Appendix A

A brief introduction to interpolation theory

A.1 Interpolation spaces

This appendix is devoted to present an elementary treatment of the interpolation theory. This theory has a wide range of applications to partial differential operators and partial differential equations. We have used interpolation techniques in Chapter 3. In particular, Theorem 3.1.2 relies on Theorem A.2.7 and both have been proved in [6]. The most known and useful families of interpolation spaces are the real and the complex interpolation spaces.
Let X, Y be two real or complex Banach spaces. By $X = Y$ we mean that X and Y have the same elements with equivalence of the norms. By $Y \hookrightarrow X$ we mean that Y is continuously embedded in X.
Suppose that $Y \hookrightarrow X$; we say that D is an intermediate space between X and Y if

$$Y \hookrightarrow D \hookrightarrow X.$$

An interpolation space between X and Y is any intermediate space such that for every $T \in \mathcal{L}(X)$, whose restriction to Y belongs to $\mathcal{L}(Y)$, the restriction to D belongs to $\mathcal{L}(D)$. Another important class of intermediate spaces are the space of class J_α.

Definition A.1.1. *An intermediate space D between X and Y is said to be of class J_α if there exists a constant $C > 0$ such that*

$$\|y\|_D \le C\|y\|_Y^\alpha \|y\|_X^{1-\alpha}, \quad y \in Y.$$

In this case we write $D \in J_\alpha(X, Y)$.

A.1.1 Some interpolation estimates

In the next section some important examples of interpolatory inclusion are shown. First we prove a useful interpolation estimate which allows us to estimate the L^p norm of the gradient of a function with respect to the L^p norm of the function and of its second derivatives. For a more general statement see [1, Theorem 4.17].

Proposition A.1.2. *Let $1 \le p < \infty$, then $W^{1,p}(\mathbf{R}^n)$ is of class $J_{1/2}$ between $L^p(\mathbf{R}^n)$ and $W^{2,p}(\mathbf{R}^n)$. In other words*

$$\|Du\|_{L^p(\mathbf{R}^n)} \le c\|D^2u\|^{1/2}_{L^p(\mathbf{R}^n)}\|u\|^{1/2}_{L^p(\mathbf{R}^n)} \tag{A.1}$$

for $u \in W^{2,p}(\mathbf{R}^n)$ and some constant $c > 0$.

Proof. We first consider the one-dimensional case. Let $u \in C_c^\infty(\mathbf{R})$ and $x \in \mathbf{R}$; then for $h > 0$

$$u(x+h) = u(x) + hu'(x) + \int_0^h (h-s)u''(s+x)ds$$

hence

$$u'(x) = \frac{u(x+h)-u(x)}{h} - \frac{1}{h}\int_0^h (h-s)u''(s+x)ds.$$

Taking the L^p norm we get

$$\begin{aligned}\|u'\|_{L^p(\mathbf{R})} &\le \frac{2}{h}\|u\|_{L^p(\mathbf{R})} + \frac{1}{h}\int_0^h (h-s)\|u''(s+\cdot)\|_{L^p(\mathbf{R})}ds \\ &= \frac{2}{h}\|u\|_{L^p(\mathbf{R})} + \frac{h}{2}\|u''\|_{L^p(\mathbf{R})}\end{aligned}$$

Let $\varepsilon = \frac{2}{h}$ then

$$\|u'\|_{L^p(\mathbf{R})} \le \varepsilon\|u\|_{L^p(\mathbf{R})} + \frac{1}{\varepsilon}\|u''\|_{L^p(\mathbf{R})}. \tag{A.2}$$

Now, let $u \in C_c^\infty(\mathbf{R}^n)$, then by (A.2) we get

$$\int_{\mathbf{R}} |D_iu|^p dx_i \le 2^{p-1}\left(\varepsilon^p \int_{\mathbf{R}} |D_{ii}u|^p dx_i + \frac{1}{\varepsilon^p}\int_{\mathbf{R}} |u|^p dx_i\right),$$

and by Fubini's Theorem

$$\int_{\mathbf{R}^n} |D_iu|^p dx \le 2^{p-1}\left(\varepsilon^p \int_{\mathbf{R}^n} |D_{ii}u|^p dx + \frac{1}{\varepsilon^p}\int_{\mathbf{R}^n} |u|^p dx\right)$$

therefore

$$\|D_iu\|_{L^p(\mathbf{R}^n)} \le c\left(\varepsilon\|D^2u\|_{L^p(\mathbf{R}^n)} + \frac{1}{\varepsilon}\|u\|_{L^p(\mathbf{R}^n)}\right) \tag{A.3}$$

holds for every $\varepsilon > 0$ and some constant c depending only on p. Minimizing (A.3) on ε, we get

$$\|Du\|_{L^p(\mathbf{R}^n)} \le 2c\|D^2u\|^{1/2}_{L^p(\mathbf{R}^n)}\|u\|^{1/2}_{L^p(\mathbf{R}^n)}$$

for every $u \in C_c^\infty(\mathbf{R}^n)$. Finally the estimate can be extended by density to $W^{2,p}(\mathbf{R}^n)$. □

