Big)^{1/p'} \\ &= c_3 t^{-\frac{n(p'-1)}{2p'}}\|f\|_{L^p(\Omega)} = c_3 t^{-\frac{n}{2p}}\|f\|_{L^p(\Omega)}.\end{aligned}$$

□

B.2 Gaussian lower bounds

This section is devoted to obtain Gaussian lower bounds for $p(t,x,y)$. Such lower bounds in the symmetric case can be deduced from Gaussian upper bounds and Hölder continuity of the kernel.

Remark B.2.1. One can easily observe that if some Gaussian lower bounds are established for p', the same hold for p, more precisely $p(t,x,y) \ge e^{-\omega t}p'(t,x,y)$. Indeed, since $c \in L^\infty(\Omega)$, then there exists $\omega > 0$ such that $-\omega \le c(x) \le \omega$ a.e. $x \in \Omega$. Let $f \ge 0$ in Ω and consider u and v solutions respectively of the problems

$$\begin{cases} \partial_t u = \operatorname{div}(A\cdot Du) + B\cdot Du & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = f(x) & \text{in } \Omega \\ \langle ADu,\nu\rangle = 0 & \text{in } (0,+\infty)\times\partial\Omega. \end{cases} \tag{B.5}$$

and

$$\begin{cases} \partial_t v = \operatorname{div}(A\cdot Dv) + B\cdot Dv + cv & \text{in } (0,+\infty)\times\Omega \\ v(0,x) = f(x) & \text{in } \Omega \\ \langle ADv,\nu\rangle = 0 & \text{in } (0,+\infty)\times\partial\Omega. \end{cases} \tag{B.6}$$

By the maximum principle we deduce that $u \ge 0$. We want to prove that $v \ge e^{-\omega t}u$, hence $p(t,x,y) \ge e^{-\omega t}p'(t,x,y)$ as announced. The problem satisfied by $z = v - w$, with $w = e^{-\omega t}u$, is

$$\begin{cases} \partial_t z - \mathcal{A}z = (c+\omega)w \ge 0 & \text{in } (0,+\infty)\times\Omega \\ z(0,x) = 0 & \text{in } \Omega \\ \langle ADz,\nu\rangle = 0 & \text{in } (0,+\infty)\times\partial\Omega. \end{cases} \tag{B.7}$$

Thus applying again the maximum principle we deduce $z \ge 0$, i.e.

$$p(t,x,y) \ge p'(t,x,y)e^{-\omega t}.$$

As a consequence of Remark B.2.1 we can restrict the study to the operator $\mathcal{A}' = \mathcal{A} - c$ and our aim will be to deduce Gaussian lower bound for p'.

B.2.1 The symmetric case

We first consider the symmetric case and show lower bounds for p_0 (more details are contained in [34]). Under our assumptions on the coefficients, p_0 is Hölder continuous, that is

$$|p_0(t,x,y) - p_0(t,x',y)| \le kt^{-n/2-\gamma/2}|x-x'|^\gamma, \quad \text{for all } x, x', y \in \Omega \tag{B.8}$$

for some $\gamma > 0$ and $k > 0$ independent on y. Moreover it satisfies the Gaussian upper bound in Theorem B.1.1 and the conservation property holds: $\int_\Omega p_0(t,x,y)\,dy = 1$ for all $t > 0$ and $x \in \Omega$.
The first step shows that an on-diagonal lower bound can be deduced from a Gaussian upper bound and the conservation property.

Proposition B.2.2. *There exists a constant $C > 0$ such that for all $t > 0$ and a.e. $x \in \Omega$*

$$p_0(t,x,x) \ge Ct^{-n/2} \tag{B.9}$$

Proof. Fix $\delta > 0$; we have

$$\begin{aligned}
\int_{\Omega\setminus B(x,\delta\sqrt{t})} p_0(t,x,y)\,dy &\le c_1 t^{-n/2} \int_{\Omega\setminus B(x,\delta\sqrt{t})} e^{-\frac{b}{2}\frac{|x-y|^2}{t}} e^{-\frac{b}{2}\frac{|x-y|^2}{t}}\,dy \\
&\le c_1 t^{-n/2} e^{-\frac{b}{2}\delta^2} \int_{\mathbf{R}^n} e^{-\frac{b}{2}\frac{|x-y|^2}{t}}\,dy \\
&\le k e^{-\frac{b}{2}\delta^2}.
\end{aligned}$$

