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Estimating Technical Efficiency through Reduced Rank Regression

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Abstract: *In this paper we develop a statistical approach for verifying the possibility of substituting joint production frontier with single-output specification by means of Reduced Rank Regression (RRR) Our work introduces the multivariate model in the RRR framework which allows us to verify the unitary rank of the regression coefficient matrix. If the rank is one it is possible to express the production frontier in terms of aggregated output through an econometric model. Firm-specific efficiency is also measured.*

Keywords: Stochastic Frontier Analysis, Reduced Rank Regression

1. Introduction

Considering a multi-output production, the statistical technique of linear regression analysis may be applied in measuring technical efficiency. Since outputs are generally inter-related, it may be feasible to use an empirical linear relationship to predict the approximate value of a certain response from knowledge of the others. Such relationships can be exploited in the multivariate regression model by imposing constraints on the rank of the coefficients matrix. An interpretation of the reduced rank regression (RRR) model in multi-output stochastic frontiers consists of considering the k inputs which are used to produce p outputs using $r\leq min(p,k)$ productive processes. In the special case r=1, it exists just one productive process.

In most multi-output technology analyses, the starting point is the joint production frontier, $F(\mathbf{y},\mathbf{x})=0$, where $\mathbf{y}$ is a $p\times 1$ vector of outputs and $\mathbf{x}$ is a $k\times 1$ vector of inputs. Since generally it is assumed that the transformation function is separable $f(\mathbf{y})= g(\mathbf{x})$, even in presence of multi-output production the single output frontier is used. To our knowledge, there is a very little empirical literature which directly estimates the multi-output transformation function using econometric approaches. Some references are Adams *et al.* (1997, 1999), Löthgren (1997), and Fernández *et al.* (2000, 2002, 2005).

In this paper we suggest to verify the possibility of substituting joint production frontier with single-output specification by means of Reduced Rank Regression (RRR).

If a single output production frontier specification is reasonable, technical efficiency can be estimated in the framework of stochastic frontier (SF) approach (Aigner et. al 1977- Meeusen and van den Broek 1977).

2. Reduced Rank Regression

Let us assume the following multivariate linear model

$$\tilde{\mathbf{y}} = \boldsymbol{\beta}_0 + \mathbf{C}\tilde{\mathbf{x}} + \boldsymbol{\varepsilon} \tag{1}$$

where $\tilde{\mathbf{y}} = \log(\mathbf{y}) = (\log y_1,\ldots,\log y_p)'$ is the $p\times 1$ vector of the logarithm of outputs, $\tilde{\mathbf{x}} = \log(\mathbf{x}) = (\log x_1,\ldots,\log x_k)'$ is the $k\times 1$ vector of function of log-transformed input $\mathbf{x}$, $\boldsymbol{\beta}_0$ is the $p\times 1$ vector of intercepts, $\mathbf{C}$ is the $p\times k$ regression coefficients matrix, and $\boldsymbol{\varepsilon}$ is the $p\times 1$ vectors of disturbances with zero mean, $E(\boldsymbol{\varepsilon}) = \mathbf{0}$, and covariance matrix $\boldsymbol{\Sigma}_{\varepsilon\varepsilon}$.

If we assume that the coefficient matrix $\mathbf{C}$ is not of full rank, i.e. $r=rank(\mathbf{C})$, then $\mathbf{C}$ can be written as a product of two full ranks lower dimensional matrices:

$$\mathbf{C} = \mathbf{\Gamma B} \tag{2}$$

where $\mathbf{\Gamma}$ is $p \times r$, $\mathbf{B}$ is $r \times k$ and $rank(\mathbf{B}) = rank(\mathbf{\Gamma}) = r < min(p,k)$.
Under this assumption, the regression model in equation (1) can then be written as

$$\tilde{\mathbf{y}} = \boldsymbol{\beta}_0 + \mathbf{\Gamma B}\tilde{\mathbf{x}} + \boldsymbol{\varepsilon} \tag{3}$$