A.2 Real interpolation spaces

Let X, Y be Banach spaces, with $Y \hookrightarrow X$ (in general this is not required; however, this simplifying assumption is satisfied in the case we are investigating). We describe briefly the K-method used to construct a family of intermediate spaces between X and Y, called real interpolation spaces and denoted by $(X,Y)_{\theta,p}$, where $0 < \theta \le 1$ and $1 \le p \le \infty$. Let I be any interval contained in $(0,+\infty)$, $1 \le p < \infty$. We denote by $L^p_*(I)$ the Lebesgue space L^p with respect to the measure $\frac{dt}{t}$ in I. If $p = \infty$, $L^\infty_*(I) = L^\infty(I)$. We set $1/\infty = 0$.

Definition A.2.1. *For every $x \in X$ and $t > 0$, set*

$$K(t,x,X,Y) = \inf_{x=a+b, a\in X, b\in Y} (\|a\|_X + t\|b\|_Y).$$

Now we define a family of intermediate spaces by means of the function K.

Definition A.2.2. *Let $0 < \theta \le 1$, $1 \le p \le \infty$, set*

$$(X,Y)_{\theta,p} = \{x \in X : \, t \mapsto t^{-\theta} K(t,x,X,Y) \in L^p_*(0,+\infty)\}$$

with

$$\|x\|_{\theta,p} = \|t^{-\theta} K(t,x,X,Y)\|_{L^p_*((0,+\infty))}$$

and

$$(X,Y)_\theta = \{x \in X : \, \lim_{t\to 0^+} t^{-\theta} K(t,x,X,Y) = 0\}$$

Definition (A.2.2) concerns only the behavior of $t^{-\theta}K(t,x,X,Y)$ as $t \to 0$, since $K(\cdot,x,X,Y)$ is bounded. Moreover since $K(t,x,X,Y) \ge \min\{1,t\}K(1,x,X,Y)$, for $\theta = 1$ we deduce that

$$(X,Y)_{1,p} = \{0\}, \quad p < \infty.$$

Therefore, henceforth we consider the cases $(\theta,p) \in (0,1) \times [1,+\infty]$ and $(\theta,p) = (1,\infty)$. Such spaces are called *real interpolation spaces*. One can prove that $\|x\|_{(X,Y)_{\theta,p}}$ is a norm in $(X,Y)_{\theta,p}$ and that the following results hold (see [31] for their proof).

Proposition A.2.3. *For all $(\theta,p) \in (0,1) \times [1,+\infty]$ and $(\theta,p) = (1,\infty)$, $(X,Y)_{\theta,p}$ is a Banach space. For all $\theta \in (0,1)$, $(X,Y)_\theta$ is a Banach space, endowed with the norm of $(X,Y)_{\theta,\infty}$.*

The spaces $(X,Y)_{\theta,p}$ and $(X,Y)_\theta$ are of class $J_\theta(X,Y)$ for every $p \in [1,\infty]$. They are actually interpolation spaces, as they enjoy the following property.

Theorem A.2.4. *Let X_i, Y_i be Banach spaces such that $Y_i \hookrightarrow X_i$ for $i = 1,2$. Let $T \in \mathcal{L}(X_1,X_2) \cap \mathcal{L}(Y_1,Y_2)$. Then for every $\theta \in (0,1)$ and $p \in [1,\infty]$, we have*

$$T \in \mathcal{L}((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p}) \cap \mathcal{L}((X_1,Y_1)_\theta,(X_2,Y_2)_\theta)$$

and

$$\|T\|_{\mathcal{L}((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p})} \le (\|T\|_{\mathcal{L}(X_1,X_2)_{\theta,p}})^{1-\theta}(\|T\|_{\mathcal{L}(Y_1,Y_2)_{\theta,p}})^\theta$$

Finally we state without proof the duality theorem for the real method. A proof of it can be found in [46, Section 1.11.2].

Theorem A.2.5. *(Dual space) Let Y dense in X. If $0<\theta<1$ then for $1\le p<\infty$*

$$(X,Y)'_{\theta,p}=(Y',X')_{1-\theta,p'},\quad \frac{1}{p}+\frac{1}{p'}=1,$$

and for $p=\infty$

$$(X,Y)'_{\theta,\infty}=(Y',X')_{1-\theta,1}. \tag{A.4}$$

A.2.1 Examples

We close this section with concrete examples of some interpolation spaces. For $\theta\in(0,1)$, $p\in[1,\infty)$, $W^{\theta,p}(\mathbf{R}^n)$ is the space of all $f\in L^p(\mathbf{R}^n)$ such that

$$[f]_{W^{\theta,p}}=\Big(\int_{\mathbf{R}^n\times\mathbf{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{\theta p+n}}dxdy\Big)^{1/p}<\infty.$$