Now, for δ large enough, $ke^{-\frac{b}{2}\delta^2} \le \frac{1}{2}$, thus a.e. $x \in \Omega$

$$\begin{aligned}
\int_{\Omega\cap B(x,\delta\sqrt{t})} p_0(t,x,y)\,dy &= 1 - \int_{\Omega\setminus B(x,\delta\sqrt{t})} p_0(t,x,y)\,dy \\
&\ge \frac{1}{2}.
\end{aligned}$$

It follows by the semigroup property and the symmetry of p_0 that

$$\begin{aligned}
p_0(t,x,x) &= \int_\Omega p_0(t/2,x,y)p_0(t/2,y,x)\,dy \\
&= \int_\Omega |p_0(t/2,x,y)|^2\,dy \\
&\ge \int_{\Omega\cap B(x,\delta\sqrt{t})} |p_0(t/2,x,y)|^2\,dy \\
&\ge \frac{1}{|\Omega\cap B(x,\delta\sqrt{t})|}\left(\int_{\Omega\cap B(x,\delta\sqrt{t})} p_0(t/2,x,y)\,dy\right)^2 \\
&\ge \frac{1}{4|B(x,\delta\sqrt{t})|} \ge Ct^{-n/2}
\end{aligned}$$

for some positive constant C. □

The following step consists in deducing an off-diagonal Gaussian lower bound from the on-diagonal one, by exploiting the Hölder continuity of p_0.

Proposition B.2.3. *There exist positive constants C and η such that*

$$p_0(t,x,y) \geq Ct^{-n/2} \tag{B.10}$$

for all $x, y \in \Omega$ and $t > 0$, sufficiently small such that $|x-y| \leq \eta\sqrt{t}$.

Proof. Since by (B.8)

$$|p_0(t,x,y) - p_0(t,x',y)| \leq kt^{-\frac{n}{2}-\frac{\gamma}{2}}|x-x'|^\gamma$$

for all $x, x', y \in \Omega$ we have

$$p_0(t,x,y) \geq p_0(t,y,y) - kt^{-n/2-\gamma/2}|x-y|^\gamma$$

Thus, using estimate (B.9),

$$\begin{aligned} p_0(t,x,y) &\geq Ct^{-n/2} - kt^{-n/2}\Big(\frac{|x-y|}{\sqrt{t}}\Big)^\gamma \\ &= Ct^{-n/2}(1 - \Big(\frac{|x-y|}{\sqrt{t}}\Big)^\gamma) \\ &\geq Ct^{-n/2} \end{aligned}$$

for $|x-y| \leq \frac{1}{2}\sqrt{t}$, which shows (B.10). □

Let us now extend the previous estimate to arbitrary x, y in Ω.

Theorem B.2.4. *Let $p_0(t,x,y)$ be the heat kernel of $\mathcal{A}_0$. There exist constants $c_0, C_0 > 0$ such that*

$$p_0(t,x,y) \geq C_0 t^{-n/2} e^{-c_0\frac{|x-y|^2}{t}} \tag{B.11}$$

for all $x, y \in \Omega$ and $t > 0$.

Proof. Let $x, y \in \Omega$. Fix $N \in \mathbf{N}$ and consider a finite sequence of points x_i, $0 \leq i \leq N$ in Ω such that $x_0 = x$, $x_N = y$, $[x_i, x_{i+1}] \subset \Omega$ and $|x_i - x_{i+1}| \leq K\frac{|x-y|}{N} =: r$ for all $i = 0, \dots, N-1$. Then by the semigroup property and the positivity of $p_0(t,x,y)$, we have

$$\begin{aligned} p_0(t,x,y) &= \int_\Omega \cdots \int_\Omega p_0(\frac{t}{N}, x, z_1) p_0(\frac{t}{N}, z_1, z_2) \dots p_0(\frac{t}{N}, z_{N-1}, y)\, dz_1 \dots dz_{N-1} \\ &\geq \int_{B(x_1,r)\cap\Omega} \cdots \int_{B(x_{N-1},r)\cap\Omega} p_0(\frac{t}{N}, x, z_1) p_0(\frac{t}{N}, z_1, z_2) \dots p_0(\frac{t}{N}, z_{N-1}, y)\, dz_1 \dots dz_{N-1} \end{aligned}$$

Let us observe that if $z_i \in B(x_i, r)$ and $z_{i+1} \in B(x_{i+1}, r)$ (where we have set $z_0 = x$ and $z_N = y$), then it holds that

$$|z_i - z_{i+1}| \leq |x_i - x_{i+1}| + 2r \leq (K+2)r \quad i = 0, \dots, N-1.$$