Denoting with $\mathbf{\Sigma}_{\tilde{y}\tilde{x}}$ the cross-covariance matrix between $\tilde{y}$ and $\tilde{\mathbf{x}}$ and assuming $\mathbf{\Sigma}_{\tilde{x}\tilde{x}}$ is non singular, then the estimators of $\mathbf{\Gamma}$ and $\mathbf{B}$ are given by

$$\hat{\mathbf{\Gamma}} = \mathbf{G}^{-1/2}\hat{\mathbf{V}} \tag{4}$$

$$\hat{\mathbf{B}} = \hat{\mathbf{V}}'\mathbf{G}^{1/2}\hat{\mathbf{\Sigma}}_{\tilde{y}\tilde{x}}\hat{\mathbf{\Sigma}}_{\tilde{x}\tilde{x}}^{-1} \tag{5}$$

where $\hat{\mathbf{\Sigma}}_{\tilde{y}\tilde{x}}$ is the estimator of the covariance matrix $\mathbf{\Sigma}_{\tilde{y}\tilde{x}}$, $\hat{\mathbf{\Sigma}}_{\tilde{x}\tilde{x}}$ is the estimator of the covariance matrix of $\tilde{\mathbf{x}}$, $\hat{\mathbf{V}} = \left[\hat{\mathbf{V}}_1, \ldots, \hat{\mathbf{V}}_r\right]$ and $\hat{\mathbf{V}}_j$ (*j=1,..,r*) is the *j*-th eigenvector corresponding to the largest eigenvalue of matrix $\mathbf{G}^{1/2}\hat{\mathbf{\Sigma}}_{\tilde{y}\tilde{x}}\hat{\mathbf{\Sigma}}_{\tilde{x}\tilde{x}}^{-1}\hat{\mathbf{\Sigma}}_{\tilde{x}\tilde{y}}\mathbf{G}^{1/2}$. Notice that $\mathbf{G}$ is a positive-definite matrix which, in general, corresponds to $\tilde{\mathbf{\Sigma}}_{\varepsilon\varepsilon}^{-1}$ or $\hat{\mathbf{\Sigma}}_{\tilde{y}\tilde{y}}^{-1}$.

3. Reduced Rank Regression with rank(C)=1

If the coefficient matrix $\mathbf{C}$ has rank equal to 1, *rank(*$\mathbf{C}$*)*=1, then $\mathbf{C}$ can be written by means of two vectors, $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$, of dimension $p \times 1$ and $k \times 1$, respectively, such that $\mathbf{C} = \boldsymbol{\gamma\beta}'$. This refers to a situation where the k inputs are used to produce p outputs, but the inputs are assumed to be combined through one single productive process.
Accordingly, the multivariate regression model can be written as

$$\tilde{\mathbf{y}} = \boldsymbol{\beta}_0 + \boldsymbol{\gamma\beta}'\tilde{\mathbf{x}} + \boldsymbol{\varepsilon} \tag{6}$$

This model suggests the possibility of expressing the production equivalence surfaces in terms of aggregated output and also provides a link to the multivariate regression models of Adams *et al.* (1997, 1999) and Fernández *et al.* (2000, 2002, 2005).
The model originally proposed in Fernández *et al.* (2000) is

$$\left(\sum_{j=1}^{p} \alpha_j^q y_j^q\right)^{1/q} = e^{\ddot{\beta}_0} g\left(\mathbf{x};\boldsymbol{\beta}\right) e^{u-z} \tag{7}$$

where $\sum_{j=1}^{p} \alpha_j = 1$, q>1, u is a noise term, z is one-sided error term reflecting technical inefficiency and $g\left(\mathbf{x};\boldsymbol{\beta}\right)$ is parameterised function of the inputs. Whereas, the model used in Adams *et al.* (1999) is

$$\prod_{j=1}^{p} y_j^{1/\gamma_j} = e^{\ddot{\beta}_0} g\left(\mathbf{x};\boldsymbol{\beta}\right) e^{u-z} \tag{8}$$

Let us assume that the stochastic component in model (1) can be written as $\boldsymbol{\varepsilon} = \mathbf{u} - z\boldsymbol{\gamma}$ where $\mathbf{u} \approx N\left(\mathbf{0}, \mathbf{\Sigma}\right)$ is the two-sided error term of each output, and $z \approx N^+\left(0, \sigma_z^2\right)$ is an erratic component linked to the efficiency of the aggregated output distributed as a half-normal on the non-negative part of the real number line.
We assume that the two-sided error term and the one-sided error term are each identically, independently distributed (*iid*).
Furthermore, let us assume that $\tilde{\boldsymbol{\beta}}_0 = \dot{\boldsymbol{\gamma}} \circ \boldsymbol{\beta}_0$ and $\mathbf{z} = z\boldsymbol{\gamma}$, where $\dot{\boldsymbol{\gamma}}$ is the vector with generic element $1/\gamma_j$ (*j=1,…,p*) and $\circ$ is the Hadamard product.