It is endowed with the norm $\|\cdot\|_{L^p}+[\cdot]_{W^{\theta,p}}$. When $\theta>1$ is not integer, let $[\theta]$ and $\{\theta\}$ be the integral and fractional parts of θ. Then $W^{\theta,p}(\mathbf{R}^n)$ consists of the functions $f\in W^{[\theta],p}(\mathbf{R}^n)$ such that

$$\sum_{|\alpha|=[\theta]}[D^\alpha f]_{W^{\{\theta\},p}}$$

is finite. Analogously in this case we consider the space $W^{\theta,p}$ normed by

$$\|\cdot\|_{W^{[\theta],p}}+\sum_{|\alpha|=[\theta]}[D^\alpha\cdot]_{W^{\{\theta\},p}}.$$

Example 2. *For $0<\theta<1$, $1\le p<\infty$ we have*

$$(C_b(\mathbf{R}^n),C_b^1(\mathbf{R}^n))_{\theta,\infty}=C_b^\theta(\mathbf{R}^n)$$

$$(L^p(\mathbf{R}^n),W^{1,p}(\mathbf{R}^n))_{\theta,p}=W^{\theta,p}(\mathbf{R}^n),$$

with equivalence of the respective norms.

Example 3. *Let $0\le\theta_1<\theta_2\le 1$, $0<\theta<1$, $1\le p<\infty$. Then*

$$(W^{\theta_1,p}(\mathbf{R}^n),W^{\theta_2,p}(\mathbf{R}^n))_{\theta,p}=W^{(1-\theta)\theta_1+\theta\theta_2,p}(\mathbf{R}^n).$$

If Ω is an open set in $\mathbf{R}^n$ with uniformly C^1 boundary, then

$$(W^{\theta_1,p}(\Omega),W^{\theta_2,p}(\Omega))_{\theta,\infty}=W^{(1-\theta)\theta_1+\theta\theta_2,p}(\Omega). \tag{A.5}$$

Example 4. *For $0<\theta<1$, $1\le p,q<\infty$, $m\in\mathbf{N}$,*

$$(L^p(\mathbf{R}^n),W^{m,p}(\mathbf{R}^n))_{\theta,q}=B^{m\theta}_{p,q}(\mathbf{R}^n).$$

Here $B^s_{p,q}(\mathbf{R}^n)$ *is the Besov space defined as follows: if* s *is not an integer, let* $[s]$ *and* $\{s\}$ *be the integer and the fractional parts of* s*, respectively. Then* $B^s_{p,q}(\mathbf{R}^n)$ *consists of the functions* $f \in W^{[s],p}(\mathbf{R}^n)$ *such that*

$$[f]_{B^s_{p,q}} = \sum_{|\alpha|=[s]} \Big(\int_{\mathbf{R}^n} \frac{dh}{|h|^{n+\{s\}q}} \Big(\int_{\mathbf{R}^n} |D^\alpha f(x+h) - D^\alpha f(x)|^p dx \Big)^{q/p} \Big)^{1/q}$$

is finite. In particular, for $p = q$ *we have* $B^s_{p,p}(\mathbf{R}^n) = W^{s,p}(\mathbf{R}^n)$*. If* $s = k \in \mathbf{N}$*, then* $B^k_{p,q}(\mathbf{R}^n)$ *consists of the functions* $f \in W^{k-1,p}(\mathbf{R}^n)$ *such that*

$$[f]_{B^k_{p,q}} = \sum_{|\alpha|=k-1} \Big(\int_{\mathbf{R}^n} \frac{dh}{|h|^{n+q}} \Big(\int_{\mathbf{R}^n} |D^\alpha f(x+2h) - 2D^\alpha f(x+h) + D^\alpha f(x)|^p dx \Big)^{q/p}$$

is finite.

For a complete proof of Examples above see [46, Sections 2.3, 2.4].

Corollary A.2.6. *For* $0 < \theta < 1/2$*,* $1 \le p < \infty$*, we have*

$$(L^p(\mathbf{R}^n), W^{2,p}(\mathbf{R}^n))_{\theta,p} = W^{2\theta,p}(\mathbf{R}^n)$$

with equivalence of the respective norms.

In the following result we characterize the interpolation space between $L^1(\Omega)$ and a subspace of $W^{1,1}(\Omega)$ which takes into account in a suitable way the boundary conditions that are to be imposed in the parabolic of our interest.

Theorem A.2.7. *Let* Ω *be a subset of* $\mathbf{R}^n$ *with uniformly* C^2 *boundary; then for every* $\theta \in (0, 1/2)$ *we have*

$$(L^1(\Omega), W^{2,1}(\Omega) \cap W^{1,1}_{A,\nu}(\Omega))_{\theta,1} = W^{2\theta,1}(\Omega) \tag{A.6}$$

where $\nu(x)$ *denotes the external normal to* $\partial\Omega$ *at* x*,* A *is the matrix in* (2.106) *and* $W^{1,1}_{A,\nu}(\Omega)$ *is the closure of* $\{u \in C^1(\overline{\Omega}) \,|\, \langle A(x) \cdot \nabla u, \nu(x)\rangle = 0$ *for* $x \in \partial\Omega\}$ *with respect to the topology of* $W^{1,1}(\Omega)$*.*

