Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v10n1p206

The gamma log-logistic Weibull distribution: model, properties and application

By Foya et al.

Published: 26 April 2017

Electronic Journal of Applied Statistical Analysis
Vol. 10, Issue 01, April 2017, 206-241
DOI: 10.1285/i20705948v10n1p206

The gamma log-logistic Weibull distribution: model, properties and application

Susan Foya[a], Broderick O. Oluyede*[b], Adeniyi F. Fagbamigbe[a], and Boikanyo Makubate[a]

[a]*Botswana International University of Science and Technology, Department of Mathematics and Statistical Sciences, Palapye, BW*
[b]*Georgia Southern University, Department of Mathematical Sciences, Statesboro, GA, 30460*

Published: 26 April 2017

In this paper, a new generalized distribution called the gamma log-logistic Weibull (GLLoGW) distribution is proposed and studied. The GLLoGW distribution include the gamma log-logistic, gamma log-logistic Rayleigh, gamma log logistic exponential, log-logistic Weibull, log-logistic Rayleigh, log-logistic exponential, log-logistic as well as other special cases as sub-models. Some mathematical properties of the new distribution including moments, conditional moments, mean and median deviations, Bonferroni and Lorenz curves, distribution of the order statistics and Rényi entropy are derived. Maximum likelihood estimation technique is used to estimate the model parameters. A Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimators is presented and an application to real dataset to illustrate the usefulness of the model is given.

Keywords: Gamma distribution, Log-logistic Weibull distribution, Weibull distribution, Maximum likelihood estimation.

1 Introduction

Motivated by the various applications of log-logistic and Weibull distributions in finance and actuarial sciences, as well as in reliability and economics, we construct a new class of

*Corresponding author:boluyede@georgiasouthern.edu

ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

log-logistic Weibull-type distribution called the gamma log-logistic Weibull (GLLoGW) distribution and apply the model to real lifetime data.

For any baseline cumulative distribution function (cdf) $F(x)$, and $x \in \mathbf{R}$, Zografos and Balakrishnan (2009) defined the distribution (when $\theta = 1$) with probability density function (pdf) $g(x)$ and cdf $G(x)$ as follows

$$g(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}}[-\log(\overline{F}(x))]^{\alpha-1}(1-F(x))^{(1/\theta)-1}f(x), \tag{1}$$

and

$$G(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}}\int_0^{-\log(\overline{F}(x))} t^{\alpha-1}e^{-t/\theta}dt = \frac{\gamma(-\theta^{-1}\log(\overline{F}(x)),\alpha)}{\Gamma(\alpha)}, \tag{2}$$

respectively, for $\alpha, \theta > 0$, where $g(x) = dG(x)/dx$, $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt$ denotes the gamma function, and $\gamma(z,\alpha) = \int_0^z t^{\alpha-1}e^{-t}dt$ denotes the incomplete gamma function. The class of distributions for the special case of $\theta = 1$, is referred to as the ZB-G family of distributions. Also, (when $\theta = 1$), Ristic and Balakrishnan (2012) proposed an alternative gamma-generator defined by the cdf and pdf

$$G_2(x) = 1 - \frac{1}{\Gamma(\alpha)\theta^{\alpha}}\int_0^{-\log(F(x))} t^{\alpha-1}e^{-t/\theta}dt, \quad \alpha > 0, \tag{3}$$

and

$$g_2(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}}[-\log(F(x))]^{\alpha-1}(F(x))^{(1/\theta)-1}f(x), \tag{4}$$

respectively. Note that if $\theta = 1$ and $\alpha = n+1$, in equation (1), we obtain the cdf and pdf of the upper record values U given by

$$G_U(u) = \frac{1}{n!}\int_0^{-\log(1-F(u))} y^n e^{-y}dy, \quad \text{and} \quad g_U(u) = f(u)[-\log(1-F(u))]^n/n!. \tag{5}$$

Similarly, from equation (4), the pdf of the lower record values is given by

$$g_L(t) = f(t)[-\log(F(t))]^n/n!. \tag{6}$$

In this paper, we consider the generalized family of distributions given in equation (4) via log-logistic Weibull distribution. In addition to the motivations provided by Ristic and Balakrishnan (2012), its is also the case that generalization of the log-logistic Weibull distribution via the gamma-generator also establishes the relationship between the distributions in equations (1) and (4), and weighted distributions in general. See Oluyede et al. (2014) for additional details.

Ristic and Balakrishnan (2012), provided motivations for the new family of distributions given in equation (3) when $\theta = 1$, that is for $n \in \mathbf{N}$, equation (3) above is the cdf of the n^{th} lower record value of a sequence of i.i.d. variables from a population with density $f(x)$. Ristic and Balakrishnan (2012) used the exponentiated exponential (EE) distribution with cdf $F(x) = (1-e^{-\beta x})^{\alpha}$, where $\alpha > 0$ and $\beta > 0$, in equation (4) to obtained and study the gamma-exponentiated exponential (GEE) model. See references therein for additional results on the GEE model. Pinho et al. (2012) presented the statistical properties of the gamma-exponentiated Weibull distribution. In this note, we obtain a natural extension for log-logistic Weibull distribution, which we refer to as the gamma log-logistic Weibull (GLLoGW) distribution.

This paper is organized as follows. In section 2, some basic results, the gamma-LLoGW (GLLoGW) distribution, series expansion and its sub-models, hazard and reverse hazard functions and the quantile function are presented. The moments and moment generating function, mean and median deviations are given in section 3. Section 4 contains some additional useful results on the distribution of order statistics and Rényi entropy. In section 5.1, results on the estimation of the parameters of the GLLoGW distribution via the method of maximum likelihood are presented. A Monte Carlo simulation study is conducted to examine the bias and mean square error of the maximum likelihood estimators in section 6. An application is given in section 7, followed by some concluding remarks.

2 GLLoGW Distribution, Expansion of Density, Sub-models, Hazard and Reverse Hazard Functions and Quantile Function

In this section, the GLLoGW distribution, series expansion of its pdf, some sub-models, quantile function, hazard and reverse hazard functions as well some graphs are presented. Now, consider the log-logistic Weibull (LLoGW) distribution (Oluyede et al. (2016)) with the cdf

$$F_{LLoGW}(x) = 1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}, \tag{7}$$

for c, α, $\beta > 0$ and $x \geq 0$. The corresponding LLoGW pdf is given by

$$\begin{aligned} f_{LLoGW}(x) &= e^{-\alpha x^\beta}(1 + x^c)^{-1}\left\{\alpha\beta x^{\beta-1} + \frac{cx^{c-1}}{(1 + x^c)}\right\} \\ &= (1 + x^c)^{-1} e^{-\alpha x^\beta}\left[(1 + x^c)^{-1} cx^{c-1} + \alpha\beta x^{\beta-1}\right], \end{aligned} \tag{8}$$

c, α, $\beta > 0$, and $x \geq 0$. By inserting LLoGW distribution in equation (3), the cdf $G_{GLLoGW}(x) = G(x)$ of the GLLoGW distribution is obtained as follows:

$$\begin{aligned} G_{GLLoGW}(x) &= 1 - \frac{1}{\Gamma(\delta)\theta^\delta}\int_0^{-\log[1-(1+x^c)^{-1}e^{-\alpha x^\beta}]} t^{\delta-1}e^{-t/\theta}dt \\ &= 1 - \frac{\gamma(-\theta^{-1}\log[1 - (1 + x^c)^{-1}e^{-\alpha x^\beta}], \delta)}{\Gamma(\delta)}, \end{aligned} \tag{9}$$

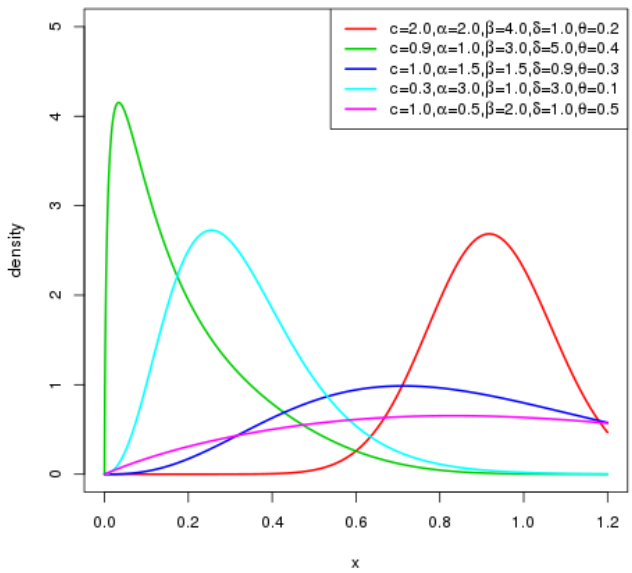

Figure 1: Plots of GLLoGW Density Function

where $x > 0$, $c, \alpha, \beta, \delta, \theta > 0$, and $\gamma(x, \delta) = \int_0^x t^{\delta-1}e^{-t}dt$ is the lower incomplete gamma function. The corresponding GLLoGW pdf is given by

$$\begin{aligned} g_{GLLoGW}(x) &= \frac{1}{\Gamma(\delta)\theta^{\delta}}(1+x^c)^{-1}e^{-\alpha x^{\beta}}[(1+x^c)^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}] \\ &\times \left(-\log[1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}]\right)^{\delta-1} \\ &\times [1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}]^{(1/\theta)-1}. \end{aligned} \tag{10}$$

The graph of pdf for some combinations of values of the model parameters are given in Figure 1. The plots indicate that the GLLoGW pdf can be left or right skewed.