From these assumptions, it follows that $\boldsymbol{\beta}_0 = \boldsymbol{\gamma} \circ \tilde{\boldsymbol{\beta}}_0$, and dividing each output y_j, *j=1,…,p,* in (1) by the corresponding weight, γ_j, we have

$$\dot{\boldsymbol{\gamma}} \odot \tilde{\mathbf{y}} = \tilde{\boldsymbol{\beta}}_0 + \mathbf{i}_p \boldsymbol{\beta}' \tilde{\mathbf{x}} - z\mathbf{i}_p + \dot{\boldsymbol{\gamma}} \odot \mathbf{u} \tag{9}$$

Since $\tilde{\mathbf{y}} = \log(\mathbf{y})$, by applying the exponential transform on each member and raising all to a power of q, we obtain

$$\dot{\mathbf{y}}^q = e^{q\tilde{\boldsymbol{\beta}}_0} e^{q\boldsymbol{\beta}'\tilde{\mathbf{x}}} e^{q\dot{\mathbf{u}} - qz} \tag{10}$$

Then, pre-multiplying both members in (10) by $\dot{\boldsymbol{\alpha}}^{q\prime} = \left(\alpha_1^{q/\gamma_1} \quad \dots \quad \alpha_p^{q/\gamma_p} \right)$ we have

$$\dot{\boldsymbol{\alpha}}^{q\prime} \dot{\mathbf{y}}^q = \dot{\boldsymbol{\alpha}}^{q\prime} e^{q\tilde{\boldsymbol{\beta}}_0} e^{q\boldsymbol{\beta}'\tilde{\mathbf{x}}} e^{q\dot{\mathbf{u}} - qz} \tag{11}$$

Finally, let us assume that $\tilde{\boldsymbol{\beta}}_0 = \frac{1}{q} \log \frac{\boldsymbol{\alpha}^q}{\left\| \boldsymbol{\alpha}^q \right\|} + \ddot{\boldsymbol{\beta}}_0$ where $\frac{\boldsymbol{\alpha}^q}{\left\| \boldsymbol{\alpha}^q \right\|}$ is the Moore-Penrose generalized inverse of $\boldsymbol{\alpha}^q$ and $\ddot{\boldsymbol{\beta}}_0$ is a $p \times 1$ unitary vector multiplied by a scalar.

This assumption corresponds to the hypothesis that, given the aggregate output, in the multivariate regression model the differences between the observed outputs depend on the scale factors α_j for $j=1,\dots,p$, on the output weights γ_j for $j=1,\dots,p$ and on the parameter q. In fact, under this assumption the regression model corresponds to

$$\begin{pmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_p \end{pmatrix} = \begin{pmatrix} \frac{\gamma_1}{q} \log \frac{\alpha_1^q}{\left\| \boldsymbol{\alpha}^q \right\|} + \ddot{\beta}_0 \gamma_1 \\ \vdots \\ \frac{\gamma_p}{q} \log \frac{\alpha_p^q}{\left\| \boldsymbol{\alpha}^q \right\|} + \ddot{\beta}_0 \gamma_p \end{pmatrix} + \begin{pmatrix} \gamma_1 \beta_1 & \dots & \gamma_1 \beta_k \\ \vdots & \vdots & \vdots \\ \gamma_p \beta_1 & \dots & \gamma_p \beta_k \end{pmatrix} \begin{pmatrix} \tilde{x}_1 & \dots & \tilde{x}_k \end{pmatrix} - z \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_p \end{pmatrix} + \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix} \tag{12}$$

In this model the linear combination $\boldsymbol{\beta}'\tilde{\mathbf{x}}$ represents the production function that is common to all the endogenous variables; then the relative importance of the production function is provided by the parameters $\boldsymbol{\gamma}$ that also represents the idiosyncratic component with respect to each output both for the model intercept and the one sided error term z.