Proof. We define for an open and regular set $\omega \subset \mathbf{R}^n$ the space

$$X^A_\theta(\omega) = (L^1(\omega), W^{2,1}(\omega) \cap W^{1,1}_{A,\nu}(\omega))_{\theta,1}$$

endowed with the norm

$$\|u\|_{X^A_\theta(\omega)} := \int_0^{+\infty} \frac{K(t,u)}{t^{1+\theta}} dt, \quad K(t,u) := \inf_{\substack{a+b=u \\ a \in L^1(\omega) \\ b \in W^{2,1}(\omega) \cap W^{1,1}_{A,\nu}(\omega)}} \big(\|a\|_{L^1(\omega)} + t\|b\|_{W^{2,1}(\omega)} \big) \tag{A.7}$$

We want to prove that $X^A_\theta(\Omega) = W^{2\theta,1}(\Omega)$ for $\theta \in (0, 1/2)$. For the result in the case when $\omega = \mathbf{R}^n$ we refer to [9, Theorem 4.3.6].

We divide this proof in two steps, in the first one we prove that $X_\theta^I(\mathbf{R}^n_+) = W^{2\theta,1}(\mathbf{R}^n_+)$, where I is the identity matrix. In the second one, we use a local change of coordinates and the regularity of the domain Ω to conclude.

First step

We want to prove that

$$(L^1(\mathbf{R}^n_+), W^{2,1}(\mathbf{R}^n_+) \cap W^{1,1}_N(\mathbf{R}^n_+))_{\theta,1} = W^{2\theta,1}(\mathbf{R}^n_+) \tag{A.8}$$

where $W^{1,1}_N(\mathbf{R}^n_+)$ denote the space $W^{1,1}_{I,-e_n}(\mathbf{R}^n_+)$.

Fix $\theta \in (0, 1/2)$ and consider T the operator that to any function $u : \mathbf{R}^n_+ \to \mathbf{R}$ associates

$$Tu = \tilde{u}(x) := \begin{cases} u(x_1, \ldots, x_n) & \text{if } x_n \geq 0 \\ u(x_1, \ldots, -x_n) & \text{if } x_n < 0. \end{cases} \tag{A.9}$$

As it is easily seen $T \in \mathcal{L}(L^1(\mathbf{R}^n_+), L^1(\mathbf{R}^n)) \cap \mathcal{L}(W^{1,1}_N(\mathbf{R}^n_+) \cap W^{2,1}(\mathbf{R}^n_+), W^{2,1}(\mathbf{R}^n))$; therefore applying Theorem A.2.4 we get

$$T \in \mathcal{L}\big((L^1(\mathbf{R}^n_+), W^{1,1}_N(\mathbf{R}^n_+) \cap W^{2,1}(\mathbf{R}^n_+))_{\theta,1}, (L^1(\mathbf{R}^n), W^{2,1}(\mathbf{R}^n))_{\theta,1}\big).$$

As a consequence we deduce that if $u \in (L^1(\mathbf{R}^n_+), W^{1,1}_N(\mathbf{R}^n_+) \cap W^{2,1}(\mathbf{R}^n_+))_{\theta,1}$ then $Tu \in W^{2\theta,1}(\mathbf{R}^n)$ con $2\|u\|_{W^{2\theta,1}(\mathbf{R}^n_+)} = \|\tilde{u}\|_{W^{2\theta,1}(\mathbf{R}^n_+)} \leq \|u\|_{X^I_\theta(\mathbf{R}^n_+)}$, hence $u \in W^{2\theta,1}(\mathbf{R}^n_+)$.
Conversely let $u \in W^{2\theta,1}(\mathbf{R}^n_+)$; then the function $\tilde{u}$ defined in the same way of (A.9) belongs to $W^{2\theta,1}(\mathbf{R}^n)$; indeed

$$\begin{aligned}
[\tilde{u}]_{W^{2\theta,1}(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} dx \int_{\mathbf{R}^n} \frac{|\tilde{u}(x) - \tilde{u}(y)|}{|x-y|^{n+2\theta}} dy \\
&= 2[u]_{W^{2\theta,1}(\mathbf{R}^n_+)} + 2\int_{\mathbf{R}^{n-1}\times\mathbf{R}_+} dx \int_{\mathbf{R}^{n-1}\times\mathbf{R}_-} \frac{|\tilde{u}(x) - \tilde{u}(y)|}{|x-y|^{n+2\theta}} dy \\
&\leq 4[u]_{W^{2\theta,1}(\mathbf{R}^n_+)}.
\end{aligned}$$