If a random variable X has the gamma log-logistic Weibull density, we write $X \sim GLLoGW(c, \alpha, \beta, \delta, \theta)$. Herein, we set $\theta = 1$, for convenience, and ease of computation. Let $y = [1+x^c]^{-1}e^{-\alpha x^{\beta}}$, then using the series representation $-\log(1-y) = \sum_{i=0}^{\infty}\frac{y^{i+1}}{i+1}$, we have

$$\left[-\log(1-y)\right]^{\delta-1} = y^{\delta-1}\left[\sum_{m=0}^{\infty}\binom{\delta-1}{m}y^m\left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m\right],$$

and applying the result on power series raised to a positive integer, with $a_s = (s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty}a_s y^s\right)^m = \sum_{s=0}^{\infty}b_{s,m}y^s, \tag{11}$$

where $b_{s,m}=(sa_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]a_l b_{s-l,m}$, and $b_{0,m}=a_0^m$, (Gradshteyn and Ryzhik (2000)), the GLLoGW pdf can be written as

$$\begin{aligned}
g_{GLLoGW}(x) &= \frac{1}{\Gamma(\delta)}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\delta-1}{m}b_{s,m}y^{m+s+\delta}((1+x^c)^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}) \\
&= \frac{1}{\Gamma(\delta)}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\delta-1}{m}b_{s,m}[(1+x^c)^{-(m+s+\delta)}e^{-\alpha(m+s+\delta)x^{\beta}} \\
&\times ([1+x^c]^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}) \\
&= \frac{1}{\Gamma(\delta)}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\delta-1}{m}\frac{b_{s,m}}{m+s+\delta}(m+s+\delta) \\
&\times [(1+x^c)^{-(m+s+\delta)}e^{-\alpha(m+s+\delta)x^{\beta}}]([1+x^c]^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}) \\
&= \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\delta-1}{m}\frac{b_{s,m}}{(m+s+\delta)\Gamma(\delta)} \\
&\times f_{BW}(x;c,m+s+\delta,\alpha(m+s+\delta),\beta),
\end{aligned}$$

where $f_{BW}(x;c,m+s+\delta,\alpha(m+s+\delta),\beta)$ is the Burr XII-Weibull (BW) pdf with parameters c, $m+s+\delta$, $\alpha(m+s+\delta)$ and β. Let $D=\{(m,s)\in\mathbf{Z}_+^2\}$, then the weights in the GLLoGW pdf are

$$w_\nu=\binom{\delta-1}{m}\frac{b_{m,s}}{(m+s+\delta)\Gamma(\delta)},$$

and

$$g_{GLLoGW}(x)=\sum_{\nu\in D}w_v f_{BW}(x;c,m+s+\delta,\alpha(m+s+\delta),\beta). \tag{12}$$

It follows therefore that the GLLoGW density is a linear combination of the Burr XII-Weibull pdfs. The statistical and mathematical properties can be readily obtained from those of the Burr XII-Weibull distribution.

Note that $g_{GLLoGW}(x)$ is a weighted pdf, (see Oluyede (1999) and references therein for additional details) with the weight function

$$w(x)=[-\log(1-F(x))]^{\alpha-1}[1-F(x)]^{\frac{1}{\theta}-1}, \tag{13}$$

that is,

$$\begin{aligned}
g_{GLLoGW}(x) &= \frac{[-\log(1-F(x))]^{\alpha-1}[1-F(x)]^{\frac{1}{\theta}-1}}{\theta^{\alpha}\Gamma(\alpha)}f(x) \\
&= \frac{w(x)f(x)}{E_F(w(X))},
\end{aligned}$$

where $0<E_F[[-\log(1-F(X))]^{\alpha-1}[1-F(X)]^{\frac{1}{\theta}-1}]=\theta^{\alpha}\Gamma(\alpha)<\infty$, is the normalizing constant. Similarly,

$$g_2(x) = \frac{[-\log(F(x))]^{\alpha-1}[F(x)]^{\frac{1}{\theta}-1}}{\theta^{\alpha}\Gamma(\alpha)}f(x)=\frac{w_2(x)f(x)}{E_F(w_2(X))},$$

where $0 < E_F(w_2(X)) = E_F([-\log(F(X))]^{\alpha-1}[F(X)]^{\frac{1}{\theta}-1}) = \theta^{\alpha}\Gamma(\alpha) < \infty$.

2.1 Some Sub-models of the GLLoGW Distribution

There are several useful sub-models that can be readily obtained from the GLLoGW distribution. Some of the sub-models of the GLLoGW distribution are listed below:

- If $\beta = 1$, we obtain the gamma log-logistic exponential (GLLoGE) distribution.
- If $\beta = 2$, we have the gamma log-logistic Rayleigh (GLLoGR) distribution.
- When $\alpha \to 0^+$, we have the gamma log-logistic (GLLoG) distribution.
- If $\beta = \delta = \theta = 1$, then the GLLoGW cdf reduces to the two-parameter log-logistic exponential (LLoGE) distribution given by

$$G(x) = 1 - (1 + x^c)^{-1} \exp(-\alpha x), \tag{14}$$

  for $c, \alpha > 0$, and $x \geq 0$. We denote this model by $GLLoGW(c, \alpha, 1, 1, 1)$.

- If $\beta = 2$, $\delta = \theta = 1$, then the GLLoGW cdf reduces to the two-parameter log-logistic Rayleigh (LLoGR) distribution given by

$$G(x) = 1 - (1 + x^c)^{-1} \exp(-\alpha x^2), \tag{15}$$

  for $c, \alpha > 0$, and $x \geq 0$. We denote this model by $GLLoGW(c, \alpha, 2, 1, 1)$.

- If $\theta = 1$, then the GLLoGW cdf reduces to the four parameter distribution with cdf given by

$$G(x) = \frac{1}{\Gamma(\delta)}\gamma(-\log(1 - (1 + x^c)^{-1} \exp(-\alpha x^{\beta})), \delta), \tag{16}$$

  for c, δ, α, $\beta > 0$, and $x \geq 0$. This model is denoted by $GLLoGW(c, \alpha, \beta, \delta, 1)$.

- If $c = \theta = 1$, then the GLLoGW cdf reduces to the three parameter distribution with cdf given by

$$G(x) = \frac{1}{\Gamma(\delta)}\gamma(-\log(1 - (1 + x)^{-1} \exp(-\alpha x^{\beta})), \delta), \tag{17}$$

  for δ, α, $\beta > 0$, and $x \geq 0$. This model is denoted by $GLLoGW(1, \alpha, \beta, \delta, 1)$.

- If $c = \beta = \theta = 1$, then the GLLoGW cdf reduces to the two parameter distribution given by

$$G(x) = \frac{1}{\Gamma(\delta)}\gamma(-\log(1 - (1 + x)^{-1} \exp(-\alpha x)), \delta), \tag{18}$$

  for δ, $\alpha > 0$, and $x \geq 0$. We denote this model by $GLLoGW(1, \alpha, 1, \delta, 1)$.