If $(K+2)|x-y| \le \eta\sqrt{t}$ (η as in Proposition B.2.3) then $|x-y| \le \eta\sqrt{t}$. In this case (B.11) follows from (B.9) and Proposition B.2.3.
If $(K+2)|x-y| > \eta\sqrt{t}$, we choose $N \ge 2$ to be the smallest integer such that

$$(K+2)\frac{|x-y|}{\sqrt{N}} \le \eta\sqrt{t}.$$

this yields that $|z_i - z_{i+1}| \le (K+2)\frac{|x-y|}{N} \le \eta\sqrt{\frac{t}{N}}$ for $i = 0, \dots, N-1$. Then using Proposition B.2.3 in the above integrals we get

$$\begin{aligned} p_0(t,x,y) &\ge \int_{B(x_1,r)\cap\Omega} \dots \int_{B(x_{N-1},r)\cap\Omega} p_0(\frac{t}{N}, x, z_1) \dots p_0(\frac{t}{N}, z_{N-1}, y)\, dz_1 \dots dz_{N-1} \\ &\ge C^N \Big[\Big(\frac{t}{N}\Big)^{-\frac{n}{2}}\Big]^N \int_{B(x_1,r)\cap\Omega} \dots \int_{B(x_{N-1},r)\cap\Omega} dz_1 \dots dz_{N-1} \\ &\ge k(n,\Omega) C^N \Big[\Big(\frac{t}{N}\Big)^{-\frac{n}{2}}\Big]^N \Big[\Big(\frac{t}{N}\Big)^{\frac{n}{2}}\Big]^{N-1} \ge k(n,\Omega) e^{-C'N}\Big(\frac{t}{N}\Big)^{-\frac{n}{2}}, \end{aligned} \tag{B.12}$$

where we have used the regularity of Ω in order to say that there exists a constant $k(n,\Omega) > 0$ such that for all $x \in \overline{\Omega}$, $|\Omega \cap B(x,r)| \ge k(n,\Omega)|B(x,r)|$. Finally, by definition of N, we have $N - 1 \le K_\gamma \frac{|x-y|^2}{t}$, thus from (B.12)

$$p_0(t,x,y) \ge C_0 t^{-n/2} e^{-c_0\frac{|x-y|^2}{t}}.$$

This concludes the proof. □

B.2.2 The non-symmetric case

Notice that in the proof of Proposition B.2.3 symmetry has not been used. Therefore if $p'(t,\cdot,y)$ is Hölder continuous and $p'(t,x,x) \ge ct^{-n/2}$, using an argument similar to Proposition B.2.3 and Theorem B.2.4, we get Gaussian lower bound for $p'(t,x,y)$, too. Moreover Theorem B.2.4 holds also without assumptions of symmetry and Hölder continuity. Its proof uses only estimate (B.10).

Let us show that the $L^1 \to L^\infty$ norm of the difference $e^{-t\mathcal{A}'} - e^{-t\mathcal{A}_0}$ is relatively small. Now, we prove a result which allows us to conclude without assuming Hölder continuity for p'.

Proposition B.2.5. *There exists $C > 0$ such that*

$$\|e^{-t\mathcal{A}'} - e^{-t\mathcal{A}_0}\|_{\mathcal{L}(L^1,L^\infty)} \le Ct^{-\frac{n}{2}+\frac{1}{2}} \tag{B.13}$$

Proof. The integral representation of the solution gives that

$$\begin{aligned} e^{-t\mathcal{A}'} - e^{-t\mathcal{A}_0} &= \int_0^t e^{-(t-s)\mathcal{A}'} B\cdot De^{-s\mathcal{A}_0} ds \\ &= \int_0^{t/2} e^{-(t-s)\mathcal{A}'} B\cdot De^{-s\mathcal{A}_0} ds + \int_{t/2}^t e^{-(t-s)\mathcal{A}'} B\cdot De^{-s\mathcal{A}_0} ds \end{aligned}$$