Thus, since $X_\theta(\mathbf{R}^n) = W^{2\theta,1}(\mathbf{R}^n)$ for $\theta \in (0, 1/2)$, there exist $v_1 \in L^1(\mathbf{R}^n)$ and $v_2 \in W^{2,1}(\mathbf{R}^n)$ such that $\tilde{u} = v_1 + v_2$ and $t^{-\theta}K(t, \tilde{u}) \in L^1_*(0, +\infty)$. Now, let $g \in C^\infty_c(\mathbf{R}^n)$ with $D_n g = 0$ in $x_n = 0$, then $\tilde{u}$ can be represented as the sum $(v_1 + v_2 - g) + g =: w + g$ with $w \in L^1(\mathbf{R}^n)$, $g \in W^{2,1}(\mathbf{R}^n)$. If we consider the restriction of w and g in $\mathbf{R}^n_+$ we get that $u = \tilde{u}_{|\mathbf{R}^n_+} = w_{|\mathbf{R}^n_+} + g_{|\mathbf{R}^n_+}$ with $w_{|\mathbf{R}^n_+} \in L^1(\mathbf{R}^n_+)$, $g_{|\mathbf{R}^n_+} \in W^{2,1}(\mathbf{R}^n_+) \cap W^{1,1}_N(\mathbf{R}^n_+)$ and $t^{-\theta}K(t, u) \in L^1_*(0, +\infty)$ since $K(t, u) \leq K(t, \tilde{u})$ for all $t \in (0, \infty)$. Thus (A.8) is proved.

Second step

Now we consider the same partition of unity $\{\eta_h\}_h$ associated with the covering $\{U_h\}_h$ of Ω considered in the proof of Proposition 3.1.1. Then, for a given function u defined in Ω, writing u as $\sum_{h=0}^{+\infty} u\eta_h$, we can prove that $u\eta_0 \in X_\theta(\Omega)$ if and only if $u\eta_0 \in W^{2\theta,1}(\Omega)$. For every $h \geq 1$ we can find $\psi_h : B_+(0) \to U_h \cap \Omega$ such that $d(\psi_h)_x(a(x)\nu(x)) = -e_n$, and prove that $v_h := u\eta_h \circ \psi_h$ belongs to $X_\theta(\mathbf{R}^n_+)$ if and only if belongs to $W^{2\theta,1}(\mathbf{R}^n_+)$, by which $u\eta_h \in X_\theta(\Omega)$ if and only if $u\eta_h \in W^{2\theta,1}(\Omega)$. Now in order to conclude we have to show that $u \in X_\theta(\Omega)$ if and only if $u \in W^{2\theta,1}(\Omega)$. Notice that the result is immediate if Ω is bounded, since in that case the covering $\{U_h\}_h$ is finite.

Suppose first that $u \in X_\theta(\Omega)$. Since $X_\theta(\Omega)$ continuously embeds in $L^1(\Omega)$, it is sufficient to estimate the seminorm $[u]_{W^{2\theta,1}(\Omega)}$. Moreover, since $u \in X_\theta(\Omega)$ we also have that $u\eta_h \in X_\theta(\Omega)$ for each $h \in \mathbf{N}$. Notice that, for fixed $x \in U_h$, $y \in U_k$ there exists $I_{hk} \subset \mathbf{N}$ such that

$$u(x) - u(y) = \sum_{i \in I_{hk}} u(x)\eta_i(x) - u(y)\eta_i(y)$$

where either $\operatorname{supp}(\eta_i) \cap U_h \neq \emptyset$ or $\operatorname{supp}(\eta_i) \cap U_k \neq \emptyset$. Since $\{U_h\}_h$ has a bounded overlapping κ, then $\#(I_{hk}) \leq 2\kappa$. Then

$$\begin{aligned} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x-y|^{n+2\theta}}\, dxdy &\leq \sum_{h,k=1}^{\infty} \int_{U_h} dx \int_{U_k} \frac{|\sum_{i\in I_{hk}}(u(x)\eta_i(x) - u(y)\eta_i(y))|}{|x-y|^{n+2\theta}} dy \\ &\leq \sum_{h,k=1}^{\infty} \sum_{i \in I_{hk}} \int_{U_h} dx \int_{U_k} \frac{|u(x)\eta_h(x) - u(y)\eta_h(y)|}{|x-y|^{n+2\theta}} dy\,. \end{aligned} \tag{A.10}$$

Now, we define $V_h = \bigcup_{\{j : U_j \cap U_h \neq \emptyset\}} U_j$, then there is a constant $c_\kappa > 0$ depending only on κ, the overlapping of the U_i, such that

$$\begin{aligned} &\textstyle\sum_{i\in I_{hk}} \|u\eta_i\|_{L^1(U_i)} \leq c_\kappa \|u\|_{L^1(V_h \cup V_k)} \\ &\textstyle\sum_{i\in I_{hk}} \|u\eta_i\|_{W^{2,1}(U_i)} \leq c_\kappa \overline{M} \|u\|_{W^{2,1}(V_h \cup V_k)} \end{aligned} \tag{A.11}$$

where $\overline{M} := \sup_{h\in\mathbf{N}} \|\eta_h\|_{2,\infty}$. Moreover we can write $\Omega = \bigcup_{i=1}^{\kappa} \Omega_i$ where $\Omega_i = \{x \in \Omega : \#\{j : x \in U_j\} = i\}$ and $\Omega_i \cap \Omega_k = \emptyset$ if $i \neq k$. Then

$$\begin{aligned} \sum_{h,k} \int_{V_h \cup V_k} |u|\, dx &= \sum_{h,k} \sum_{i=1}^{\kappa} \int_{(V_h \cup V_k)\cap\Omega_i} |u|\, dx \\ &= \sum_{i=1}^{\kappa} \sum_{h,k} \int_{(V_h \cup V_k)\cap\Omega_i} |u|\, dx \\ &= \sum_{i=1}^{\kappa} i \int_{\Omega_i} |u|\, dx \leq \kappa \|u\|_{L^1(\Omega)}. \end{aligned} \tag{A.12}$$