- If $c = \theta = 1$ and $\beta = 2$, then the GLLoGW cdf reduces to the two parameter model
$$G(x) = \frac{1}{\Gamma(\delta)}\gamma(-\log(1-(1+x)^{-1}\exp(-\alpha x^2)), \delta), \tag{19}$$
for δ, $\alpha > 0$, and $x \geq 0$. We denote this model by $GLLoGW(1, \alpha, 2, \delta, 1)$.
- If $\beta = \delta = \theta = 1$, we obtain the log-logistic exponential (LLoGE) distribution.
- If $\beta = 2$, and $\delta = \theta = 1$, we have the log-logistic Rayleigh (LLoGR) distribution.
- When $\alpha \to 0^+$, $\delta = 1$ and $\theta = 1$, we have the log-logistic (LLoG) distribution.
- If $c = \delta = \theta = 1$, then the GLLoGW cdf reduces to the two-parameter distribution with cdf given by
$$G(x) = 1 - (1+x)^{-1}\exp(-\alpha x^{\beta}), \tag{20}$$
for α, $\beta > 0$, and $x \geq 0$. We denote this model by $GLLoGW(1, \alpha, \beta, 1, 1)$.
- If $c = \beta = \delta = 1$ and $\theta = 1$, then the GLLoGW cdf reduces to the one-parameter distribution given by
$$G(x) = 1 - (1+x)^{-1}\exp(-\alpha x), \tag{21}$$
for $\alpha > 0$, and $x \geq 0$. We denote this model by $GLLoGW(1, \alpha, 1, 1, 1)$.
- If $c = \delta = \theta = 1$, and $\beta = 2$, then the GLLoGW cdf reduces to the one-parameter distribution given by
$$G(x) = 1 - (1+x)^{-1}\exp(-\alpha x^2), \tag{22}$$
for $\alpha > 0$, and $x \geq 0$. We denote this model by $GLLoGW(1, \alpha, 2, 1, 1)$.

2.2 Hazard and Reverse Hazard Functions

Let X be a continuous random variable with distribution function F, and probability density function (pdf) f, then the hazard function, reverse hazard function and mean residual life functions are given by $h_F(x) = f(x)/\overline{F}(x)$, $\tau_F(x) = f(x)/F(x)$, and $\delta_F(x) = \int_x^{\infty}\overline{F}(u)du/\overline{F}(x)$, respectively. The functions $h_F(x)$, $\delta_F(x)$, and $\overline{F}(x)$ are equivalent (Shaked and Shanthikumar (1994)). The hazard and reverse hazard functions of the GLLoGW distribution are given by

$$\begin{aligned} h_G(x) &= \frac{\theta^{-\delta}(-\log(1-[1+x^c]^{-1}e^{-\alpha x^{\beta}}))^{\delta-1}[(1+x^c)^{-1}e^{-\alpha x^{\beta}}]}{\gamma(\delta, -\theta^{-1}\log(1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}))} \\ &\times [(1+x^c)^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}][1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}]^{(1/\theta)-1}, \end{aligned} \tag{23}$$

and

$$\begin{aligned} \tau_G(x) &= \frac{\theta^{-\delta}(-\log(1-[1+x^c]^{-1}e^{-\alpha x^{\beta}}))^{\delta-1}[(1+x^c)^{-1}e^{-\alpha x^{\beta}}]}{\Gamma(\delta)-\gamma(\delta, -\theta^{-1}\log(1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}))} \\ &\times [(1+x^c)^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}][1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}]^{(1/\theta)-1} \end{aligned} \tag{24}$$

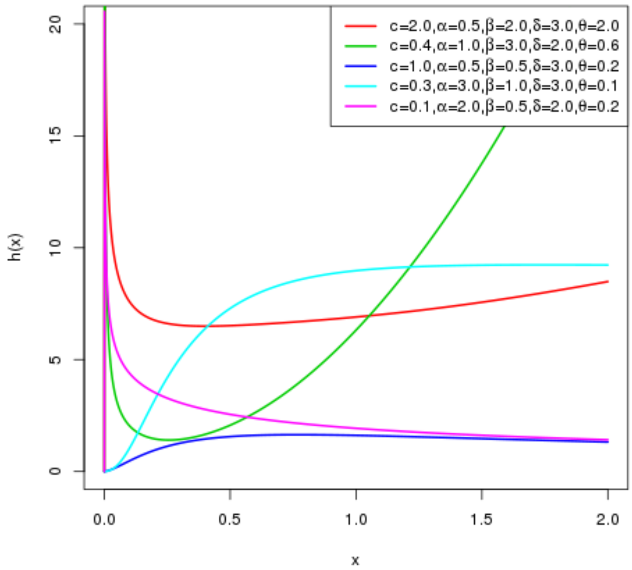

Figure 2: Plots of GLLoGW hazard function

for $x \geq 0$, $c, \alpha, \beta, \delta, \theta > 0$, respectively.

The graph of hazard function for selected parameters are given in Figure 2. The plots show various shapes including monotonically decreasing, monotonically increasing, bathtub and upside down bathtub shapes for five combinations of values of the parameters. This very attractive flexibility makes the GLLoGW hazard function useful and suitable for monotonic and non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations.

2.3 GLLoGW Quantile Function

The quantile function of GLLoGW distribution is obtained by solving the equation

$$G_{GLLoGW}(x) = u, \quad 0 < u < 1. \tag{25}$$

That is,

$$\gamma(\delta, -\theta^{-1}\log(1-(1+x^c)^{-1}e^{-\alpha x^\beta}) = (1-u)\Gamma(\delta),$$

so that

$$1-(1+x^c)^{-1}e^{-\alpha x^\beta} = e^{-\theta\gamma^{-1}((1-u)\Gamma(\delta),\delta)},$$

and

$$-\log(1+x^c) - \alpha x^\beta = \log\left(1-e^{-\theta\gamma^{-1}((1-u)\Gamma(\delta),\delta)}\right).$$

The quantile function of the GLLoGW distribution is obtained by solving the nonlinear equation

$$\alpha x^{\beta} + \log(1 + x^{c}) + \log\left[1 - e^{-\theta\gamma^{-1}((1-u)\Gamma(\delta),\delta)}\right] = 0, \tag{26}$$

by using numerical methods. Consequently, random numbers can be readily generated from GLLoGW distribution based on equation (26). Some quantile for selected parameters are given in Table 1.

Table 1: Table of Quantiles for Selected Parameters

u	(1,3,2,1,4)	(5,5,1,1,5)	(1,1,2,1,0.5)	(2,1,0.8,1.5,2)	(1,0.3,2,0.4,4)
0.1	0.00013	0.00000	0.32006	0.00038	0.01098
0.2	0.00158	0.00012	0.46170	0.00302	0.07972
0.3	0.00798	0.00047	0.57981	0.01036	0.23571
0.4	0.02444	0.00206	0.69001	0.02562	0.49030
0.5	0.05653	0.00633	0.80005	0.05325	0.83897
0.6	0.10883	0.01618	0.91625	0.09938	1.26860
0.7	0.18610	0.03681	1.04674	0.17324	1.77802
0.8	0.29790	0.07940	1.20696	0.29224	2.40096
0.9	0.47553	0.17853	1.44124	0.51027	3.27577

3 Moments, Moment Generating Function, Conditional Moments, Mean and Median Deviations

In this section, we present the moments, moment generating function, mean and median deviations for the GLLoGW distribution. Moments are very important and necessary in any statistical analysis, especially in applications. Moments can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis). These measures (moments, moment generating function, mean and median deviations) can be readily obtained for the sub-models given in section 2. For ease of computation, and without loss of generality, we set $\theta = 1$.

3.1 Moments and Moment Generating Function

Let $\alpha^{*} = \alpha(m + s + \delta)$, and $Y \sim BurrXII - Weibull(c, m + s + \delta, \alpha^{*}, \delta)$. Note that the r^{th} moment of the Burr XII-Weibull (BW) random variable Y is obtained as follows.