Now, by using (B.3), (B.4), the fact that $De^{-s\mathcal{A}_0} \in \mathcal{L}(L^p)$ for $1 < p \le 2$ and that $\|De^{-\frac{s}{2}\mathcal{A}_0}\|_{\mathcal{L}(L^p)} \le c_4 s^{-1/2}$, we get for $p > 1$ (close to 1) the following estimate

$$\begin{aligned}
&\Big\| \int_0^{t/2} e^{-(t-s)\mathcal{A}'} B \cdot De^{-s\mathcal{A}_0} f ds \Big\|_{L^\infty(\Omega)} \\
&\quad \le \|B\|_\infty \|f\|_{L^1(\Omega)} \int_0^{t/2} \|e^{-(t-s)\mathcal{A}'}\|_{\mathcal{L}(L^p,L^\infty)} \|De^{-\frac{s}{2}\mathcal{A}_0}\|_{\mathcal{L}(L^p)} \|e^{-\frac{s}{2}\mathcal{A}_0}\|_{\mathcal{L}(L^1,L^p)}\, ds \\
&\quad \le C\|f\|_{L^1(\Omega)} \int_0^{t/2} (t-s)^{-n/2p} s^{-1/2} s^{-\frac{n}{2}(1-\frac{1}{p})}\, ds \\
&\quad \le Ct^{-n/2p} \|f\|_{L^1(\Omega)} \int_0^{t/2} s^{-1/2} s^{-\frac{n}{2}(1-\frac{1}{p})}\, ds \\
&\quad = Ct^{-\frac{n}{2}+\frac{1}{2}} \|f\|_{L^1(\Omega)}
\end{aligned} \tag{B.14}$$

where $C = C(c_2, c_3, c_4, \|B\|_\infty)$. Moreover from (B.2) we have that $\|De^{-\frac{s}{2}\mathcal{A}_0}\|_{\mathcal{L}(L^q,L^\infty)} \le c_5 s^{-\frac{1}{2}-\frac{1}{q}}$. Thus, using an exponent q close to ∞ we get

$$\begin{aligned}
&\Big\| \int_{t/2}^{t} e^{-(t-s)\mathcal{A}'} B \cdot De^{-s\mathcal{A}_0} f ds \Big\|_{L^\infty(\Omega)} \\
&\qquad \le \|B\|_\infty \|f\|_{L^1(\Omega)} \int_{t/2}^{t} \|De^{-\frac{s}{2}\mathcal{A}_0}\|_{\mathcal{L}(L^q,L^\infty)} \|e^{-\frac{s}{2}\mathcal{A}_0}\|_{\mathcal{L}(L^1,L^q)} ds \\
&\qquad \le C\|f\|_{L^1(\Omega)} \int_{t/2}^{t} s^{-1/2} s^{-n/2}\, ds \\
&\qquad \le Ct^{-\frac{n}{2}+\frac{1}{2}} \|f\|_{L^1(\Omega)}
\end{aligned} \tag{B.15}$$

where $C = C(c_2, c_5, \|B\|_\infty)$. Summing up (B.14) and (B.15) we get the claim. □

As an immediate consequence we deduce a Gaussian lower bound for $p(t, x, y)$.

Theorem B.2.6. *Let $p'(t, x, y)$ be the fundamental solution of $\partial_t - \mathcal{A}'$. Then there exist positive constants C_1, c_1 such that*

$$p'(t, x, y) \ge C_1 t^{-n/2} e^{-c_1 \frac{|x-y|^2}{t}}$$

for all $x, y \in \Omega$ and $t > 0$, sufficiently small.

Proof. Since

$$\|e^{-t\mathcal{A}'} - e^{-t\mathcal{A}_0}\|_{L^1 \to L^\infty} \le Ct^{-\frac{n}{2}+\frac{1}{2}} \tag{B.16}$$

by the Dunford-Pettis theorem (see [7] for a proof) we have

$$\sup_{x,y\in\Omega} |p'(t, x, y) - p_0(t, x, y)| \le Ct^{-\frac{n}{2}+\frac{1}{2}}$$

whence, for $|x - y| \le \eta\sqrt{t}$ (η as in Proposition B.2.3) we get

$$\begin{aligned}
p'(t, x, y) &\ge p_0(t, x, y) - Ct^{-\frac{n}{2}+\frac{1}{2}} \\
&\ge Ct^{-\frac{n}{2}}(1 - \sqrt{t}) \\
&\ge Ct^{-\frac{n}{2}}
\end{aligned}$$

for $t \leq \delta_0$ independent of x, y. Thus Proposition B.2.3 is true also for $p'(t,x,y)$ and proceeding as before we deduce (B.11) also for $p'(t,x,y)$. □

From Remark B.2.1 we finally deduce the following.

Corollary B.2.7. *Let $p(t,x,y)$ be the heat kernel of $\partial_t - \mathcal{A}$. Then there exist constants $c_1, C_1 > 0$ such that*

$$p(t,x,y) \geq C_1 t^{-n/2} e^{-c_1 \frac{|x-y|^2}{t}} e^{-\omega t} \tag{B.17}$$

for all $x, y \in \Omega$ and $t > 0$ small.