Analogously,

$$\sum_{h,k} \|u\|_{W^{2,1}(V_h \cup V_k)} \leq \kappa \|u\|_{W^{2,1}(\Omega)}. \tag{A.13}$$

Since the functions $v_h := u\eta_h \circ \psi_h$ belong both to $(L^1(\mathbf{R}^n_+), W^{2,1}(\mathbf{R}^n_+) \cap W^{1,1}_{A,\nu}(\mathbf{R}^n_+))_{\theta,1}$ and $W^{2\theta,1}(\mathbf{R}^n_+)$, and in $\mathbf{R}^N_+$ the norms of $W^{2\theta,1}(\mathbf{R}^n_+)$ and $X_\theta(\mathbf{R}^n_+)$ are equivalent, we get a constant κ_0, depending only on the norm of the embedding of $X_\theta(\mathbf{R}^n_+)$ in $W^{2\theta,1}(\mathbf{R}^n_+)$ and ψ_h, such that

$$\int_{U_h} dx \int_{U_k} \frac{|u(x)\eta_i(x) - u(y)\eta_i(y)|}{|x-y|^{n+2\theta}} dy \leq \kappa_0 \int_0^{+\infty} \frac{1}{t^{1+\theta}} K(t, u\eta_i) dt \tag{A.14}$$

where K is defined in (A.7). By definition of $K(t,\cdot)$ and by (A.11) we get

$$\begin{aligned}
\sum_{i\in I_{hk}} K(t,u\eta_i) &= \sum_{i\in I_{hk}} \inf_{\substack{\tilde a+\tilde b=u\eta_i\\ \tilde a\in L^1(\Omega),\,\tilde b\in W^{2,1}(\Omega)}} \big(\|\tilde a\|_{L^1(\Omega)}+t\|\tilde b\|_{W^{2,1}(\Omega)}\big)\\
&\le \sum_{i\in I_{hk}} \inf_{\substack{a+b=u\\ a\in L^1(\Omega),\,b\in W^{2,1}(\Omega)}} \big(\|a\eta_i\|_{L^1(\Omega)}+t\|b\eta_i\|_{W^{2,1}(\Omega)}\big)\\
&\le \inf_{\substack{a+b=u\\ a\in L^1(\Omega),\,b\in W^{2,1}(\Omega)}} \sum_{i\in I_{hk}} \big(\|a\eta_i\|_{L^1(\Omega)}+t\|b\eta_i\|_{W^{2,1}(\Omega)}\big)\\
&\le \kappa_1 \inf_{\substack{a+b=u\\ a\in L^1(\Omega),\,b\in W^{2,1}(\Omega)}} \big(\|a\|_{L^1(V_h\cup V_k)}+t\|b\|_{W^{2,1}(V_h\cup V_k)}\big)
\end{aligned}$$

where κ_1 depends on κ and $\overline{M}$. Summing up on h,k we get, by (A.12) and (A.13),

$$\sum_{h,k=1}^{+\infty}\sum_{i\in I_{hk}} K(t,u\eta_i)\le \kappa_1\, K(t,u)\,.$$

Then by (A.10), (A.14) and using the last estimate we get

$$\begin{aligned}
\int_{|x-y|<\rho}\frac{|u(x)-u(y)|}{|x-y|^{n+2\theta}}\,dxdy &\le \sum_{h,k=1}^{+\infty}\sum_{i\in I_{hk}}\kappa_0\int_0^{+\infty}\frac{1}{t^{1+\theta}}K(t,u\eta_i)dt\\
&\le \kappa_0\,\kappa_1\int_0^{+\infty}\frac{1}{t^{1+\theta}}K(t,u)dt=\kappa_0\,\kappa_1\|u\|_{X_\theta(\Omega)}\,,
\end{aligned}$$

whence $X_\theta(\Omega)\subset W^{2\theta,1}(\Omega)$. To prove the reverse inclusion, consider $\{\eta_h,U_h\}_h$ as before. First of all observe that, we can estimate for each $\rho>0$

$$[u\eta_h]_{W^{2\theta,1}(\Omega)}\le \frac{c}{\rho^{n+2\theta}}\|u\|_{L^1(U_h)}+\int_{|x-y|<\rho}\frac{|u(x)\eta_h(x)-u(y)\eta_h(y)|}{|x-y|^{n+2\theta}}\,dxdy\,.$$