The r^{th} raw moment, μ'_r of the BW distribution is given by:

$$\begin{aligned}
\mu'_r = E(Y^r) &= \int_0^\infty y^r e^{-\alpha(m+s+\delta)y^\beta} (1+y^c)^{-(m+s+\delta)} \\
&\times \left(\alpha(m+s+\delta)\beta y^{\beta-1} + \frac{(m+s+\delta)cy^{c-1}}{1+y^c}\right) dy \\
&= \int_0^\infty c(m+s+\delta)y^{r+c-1}(1+y^c)^{-(m+s+\delta)-1}e^{-\alpha(m+s+\delta)y^\beta} dy \\
&+ \int_0^\infty \alpha(m+s+\delta)\beta y^{r+\beta-1}(1+y^c)^{-(m+s+\delta)}e^{-\alpha(m+s+\delta)y^\beta} dy.
\end{aligned}$$

Let $t = (1+y^c)^{-1}$, and apply the fact that $e^{-\alpha(m+s+\delta)y^\beta} = \sum_{p=0}^\infty \frac{(-1)^p[\alpha(m+s+\delta)]^p y^{p\beta}}{p!}$ to get

$$\begin{aligned}
E(Y^r) &= \sum_{p=0}^\infty \frac{(-1)^p(m+s+\delta)[\alpha(m+s+\delta)]^p}{p!} \int_0^1 t^{(m+s+\delta)-(\frac{p\beta+r}{c})-1}(1-t)^{\frac{p\beta+r}{c}} dt \\
&+ \sum_{p=0}^\infty \frac{(-1)^p\beta[\alpha(m+s+\delta)]^{p+1}}{p!c} \int_0^1 t^{(m+s+\delta)-(\frac{p\beta+\beta+r}{c})-1}(1-t)^{\frac{p\beta+\beta+r}{c}-1} dt \\
&= \sum_{p=0}^\infty \frac{(-1)^p[\alpha(m+s+\delta)]^p}{p!} \\
&\times \left[(m+s+\delta)B\left((m+s+\delta)-\frac{p\beta+r}{c}, 1+\frac{p\beta+r}{c}\right)\right. \\
&+ \left.\frac{\alpha(m+s+\delta)\beta}{c}B\left((m+s+\delta)-\frac{p\beta+\beta+r}{c}, \frac{p\beta+\beta+r}{c}\right)\right], \qquad (27)
\end{aligned}$$

for $c, m+s+\delta, \alpha(m+s+\delta), \beta > 0$, and $c > p\beta+\beta+r$. Consequently, the r^{th} moment of the GLLoGW distribution is given by

$$E(X^r) = \sum_{\nu \in D} w_\nu E(Y^r), \qquad (28)$$

where $E(Y^r)$ is given by equation (27). That is, the r^{th} moment of the GLLoGW distribution is

$$\begin{aligned}
E(X^r) &= \sum_{m,s,p=0}^\infty \binom{\delta-1}{m} \frac{b_{s,m}(-1)^p[\alpha(m+s+\delta)]^p}{\Gamma(\delta)p!} \\
&\times \left[B\left(m+s+\delta-\left(\frac{r+p\beta}{c}\right), \frac{r+p\beta+c}{c}\right)\right. \\
&+ \left.\frac{\alpha\beta}{c}B\left(m+s+\delta-\left(\frac{r+\beta+p\beta}{c}\right), \frac{r+\beta+p\beta}{c}\right)\right].
\end{aligned}$$

The moment generating function (MGF) , for $|t| < 1$, is given by:

$$M_X(t) = \sum_{\nu \in D} w_\nu M_Y(t) = \sum_{\nu \in D}\sum_{i=0}^\infty w_\nu \frac{t^i}{i!} E(Y^i).$$

The coefficients of variation (CV), Skewness (CS) and Kurtosis (CK) can be readily obtained. The variance (σ^2), Standard deviation (SD=σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\sigma^2 = \mu_2' - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1},$$

$$CS = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}}, \quad \text{and} \quad CK = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2},$$

respectively. Some moments for selected parameters values are given in Table 2 and plots are given in Figure 3 to Figure ??. Plots of skewness and kurtosis for choices of the model parameters reveal that skewness and kurtosis depend on the parameters c, β and δ.

Table 2: Table of Moments for Selected Parameters

	(1,3,2,1,4)	(5,5,1,1,5)	(1,1,2,1,0.5)	(2,1,0.8,1.5,2)	(0.2,3,2,0.5,2)
EX	0.15808	0.05698	0.84944	0.17384	0.44716
EX^2	0.07266	0.01730	0.91231	0.10972	0.39233
EX^3	0.04606	0.00860	1.15217	0.11884	0.41604
EX^4	0.03530	0.00559	1.64786	0.19367	0.50396
EX^5	0.03094	0.00436	2.60790	0.45245	0.67730
SD	0.21834	0.11857	0.43676	0.28195	0.43860
CV	1.38125	2.08097	0.51418	1.62188	0.98086
CS	1.87303	3.60512	0.63778	3.21757	0.81253
CK	6.68539	19.93772	3.32013	20.28294	2.98737

3.2 Conditional Moments

For lifetime models, it is of particular interest to find the conditional moments and the mean residual lifetime function. The r^{th} conditional moment of the GLLoGW distribution is given by

$$\begin{aligned} E(X^r|X > t) &= \frac{1}{\overline{G}_{GLLoGW}(t)} \int_t^\infty x^r g_{GLLoGW}(x)dx \\ &= \frac{1}{\overline{G}_{GLLoGW}(t)} \sum_{m,s,p=0}^{\infty} \binom{\delta-1}{m} \frac{b_{s,m}(-1)^p[\alpha(m+s+\delta)]^p}{\Gamma(\delta)p!} \\ &\times \left[B_{[1+t^c]^{-1}}\left(m+s+\delta-\left(\frac{r+p\beta}{c}\right), \frac{r+p\beta+c}{c}\right)\right. \\ &+ \left.\frac{\alpha\beta}{c} B_{[1+t^c]^{-1}}\left(m+s+\delta-\left(\frac{r+\beta+p\beta}{c}\right), \frac{r+\beta+p\beta}{c}\right)\right], \end{aligned}$$

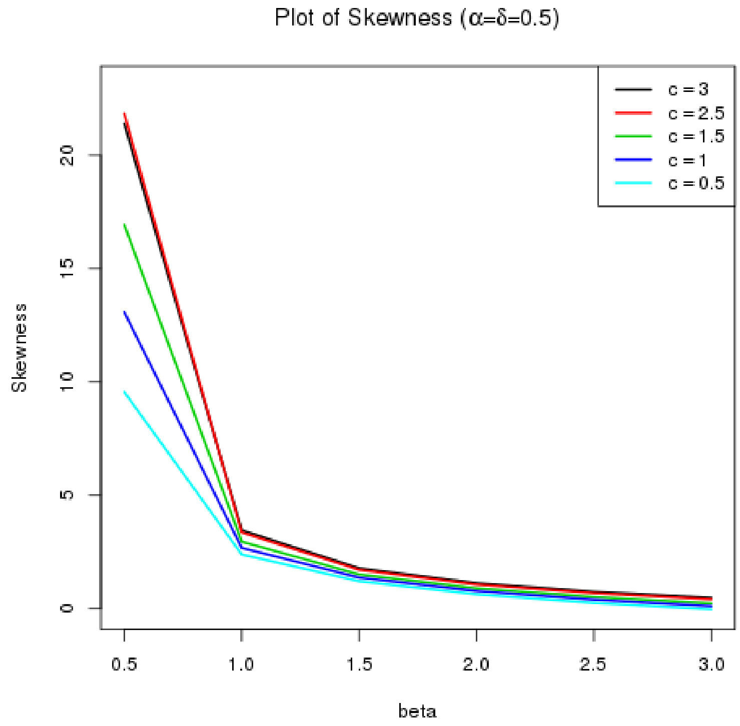

Figure 3: Plots of Skewness and Kurtosis

for $c, m+s+\delta, \alpha(m+s+\delta), \beta > 0$, and $c > p\beta+\beta+r$, where $B_y(a,b) = \int_0^y u^{a-1}(1-u)^{b-1}du$ is the incomplete beta function. The mean residual life function is given by $E(X-t|X > t) = E(X|X > t) - t = V_G(t) - t$, where $V_G(t)$ is referred to as the vitality function of the distribution function G.

3.3 Mean Deviations

The amount of scatter in a population can be measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median. If X has the GLLoGW distribution, we can derive the mean deviation about the mean μ by

$$\delta_1(X) = \int_0^\infty |x-\mu| g_{GLLoGW}(x)dx = 2\mu G_{GLLoGW}(\mu) - 2\mu + 2T(\mu),$$

