A COMPARATIVE STUDY FOR ESTIMATING THE PARAMETERS OF THE SECOND ORDER MOVING AVERAGE PROCESS

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Abstract: *Moving Average process is a representation of a time series written as a finite linear combination of uncorrelated random variables. Our main interest is to compare a classical estimation method; namely Exact Maximum Likelihood Estimation (EMLE) with the Generalized Maximum Entropy (GME) approach for estimating the parameters of the second order moving average processes. In this paper, in applying EMLE we have to find the exact likelihood function through deriving the probability density function of the series. Differentiating the function with respect to the parameters, we can obtain the exact maximum likelihood estimates. On the other hand, the idea of GME is to write the unknown parameters and error terms as the expected value of some proper probability distributions defined over some supports. We carry a simulation study to compare between the presented estimation techniques.*

Keywords: **Time series, Moving average, Exact Maximum Likelihood, Generalized maximum entropy.**

1. Introduction

In time series literature, many books and numerous articles have discussed all aspects of time series applications, a time series maybe expressed in two representations; Autoregressive or Moving Average representations.

A moving average process of order q; MA(q) may be written as a linear combination of uncorrelated random errors as follows (Wei, 1990);

$$Y_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \ . \ . \ . - \theta_q a_{t-q} = \Theta(B)\, a_t \qquad (1)$$

As in our interest the second order moving average process; MA(2) is given as follows:

$$Y_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} = (1 - \theta_1 B - \theta_2 B^2)\, a_t\ . \qquad (2)$$

The moving average process is always a stationary process, for invertibility the roots of $1 - \theta_1 B - \theta_2 B^2 = 0$ must lie outside the unit circle. Al-Talib et al. (2007) compared two classical estimation methods with GME approach for estimating the parameters of some moving average processes. Al-Rawwash et al. (2008) estimated the parameters of MA(1) using EMEL, MOM and GME approach.

2. Parameter Estimation

An important step in statistical analysis is to estimate the parameters of the model of interest. Different estimation methods have been discussed in the literature to estimate the parameters of time series models. In this paper, we are interested in estimating the parameters of the second order moving average process. To carry out this mission, we apply the Exact Maximum Likelihood Estimation Method (EMLE) and the GME approach.

2.1 Exact Maximum Likelihood Estimates

Box et al. (1994) stated the exact likelihood function formula for MA(q):

$$L(\theta, \sigma_a \mid Y) = (2\pi\sigma_a^2)^{-n/2} |D|^{-1/2} \exp\left[\frac{-\sum_{t=1-q}^{n} E^2(a_t \mid Y)}{2\sigma_a^2}\right]. \quad (4)$$

Where $D = I_q + F'L_\theta'^{-1}L_\theta^{-1}F$.

This leads to the exact likelihood function of MA(2):

$$L(\theta_1, \theta_2, \sigma_a \mid Y) = (2\pi\sigma_a^2)^{-n/2} |D|^{-1/2} \exp\left[\frac{-\sum_{t=-1}^{n} E^2(a_t \mid Y)}{2\sigma_a^2}\right]. \quad (5)$$

Therefore the exact maximum likelihood estimates of θ_1, θ_2 are obtained by differentiating (5) with respect to θ_1 and θ_2, as follows;

$$\frac{\partial l}{\partial \theta_1} = \frac{\partial}{\partial \theta_1}\left\{-\frac{1}{2}\ln[D]\right\} - \frac{\partial}{\partial \theta_1}\left\{\frac{\sum_{t=-1}^{n} E^2(a_t \mid Y)}{2\sigma_a^2}\right\}. \quad (6)$$

$$\frac{\partial l}{\partial \theta_2} = \frac{\partial}{\partial \theta_2}\left\{-\frac{1}{2}\ln[D]\right\} - \frac{\partial}{\partial \theta_2}\left\{\frac{\sum_{t=-1}^{n} E^2(a_t \mid Y)}{2\sigma_a^2}\right\}. \quad (7)$$

$\widehat{\theta}_1$ and $\widehat{\theta}_2$ can be found by equating (6) and (7) to zero and solving those equations.

2.2 Generalized Maximum Entropy

Recalling the model of MA (2);

$$y_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

In order to apply GME approach, we need to reparametrize each of the unknown parameter θ_1 and θ_2, as well as the error terms a_t (Golan et al. 1996).

Consistent with GME specification, each unknown parameter and error term should be written as a convex combination presented as the expected value of some proper discrete probability distribution over [0, 1] by a set of equally distanced discrete points with corresponding probabilities.

The reparameterization of θ_1, θ_2 and a_t are given as follows;

$$\theta_1 = A^{(1)}q^{(1)}, \ \theta_2 = A^{(2)}q^{(2)} \text{ and } a_t = V^*W^*.$$

Our objective is to recover the unknown parameters θ_1 and θ_2. The GME solution is to select $w^*, q^{(1)}$ and $q^{(2)}$ that maximizes Shannon's entropy subject to the data (Al-Nasser, 2003), that's to say, to maximize

$$\mathrm{H}(w^*, q^{(1)}, q^{(2)}) = -\sum_{t=1}^{n}\sum_{k=1}^{K} w_{tk}^* \ln w_{tk}^* - \sum_{r=1}^{R} q_r^{(1)} \ln q_r^{(1)} - \sum_{r=1}^{R} q_r^{(2)} \ln q_r^{(2)} \quad (8)$$