where $c=2|U_h|$ is a positive constant independent on h since U_h are balls with fixed radius. Adding and subtracting $u(x)\eta_h(y)$ we can estimate

$$\begin{aligned}
&\int_{|x-y|<\rho}\frac{|u(x)\eta_h(x)-u(y)\eta_h(y)|}{|x-y|^{n+2\theta}}dxdy\\
&\le \int_{\Omega\times\Omega}\Big[\mathrm{Lip}(\eta_h)\frac{|u(x)|}{|x-y|^{n-1+2\theta}}\chi_{A_{h,\rho}}(x,y)+\frac{|u(x)-u(y)|}{|x-y|^{n+2\theta}}\chi_{\Omega\times U_h}(x,y)\Big]dxdy
\end{aligned}$$

where $A_{h,\rho}=(U_h\times\Omega\cup\Omega\times U_h)\cap\{(x,y)\in\Omega\times\Omega\,:\,|x-y|<\rho\}$. Then, choosing ρ small enough in order that the ρ-enlarged sets U_h^ρ have the same overlapping as the U_h's and $A_{h,\rho}\subset U_h^\rho\times U_h^\rho$, we get

$$\|u\eta_h\|_{W^{2\theta,1}(\Omega)}\le \kappa_2\|u\|_{L^1(U_h^\rho)}+\int_{U_h}dy\int_{B(y,\rho)}\frac{|u(x)-u(y)|}{|x-y|^{n+2\theta}}dx$$

where κ_2 depends (only) on $\|\eta_h\|_{W^{1,\infty}},\theta,\rho,n$. Since the overlapping is bounded we can find two constants κ_3,κ_4 such that

$$\sum_h\|u\eta_h\|_{W^{2\theta,1}(\Omega)}\le\kappa_3\Big[\|u\|_{L^1(\Omega)}+\int_\Omega dy\int_{B(y,\rho)}\frac{|u(x)-u(y)|}{|x-y|^{n+2\theta}}dx\Big]\le\kappa_4\|u\|_{W^{2\theta,1}(\Omega)}\,.$$

Then for each $\epsilon > 0$ we can find $\tilde{a}_h \in L^1(\Omega)$, $\tilde{b}_h \in W^{2,1}(\Omega)$ such that $\tilde{a}_h + \tilde{b}_h = u\eta_h$ and $\|\tilde{a}_h\|_{L^1(\Omega)} + t\|\tilde{b}_h\|_{W^{2,1}(\Omega)} \leq K(t, u\eta_h) + \epsilon 2^{-h}$. Define $a = \sum_h \tilde{a}_h$ and $b = \sum_h \tilde{b}_h$. Then $a + b = u$ and

$$K(t,u) \leq \|a\|_{L^1(\Omega)} + t\|b\|_{W^{2,1}(\Omega)} \leq \sum_h \|\tilde{a}_h\|_{L^1(\Omega)} + t\|\tilde{b}_h\|_{W^{2,1}(\Omega)} \leq \sum_h K(t, u\eta_h) + \epsilon$$

and then $K(t,u) \leq \sum_h K(t, u\eta_h)$. Now, as before, since the functions v_h are in $W^{2\theta,1}(\mathbf{R}^n_+)$ and in $\mathbf{R}^n_+$, the norms of $W^{2\theta,1}(\mathbf{R}^n_+)$ and $X_\theta(\mathbf{R}^n_+)$ are equivalent, there exists a constant κ_5, depending only on the norm of the embedding of $W^{2\theta,1}(\mathbf{R}^n_+)$ in $X_\theta(\mathbf{R}^n_+)$ and ψ_h, such that

$$\int_0^{+\infty} \frac{1}{t^{1+\theta}} K(t, u\eta_h) dt \leq \kappa_5 \int_\Omega dx \int_\Omega \frac{|u(x)\eta_h(x) - u(y)\eta_h(y)|}{|x-y|^{n+2\theta}} dy \,. \tag{A.15}$$

Therefore there is a constant κ_6 (depending only on κ_4 and κ_5)

$$\begin{aligned} \int_0^{+\infty} \frac{1}{t^{1+\theta}} K(t,u) dt &\leq \int_0^{+\infty} \frac{1}{t^{1+\theta}} \sum_{h=1}^{+\infty} K(t, u\eta_h) dt \\ &\leq \kappa_5 \sum_{h=1}^{+\infty} \|u\eta_h\|_{W^{2\theta,1}(\Omega)} \leq \kappa_6 \|u\|_{W^{2\theta,1}(\Omega)}. \end{aligned}$$

□

A.3 Complex interpolation spaces

The complex interpolation methods were introduced by J. L. Lions in [29], A. P. Calderón in [11] and [12]. We shall follow the treatment of [46]. Let Y, X be complex Banach spaces with $Y \hookrightarrow X$ and let S be the strip $\{z = x + iy \in \mathbf{C} : 0 \leq x \leq 1\}$. By the maximum principle for holomorphic functions defined on a strip, we get that if $F : S \to X$ is holomorphic in the interior of S, continuous and bounded in S, then for each $z \in S$

$$\|F(z)\|_X \leq \max\{\sup_{t\in\mathbf{R}} \|F(it)\|_X, \sup_{t\in\mathbf{R}} \|F(1+it)\|_X\}.$$