and the median deviation about the median M by

$$\delta_2(X) = \int_0^\infty |x-M| g_{GLLoGW}(x)dx = 2T(M) - \mu,$$

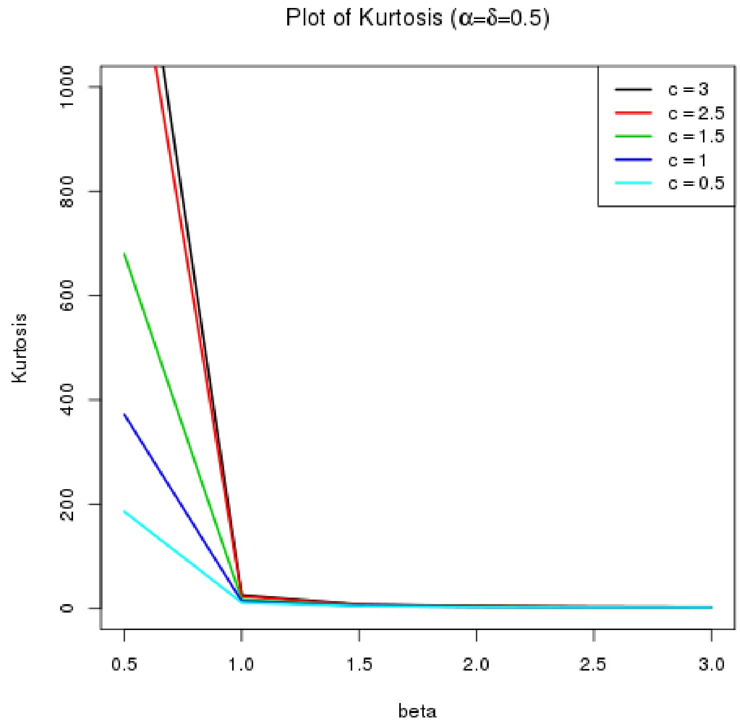

Figure 4: Plots of Skewness and Kurtosis

where $\mu = E(X)$ is given in equation (28) with $r = 1$, M the median of $G_{GLLoGW}(x)$ in equation (26) and $T(a) = \int_a^\infty x \cdot g_{GLLoGW}(x)dx$. Note that

$$\begin{aligned} T(a) &= \sum_{m,s,p=0}^{\infty} \binom{\delta-1}{m} \frac{b_{s,m}(-1)^p[\alpha(m+s+\delta)]^p}{\Gamma(\delta)p!} \\ &\times \left[B_{[1+a^c]^{-1}}\left(m+s+\delta-\left(\frac{r+p\beta}{c}\right), \frac{r+p\beta+c}{c}\right)\right. \\ &+ \left.\frac{\alpha\beta}{c} B_{[1+a^c]^{-1}}\left(m+s+\delta-\left(\frac{r+\beta+p\beta}{c}\right), \frac{r+\beta+p\beta}{c}\right)\right]. \end{aligned} \tag{29}$$

3.4 Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves are widely used tool for analyzing and visualizing income inequality. Lorenz curve, $L(p)$ can be regarded as the proportion of total income volume accumulated by those units with income lower than or equal to the volume x, and Bonferroni curve, $B(p)$ is the scaled conditional mean curve, that is, ratio of group mean income of the population. In this subsection, we present Bonferroni and Lorenz curves. Bonferroni and Lorenz curves have applications not only in economics for the study

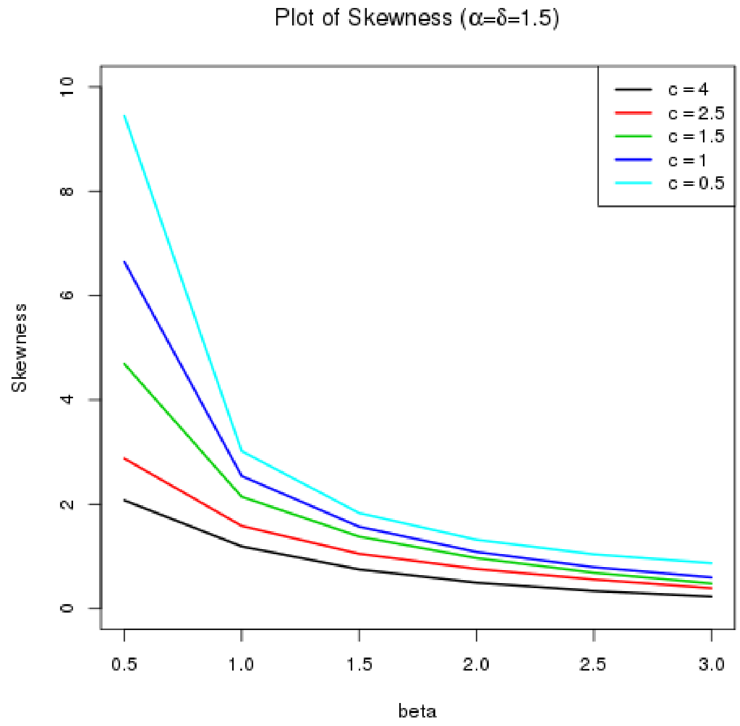

Figure 5: Plots of Skewness and Kurtosis

income and poverty, but also in other fields such as reliability, demography, insurance and medicine. Bonferroni and Lorenz curves for the GLLoGW distribution are given by

$$B(p) = \frac{1}{p\mu}\int_0^q x g_{GLLoGW}(x)dx = \frac{1}{p\mu}[\mu - T(q)],$$

and

$$L(p) = \frac{1}{\mu}\int_0^q x g_{GLLoGW}(x)dx = \frac{1}{\mu}[\mu - T(q)],$$

respectively, where $T(q) = \int_q^\infty x g_{GLLoGW}(x)dx$ is given by equation (29) with "q"in place of "a", and $q = G^{-1}_{GLLoGW}(p)$, $0 \le p \le 1$.

4 Order Statistics and Rényi Entropy

Order statistics play an important role in probability and statistics, particularly in reliability and lifetime data analysis. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty.

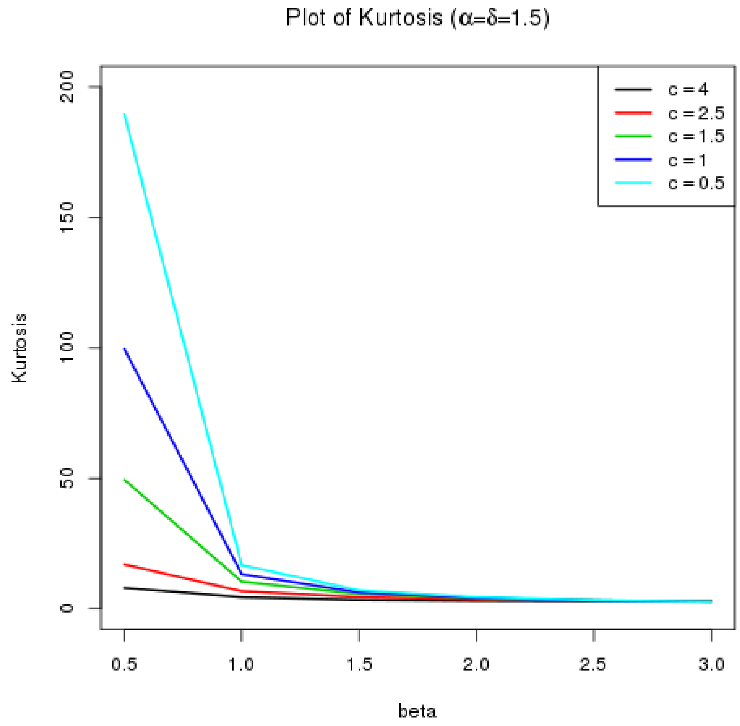

Figure 6: Plots of Skewness and Kurtosis

In this section, we present the distribution of the i^{th} order statistics and Rényi entropy for the GLLoGW distribution. We set $\theta = 1$, for ease of computation in this section.

4.1 Order Statistics

In this subsection, the pdf of the i^{th} order statistic and the corresponding moments are presented. Let $X_1, X_2, ..., X_n$ be independent and identically distributed GLLoGW random variables. The pdf of of the i^{th} order statistic for a random sample of size n for any gamma$-\overline{G}$ family with density (4) can be expressed as an infinite weighted sum of gamma$-\overline{G}$ densities. That is, using the binomial expansion

$$(1-\overline{G}_{GLLoGW}(x))^{i-1} = \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j [\overline{G}_{GLLoGW}(x)]^j,$$