From the assumption on the intercepts, it follows that

$$\dot{\boldsymbol{\alpha}}^{q\prime} \dot{\mathbf{y}}^q = e^{q\ddot{\beta}_0} e^{q\boldsymbol{\beta}'\tilde{\mathbf{x}}} e^{-qz} \dot{\boldsymbol{\alpha}}^{q\prime} \left[\left(\dot{\boldsymbol{\alpha}}^{q\prime} \right)^+ \odot e^{q\dot{\mathbf{u}}} \right] \tag{13}$$

where $\left(\dot{\boldsymbol{\alpha}}^{q\prime} \right)^+ = \left(\left(\alpha_1^{q/\gamma_1} / \left\| \boldsymbol{\alpha}^q \right\| \right)^{\frac{1}{\gamma_1}} \quad \dots \quad \left(\alpha_p^{q/\gamma_p} / \left\| \boldsymbol{\alpha}^q \right\| \right)^{\frac{1}{\gamma_p}} \right)'$.

Raising both members to the power $1/q$, we have the model

$$\left(\sum_{j=1}^{p} \alpha_j^{\frac{q}{\gamma_j}} y_j^{\frac{q}{\gamma_j}} \right)^{1/q} = e^{\beta_0} \prod_{l=1}^{k} x_l^{\beta_l} e^{-z} \left[\sum_{j=1}^{p} \frac{\boldsymbol{\alpha}_j^{\frac{q}{\gamma_j} + q}}{\left| \boldsymbol{\alpha}^q \right|} e^{\frac{q}{\gamma_j} \varepsilon_j} \right]^{1/q} \tag{14}$$

which is a more general formulation of that proposed by Fernández *et al.* (2000).

However, it is necessary to impose the constraint $\sum_{j=1}^{p} 1/\gamma_j = p$ and restrict $q \geq max(\gamma_j), j=1,\dots,p$, to ensure negativity of the elasticity of any two outputs j and l (for $j \neq l$).

Moreover, from equation (12) follows that the single output specification of the production frontier in terms of aggregate output is given by

$$\mathbf{i}'_p \dot{\boldsymbol{\gamma}} \circ \tilde{\mathbf{y}} \big/ p = \tilde{\boldsymbol{\beta}}_0 \mathbf{i}'_p \big/ p + \boldsymbol{\beta}^T \tilde{\mathbf{x}} - z + \mathbf{i}'_p \dot{\boldsymbol{\gamma}} \circ \mathbf{u} \big/ p \qquad (15)$$

which is the *log* of equation (8) with the constraint $\sum_{j=1}^{p} 1/\gamma_j = p$.

4. Technical Efficiency estimation

Given the estimates of **γ**, we can aggregate the outputs as in (15) and estimate the parameters of interests by maximizing the log–likelihood of normal half-normal distribution with respect to **β** (which include also the intercept), σ^2 ($=\sigma^2_z + \sigma^2_u$) and λ ($=\sigma_z/\sigma_u$) (Aigner et. al 1977)

$$LLF = \text{constant} - n \ln \sigma + \sum_{i=1}^{n} \ln F\left(-\frac{u_i \lambda}{\sigma}\right) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{u}{\sigma}\right)^2 \qquad (16)$$

Given the estimates of **β**, σ^2_u and σ^2_v, the technical efficiency for each firm can be obtained as in Battese e Coelli (1988):

$$\tau_i = E\left(e^{-z_i} \mid u_i\right) = \left(\frac{1 - F(\sigma_* - \mu_{*_i} / \sigma_*)}{1 - F^*(-\mu_{*_i} / \sigma_*)}\right) e^{-\mu_{*i} + \frac{1}{2}\sigma_*^2} \qquad (17)$$

where $\mu_{*i} = -\dfrac{u_i \sigma_u^2}{\sigma^2}$ e $\sigma_* = \dfrac{\sigma_u^2 \sigma_v^2}{\sigma^2}$.

The estimate τ_i given in (17) may be used for the estimation of technical efficiency in (6).

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