Definition A.3.1. *Denote by $\mathcal{H}(X,Y)$ the space consisting of all continuous and bounded functions $F : S \to X$ which are holomorphic in the interior of the strip such that $t \mapsto F(it) \in C(\mathbf{R}, X)$, $t \mapsto F(1+it) \in C(\mathbf{R}, Y)$ and such that*

$$\|F\|_{\mathcal{H}(X,Y)} = \max\{ \sup_{t\in\mathbf{R}} \|F(it)\|_X,\ \sup_{t\in\mathbf{R}} \|F(1+it)\|_Y \} < \infty.$$

By using the maximum principle, it is not hard to prove that $\mathcal{H}(X,Y)$ is a Banach space. The complex interpolation spaces are defined by means of functions in $\mathcal{H}(X,Y)$.

Definition A.3.2. *For every $\theta \in [0,1]$, we define*

$$[X,Y]_\theta = \{F(\theta) :\ F \in \mathcal{H}(X,Y)\},$$

with norm

$$\|f\|_{[X,Y]_\theta} = \inf_{F\in\mathcal{H}(X,Y),F(\theta)=f} \|F\|_{\mathcal{H}(X,Y)}$$

That $[X,Y]_\theta$ is a Banach space follows from the fact that $[X,Y]_\theta$ is isomorphic to the quotient space $\mathcal{H}(X,Y)/\mathcal{N}_\theta$ where $\mathcal{N}_\theta$ is the subset of $\mathcal{H}(X,Y)$ consisting of the functions which vanish at $z=\theta$. Since $\mathcal{N}_\theta$ is closed, the quotient space is a Banach space and so is $[X,Y]_\theta$. The Banach space $[X,Y]_\theta$ is indeed an intermediate space as the next proposition states.

Proposition A.3.3. *Let* $\theta \in (0,1)$*; then*

$$Y \hookrightarrow [X,Y]_\theta \hookrightarrow X.$$

Proof. Let $f \in Y$. The constant function $F(z) = f$ belongs to $\mathcal{H}(X,Y)$ and

$$\|F\|_{\mathcal{F}(X,Y)} = \max\{\|f\|_X, \|f\|_Y\} \le c\|f\|_Y$$

for some $c > 0$. Therefore $f = F(\theta) \in [X,Y]_\theta$ and $\|f\|_{[X,Y]_\theta} \le c\|f\|_Y$. The other embedding is a consequence of the maximum principle. Indeed if $f = F(\theta)$ with $F \in \mathcal{H}(X,Y)$ then

$$\begin{aligned}\|f\|_X &\le \max\{\sup_{t\in\mathbf{R}} \|F(it)\|_X, \sup_{t\in\mathbf{R}} \|F(1+it)\|_X\}\\ &\le c\max\{\sup_{t\in\mathbf{R}} \|F(it)\|_X, \sup_{t\in\mathbf{R}} \|F(1+it)\|_Y\}\\ &= c\|F\|_{\mathcal{H}(X,Y)}\end{aligned}$$

so that $f \in X$ and $\|f\|_X \le c\|F\|_{\mathcal{H}(X,Y)}$. □

In general $[X,Y]_\theta$ does not coincide with any $(X,Y)_{\theta,p}$. If X,Y are Hilbert spaces then the equality holds for $p=2$, that is

$$[X,Y]_\theta = (X,Y)_{\theta,2} \quad 0<\theta<1.$$

In the non Hilbertian case there are no general rules.

Two other useful facts are recalled here, one concerning the dual space of such complex interpolation spaces and the last proves that $[X,Y]_\theta$ are actually interpolation spaces.

Theorem A.3.4. *(Dual space) Let* $\theta \in (0,1)$*. If* Y *is dense in* X *and one of the two spaces* X *or* Y *is reflexive, then*

$$[X,Y]'_\theta = [Y',X']_{1-\theta}. \tag{A.16}$$

This theorem is a consequence of the results in A.P. Calderón [12]. For the proof we refer to [12].

Theorem A.3.5. *Let* $(X_1, Y_1), (X_2, Y_2)$ *be complex interpolation couples. Assume that* $T \in \mathcal{L}(X_1, X_2) \cap \mathcal{L}(Y_1, Y_2)$*, then the restriction of* $T_{|[X_1,Y_1]_\theta}$ *is in* $\mathcal{L}([X_1, Y_1]_\theta, [X_2, Y_2]_\theta)$ *for every* $\theta \in (0, 1)$*. Moreover,*

$$\|T\|_{\mathcal{L}([X_1,Y_1]_\theta,[X_2,Y_2]_\theta)} \leq (\|T\|_{\mathcal{L}[X_1,X_2]})^{1-\theta}(\|T\|_{\mathcal{L}[Y_1,Y_2]})^{\theta}.$$

For the proof and a complete analysis of these spaces we refer to [46].