Subject to

1. Normalization constrains

$$\bullet \sum_{r=1}^{R} q_r^{(1)} = 1. \qquad \bullet \sum_{r=1}^{R} q_r^{(2)} = 1. \qquad \bullet \sum_{k=1}^{K} w_{tk}^* = 1,\ t = 1,\ldots,n.$$

2. Data constrains

$$\bullet\ y_t = \sum_{k=1}^{K} v_{tk}^* w_{tk}^* + \left(\sum_{r=1}^{R} A_r^{(1)} q_r^{(1)}\right)\left(\sum_{k=1}^{K} v_{t-1,k}^* w_{t-1,k}^*\right) + \left(\sum_{r=1}^{R} A_r^{(2)} q_r^{(2)}\right)\left(\sum_{k=1}^{K} v_{t-2,k}^* w_{t-2,k}^*\right),\ t = 2,\ldots,n.$$

3. The additional constrains required for the invertibility conditions

$$\bullet \sum_{r=1}^{R} A_r^{(2)} q_r^{(2)} - \sum_{r=1}^{R} A_r^{(1)} q_r^{(1)} < 1. \qquad \bullet \sum_{r=1}^{R} A_r^{(1)} q_r^{(1)} + \sum_{r=1}^{R} A_r^{(2)} q_r^{(2)} < 1 \qquad \bullet \left|\sum_{r=1}^{R} A_r^{(2)} q_r^{(2)}\right| < 1$$

The estimates will be as follows;

$$\hat{q}_r^{(1)} = \frac{\exp\left(A_r^{(1)}\left[-\hat{\gamma}_3 + \hat{\gamma}_4 - \left(\sum_{t=1}^{n}\hat{\lambda}_t\right)\left(\sum_{k=1}^{K} v_{t-1,k}^* \hat{w}_{t-1,k}^*\right)\right]\right)}{\sum_{r=1}^{R}\exp\left(A_r^{(1)}\left[-\hat{\gamma}_3 + \hat{\gamma}_4 - \left(\sum_{t=1}^{n}\hat{\lambda}_t\right)\left(\sum_{k=1}^{K} v_{t-1,k}^* \hat{w}_{t-1,k}^*\right)\right]\right)},\ r = 1,2,\ldots,R \quad (9)$$

$$\hat{q}_r^{(2)} = \frac{\exp\left(A_r^{(2)}\left[\hat{\gamma}_3 - \hat{\gamma}_4 + \hat{\gamma}_5 + \hat{\gamma}_6 - \left(\sum_{t=1}^{n}\hat{\lambda}_t\right)\left(\sum_{k=1}^{K} v_{t-2,k}^* \hat{w}_{t-2,k}^*\right)\right]\right)}{\sum_{r=1}^{R}\exp\left(A_r^{(2)}\left[\hat{\gamma}_3 - \hat{\gamma}_4 + \hat{\gamma}_5 + \hat{\gamma}_6 - \left(\sum_{t=1}^{n}\hat{\lambda}_t\right)\left(\sum_{k=1}^{K} v_{t-2,k}^* \hat{w}_{t-2,k}^*\right)\right]\right)},\ \mathrm{r} = 1,2,\ldots,R \quad (10)$$

And

$$\hat{w}_{tk}^* = \frac{\exp\left(-\hat{\lambda}_t v_{tk}\right)}{\sum_{k=1}^{K}\exp\left(-\hat{\lambda}_t v_{tk}\right)},\ t = 1,2,\ldots,n \quad (11)$$

Equations (9), (10) and (11) can be used to form a point estimate of the unknown parameter $\hat{\theta}_1 = A^{(1)}\hat{q}^{(1)}$, $\hat{\theta}_2 = A^{(2)}\hat{q}^{(2)}$ and the unknown error $\hat{a}_t = V\hat{w}$ .in other words, $\hat{\theta}_1 = \sum_{r=1}^{R} A_r^{(1)}\hat{q}_r^{(1)}$, $\hat{\theta}_2 = \sum_{r=1}^{R} A_r^{(2)}\hat{q}_r^{(2)}$ and $\hat{a}_t = \sum_{k=1}^{K} v_k \hat{w}_{tk}$, $t = 1,2,\ldots,n$.

3 Simulation Study

A Monte Carlo experiment was conducted in order to study the performance of the presented methods; EMLE and GME. The simulation study is planed under the following assumption:

We set initial values for the unknown parameters $\theta_1 = -0.1, 0.7$ and $\theta_2 = 0.1, -0.7$, the error term a_t is generated from standard normal. We generate 50 correlated samples with MA (2) pattern, each of size = 10, 30, 50,100.

3.1 Comparison between EMLE and GME

The comparison of EMLE and GME is illustrated by the following points:

- The GME is better than EMLE as a method for estimating the parameters of MA (2) with initial values (-0.1, 0.1) despite the high value of GME at n=50 in estimating θ_1.
- The EMLE is better than GME as a method for estimating the parameters of MA (2) with initial values (-0.1, -0.7). Although GME is better at small sample size in estimating θ_1.
- The EMLE is better than GME as a method for estimating θ_1. On the other hand GME is better than EMLE for estimating θ_2 at initial values (0.7, 0.1).
- The EMLE is better than GME as a method for estimating the parameters of MA (2) with initial values (0.7, -0.7) despite the low value of GME at n=100 in estimating θ_1.

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