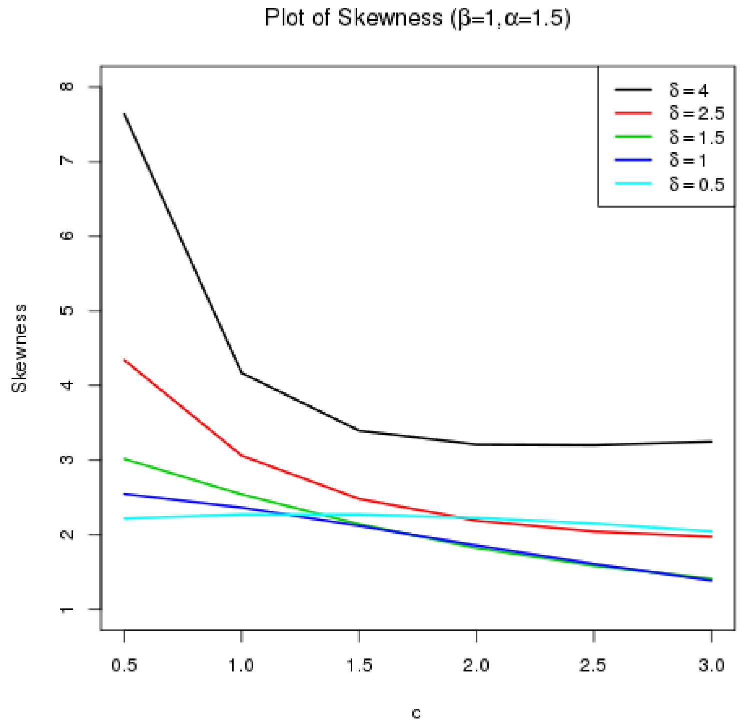

Figure 7: Plots of Skewness and Kurtosis

the pdf of the i^{th} order statistic from the GLLoGW pdf $g_{GLLoGW}(x)$ can be written as

$$\begin{aligned} g_{i:n}(x) &= \frac{n! g_{GLLoGW}(x)}{(i-1)!(n-i)!} [G_{GLLoGW}(x)]^{i-1} [1 - G_{GLLoGW}(x)]^{n-i} \\ &= \frac{n! g_{GLLoGW}(x)}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} [\overline{G}_{GLLoGW}(x)]^{n-i+j} \\ &= \frac{n! g_{GLLoGW}(x)}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} \\ &\times \left[\frac{\gamma(-\log(1-(1+x^c)^{-1} e^{-\alpha x^\beta}), \delta)}{\Gamma(\delta)} \right]^{n-i+j}. \end{aligned}$$

Now, let $0 < y = (1+x^c)^{-1} e^{-\alpha x^\beta} < 1$, $x > 0$, $c, \alpha, \beta > 0$. Using the fact that $\gamma(x, \delta) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{m+\delta}}{(m+\delta)m!}$, and setting $c_m = (-1)^m/((m+\delta)m!)$, we can write the pdf of the i^{th}

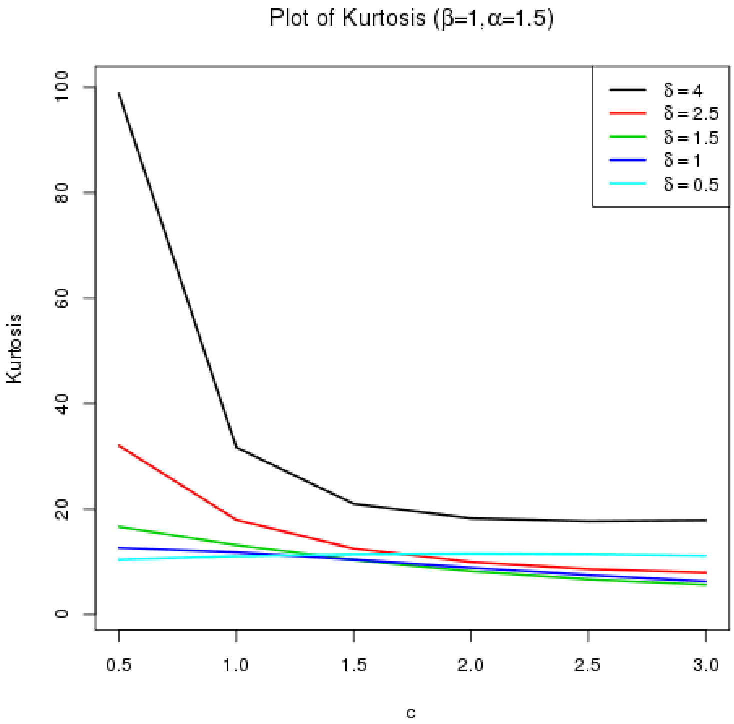

Figure 8: Plots of Skewness and Kurtosis

order statistic from the GLLoGW distribution as follows:

$$\begin{aligned} g_{i:n}(x) &= \frac{n! g_{GLLoGW}(x)}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} \binom{i-1}{j} \frac{(-1)^j}{[\Gamma(\delta)]^{n-i+j}} \\ &\times \ [-\log(1-(1+x^c)^{-1}e^{-\alpha x^\beta})]^{\delta(n-i+j)} \\ &\times \ \left[\sum_{m=0}^{\infty} \frac{(-1)^m (\log(1-(1+x^c)^{-1}e^{-\alpha x^\beta}))^m}{(m+\delta)m!} \right]^{n-i+j} \\ &= \frac{n! g_{GLLoGW}(x)}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} \binom{i-1}{j} \frac{(-1)^j}{[\Gamma(\delta)]^{n-i+j}} \\ &\times \ [-\log(1-(1+x^c)^{-1}e^{-\alpha x^\beta})]^{\delta(n-i+j)} \\ &\times \ \sum_{m=0}^{\infty} d_{m,n-i+j} (-\log(1-(1+x^c)^{-1}e^{-\alpha x^\beta}))^m, \end{aligned}$$

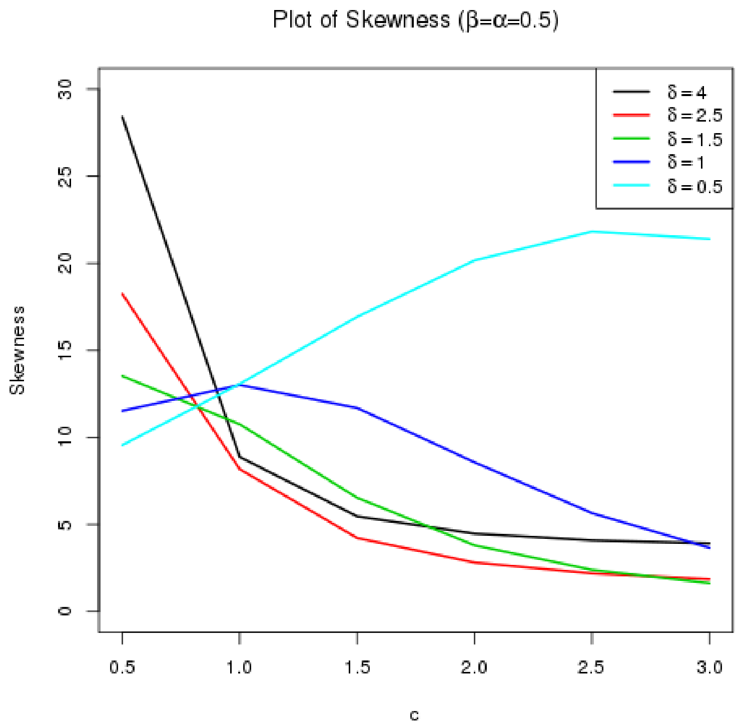

Figure 9: Plots of Skewness and Kurtosis

where $d_0 = c_0^{(n-i+j)}$, $d_{m,n-i+j} = (mc_0)^{-1}\sum_{l=1}^{m}[(n-i+j)l-m+l]c_l d_{m-l,n-i+j}$. We note that

$$\begin{aligned}
g_{i:n}(x) &= \frac{n!g_{GLLoGW}(x)}{(i-1)!(n-i)!}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\delta)]^{n-i+j}} \\
&\times [-\log(1-(1+x^c)^{-1}e^{-\alpha x^\beta})]^{\delta(n-i+j)+m} \\
&= \frac{n![-\log(1-(1+x^c)^{-1}e^{-\alpha x^\beta})]^{\delta-1}}{(i-1)!(n-i)!\Gamma(\delta)} \\
&\times (1+x^c)^{-1}e^{-\alpha x^\beta}((1+x^c)^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}) \\
&\times \sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,n-i+j}}{[\Gamma(\delta)]^{n-i+j}} \\
&\times [-\log(1-(1+x^c)^{-1}e^{-\alpha x^\beta})]^{\delta(n-i+j)+m} \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\binom{i-1}{j}\frac{(-1)^j d_{m,n-i+j}}{[\Gamma(\delta)]^{n-i+j}} \\
&\times \frac{\Gamma(\delta(n-i+j)+m+\delta)}{\Gamma(\delta(n-i+j)+m+\delta)}\frac{[-\log(1-(1+x^c)^{-1}e^{-\alpha x^\beta})]^{\delta(n-i+j)+m+\delta-1}}{\Gamma(\delta)} \\
&\times (1+x^c)^{-1}e^{-\alpha x^\beta}((1+x^c)^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}) \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\binom{i-1}{j} \\
&\times \frac{(-1)^j d_{m,n-i+j}\Gamma(\delta(n-i+j)+m+\delta)}{[\Gamma(\delta)]^{n-i+j+1}}f_{GLLoGW}(x),
\end{aligned}$$

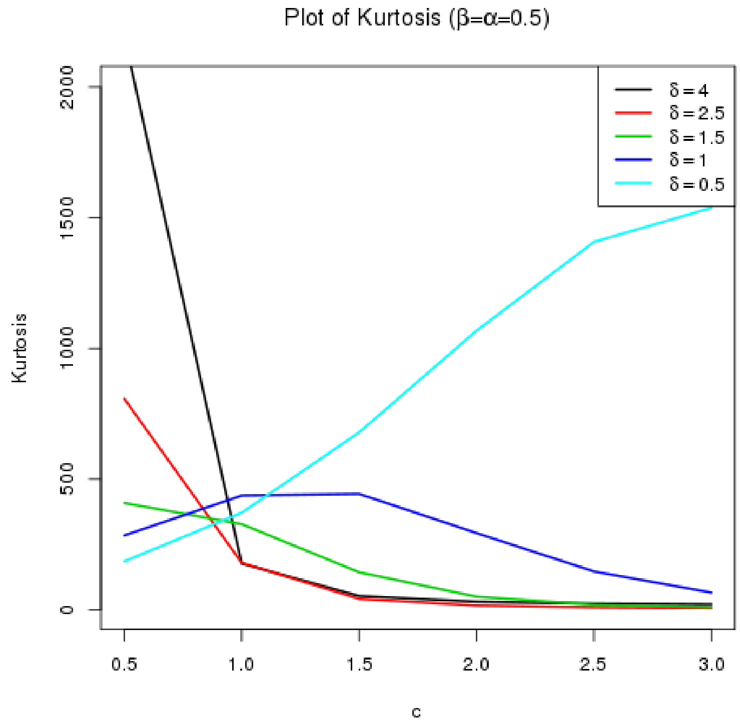

Figure 10: Plots of Skewness and Kurtosis

where

$$\begin{aligned} f_{GLLoGW}(x) &= \frac{[-\log(1-(1+x^c)^{-1}e^{-\alpha x^\beta})]^{\delta(n-i+j)+m+\delta-1}}{\Gamma(\delta(n-i+j)+m+\delta)} \\ &\times \ (1+x^c)^{-1}e^{-\alpha x^\beta}((1+x^c)^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}) \end{aligned} \quad (30)$$

is the GLLoGW pdf with parameters $c, \alpha, \beta > 0$, and shape parameter $\delta^* = \delta(n-i+j)+m+\delta > 0$. It follows therefore that the j^{th} moment of the i^{th} order statistic from the GLLoGW density is given by

$$E(X_{i:n}^j) = \sum_{\nu\in D}\sum_{j=0}^{i-1}\sum_{m,k,n=0}^{\infty} w_\nu \ell_{i,j,m} E(X^j),$$

where $E(X^j)$ is the j^{th} moment of the GLLoGW distribution given by equation (28) with the parameters c, α, β and $\delta(n-i+j)+m+\delta > 0$,

$$\ell_{i,j,m} = \frac{n!}{(i-1)!(n-i)!}\frac{(-1)^j d_{m,n-i+j}\Gamma(\delta(n-i+j)+m+\delta)}{[\Gamma(\delta)]^{n-i+j+1}}.$$

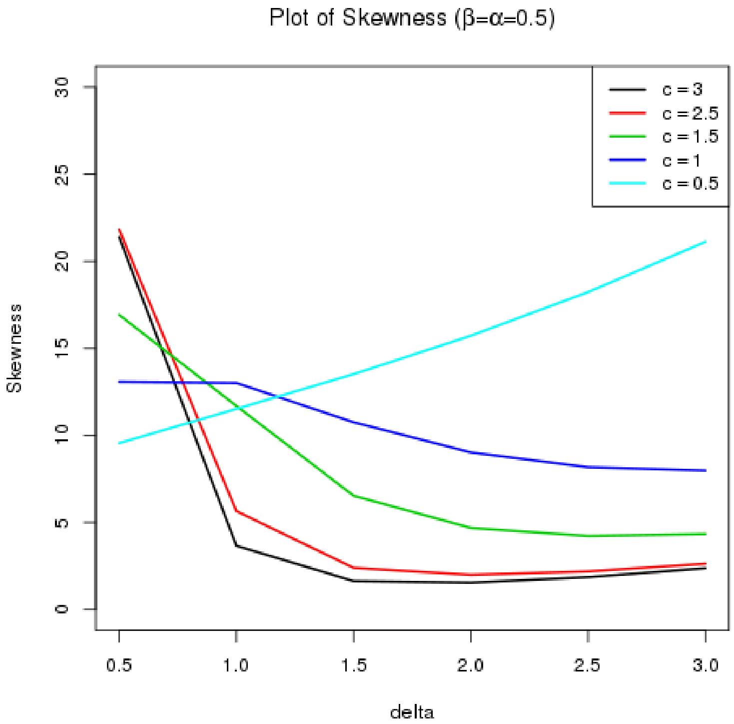

Figure 11: Plots of Skewness and Kurtosis

We note that these moments are often used in several areas including reliability, survival analysis, biometry, engineering, insurance and quality control for the prediction of future failures times from a set of past or previous failures.

4.2 Rényi Entropy

Rényi entropy (Rényi (1960)) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} \log\left(\int_0^\infty [g_{GLLoGW}(x; c, \alpha, \beta, \delta)]^v dx\right), v \neq 1, v > 0. \quad (31)$$

Rényi entropy tends to Shannon entropy as $v \to 1$. Note that

$$\begin{aligned}\int_0^\infty g^v_{GLLoGW}(x)dx &= \left(\frac{1}{\Gamma(\delta)}\right)^v \int_0^\infty (1+x^c)^{-v} e^{-v\alpha x^\beta}((1+x^c)^{-1}cx^{c-1} + \alpha\beta x^{\beta-1})^v \\ &\times \; [-\log(1-(1+x^c)^{-1}e^{-\alpha x^\beta})]^{v(\delta-1)}dx.\end{aligned}$$

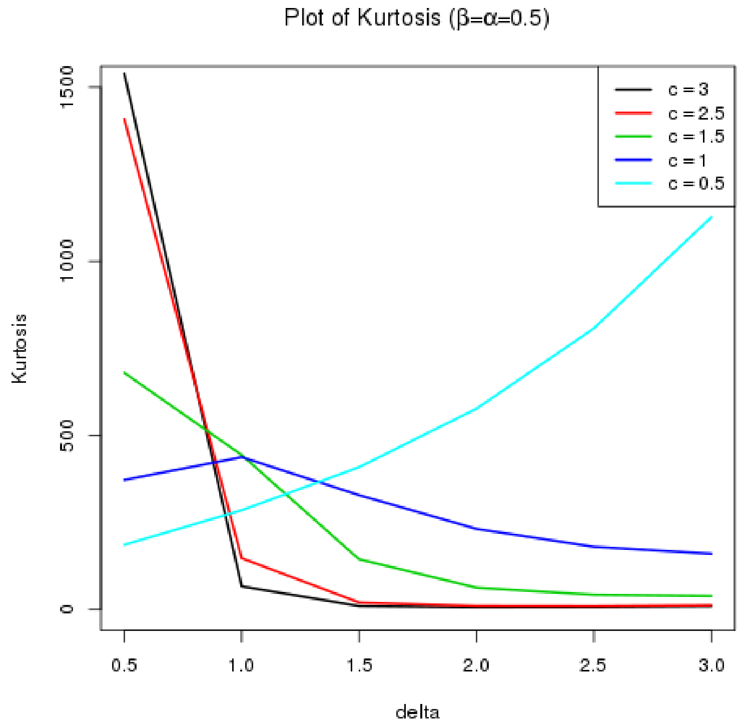

Figure 12: Plots of Skewness and Kurtosis

Let $0 < y = (1+x^c)^{-1}e^{-\alpha x^\beta} < 1$. Note that

$$((1+x^c)^{-1}cx^{c-1} + \alpha\beta x^{\beta-1})^v = \sum_{j=0}^{v}\binom{v}{j}(\alpha\beta)^{v-j}c^j x^{cj-j+(\beta-1)(v-j)}(1+x^c)^{-j},$$

and

$$[-\log(1-(1+x^c)^{-1}e^{-\alpha x^\beta})]^{v\delta-v} = \sum_{m,s=0}^{\infty}\binom{v\delta-v}{m}b_{s,m}(1+x^c)^{-(m+s+v\delta-v)}e^{-\alpha(m+s+v\delta-v)x^\beta},$$

so that

$$\begin{aligned}\int_0^\infty g^v_{GLLoGW}(x)dx &= \sum_{m,s,l=0}^{\infty}\sum_{j=0}^{v}\binom{v\delta-v}{m}\binom{v}{j}\frac{c^j(\alpha\beta)^{v-j}(-1)^l b_{s,m}[\alpha(m+s+v\delta)]^l}{l!}\\ &\times\left(\frac{1}{\Gamma(\delta)}\right)^v\int_0^\infty x^{cj+l\beta+v\beta-j\beta-v}(1+x^c)^{-(m+s+v\delta+j)}dx.\end{aligned}$$

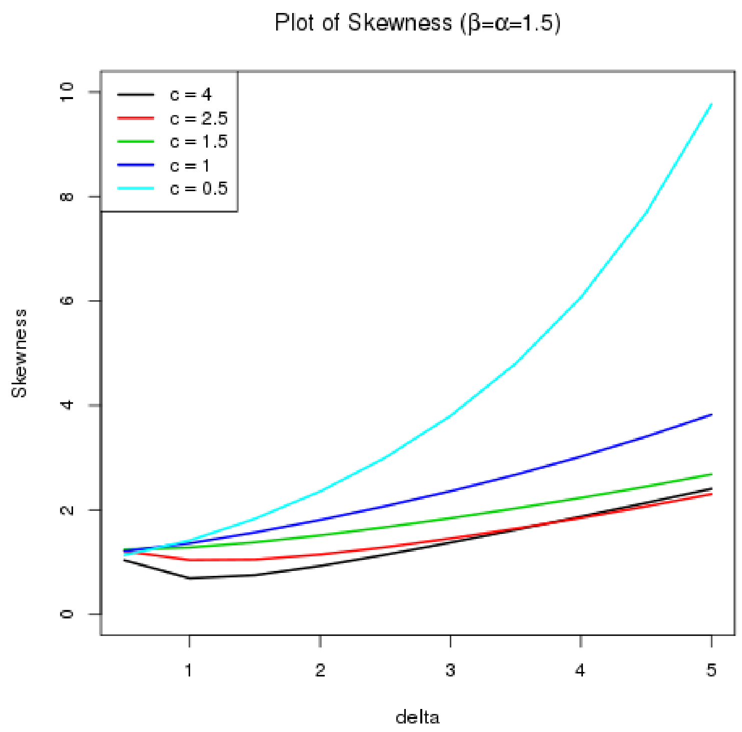

Figure 13: Plots of Skewness and Kurtosis

Note that we have applied the Taylor series expansion to $e^{-\alpha(m+s+v\delta)x^{\beta}} < 1,\ v > 0$, given below

$$e^{-\alpha(m+s+v\delta)x^{\beta}} = \sum_{l=0}^{\infty} \frac{(-1)^l[\alpha(m+s+v\delta)]^l x^{l\beta}}{l!}.$$

Now, with $y = (1+x^c)^{-1}e^{-\alpha x^{\beta}}$, we have

$$\begin{aligned}\int_0^{\infty} g^{v}_{GLLoGW}(x)dx &= \sum_{m,s,l=0}^{\infty}\sum_{j=0}^{v}\binom{v\delta-v}{m}\binom{v}{j}\frac{c^j(\alpha\beta)^{v-j}(-1)^l b_{s,m}[\alpha(m+s+v\delta)]^l}{l!}\left(\frac{1}{\Gamma(\delta)}\right)^{v}\\ &\times\ B\left(m+s+v\delta+\frac{j\beta+v-l\beta-v\beta-1}{c},\frac{cj+l\beta+v\beta-j\beta-v+1}{c}\right),\end{aligned}$$

for $v > 0$, $v \neq 1$.

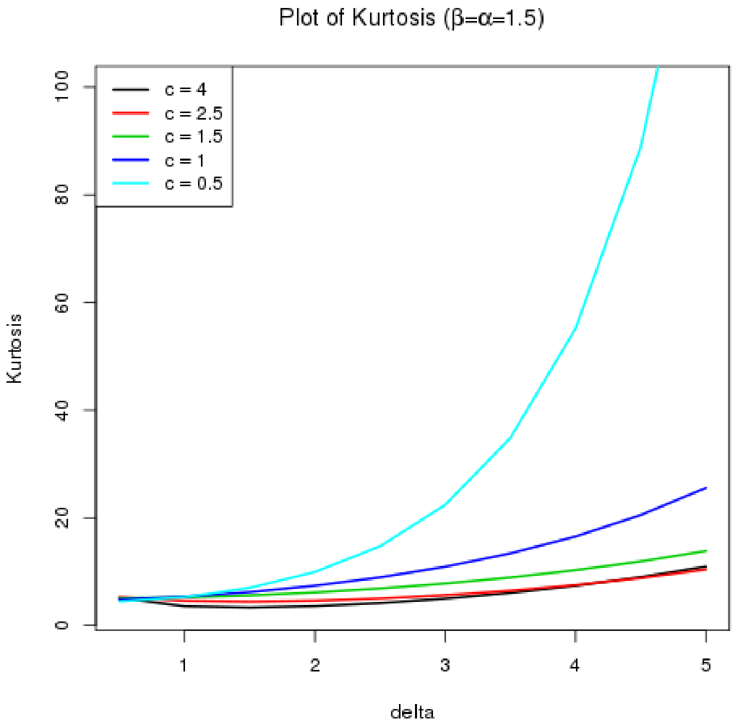

Figure 14: Plots of Skewness and Kurtosis

Consequently, Rényi entropy for the GLLoGW distribution is given by

$$\begin{aligned} I_R(v) &= \frac{1}{1-v}\log\Bigg[\sum_{m,s,l=0}^{\infty}\sum_{j=0}^{v}\binom{v\delta-v}{m}\binom{v}{j} \\ &\times \frac{c^j(\alpha\beta)^{v-j}(-1)^l b_{s,m}[\alpha(m+s+v\delta)]^l}{l!}\left(\frac{1}{\Gamma(\delta)}\right)^v \\ &\times B\left(m+s+v\delta+\frac{j\beta+v-l\beta-v\beta-1}{c},\frac{cj+l\beta+v\beta-j\beta-v+1}{c}\right)\Bigg], \end{aligned}$$

for $v>0$, $v\neq 1$.

5 Estimation and Inference

In this section, we discuss several methods of estimation of the model parameters including maximum likelihood, ordinary least squares, weighted least squares, minimum distance and maximum product of spacings. The method of maximum likelihood is presented in detail.

5.1 Maximum Likelihood Estimation

In this subsection, we present the maximum likelihood estimates of the GLLoGW model parameters. Let $X \sim GLLoGW(c, \alpha, \beta, \delta, \theta)$ and $\mathbf{\Delta} = (c, \alpha, \beta, \delta, \theta)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\mathbf{\Delta})$ based on a random sample of size n, say $x_1, x_2, ..., x_n$ from the GLLoGW distribution is given by

$$\begin{aligned}
\ell(\mathbf{\Delta}) &= -n\ln\Gamma(\delta) + (\delta-1)\sum_{i=1}^{n}\ln\left[-\ln\left(1-(1+x_i^c)^{-1}e^{-\alpha x_i^{\beta}}\right)\right] \\
&- n\delta\ln(\theta) - \sum_{i=1}^{n}\ln(1+x_i^c) - \alpha\sum_{i=1}^{n}x_i^{\beta} \\
&+ \sum_{i=1}^{n}\ln[(1+x_i^c)^{-1}cx_i^{c-1} + \alpha\beta x_i^{\beta-1}] \\
&+ \left(\frac{1}{\theta}-1\right)\sum_{i=1}^{n}\ln\left(1-(1+x_i^c)^{-1}e^{-\alpha x_i^{\beta}}\right). \qquad (32)
\end{aligned}$$

The first derivative of the log-likelihood function with respect to each component of the parameter vector $\mathbf{\Delta} = (c, \alpha, \beta, \delta, \theta)^T$ are given by

$$\begin{aligned}
\frac{\partial\ell}{\partial c} &= -(\delta-1)\sum_{i=1}^{n}\frac{x_i^c(1+x_i^c)^{-2}e^{-\alpha x_i^{\beta}}(\ln(x_i))}{\ln(1-(1+x_i^c)^{-1}e^{-\alpha x_i^{\beta}})} - \sum_{i=1}^{n}\frac{x_i^c\ln(x_i)}{1+x_i^c} \\
&+ \sum_{i=1}^{n}\frac{x_i^{c-1}(1+x_i^c)^{-1}[1+c\ln(x_i) - cx_i^c(1+x_i^c)^{-1}\ln(x_i)]}{(1+x_i^c)^{-1}cx_i^{c-1}+\alpha\beta x_i^{\beta-1}} \\
&+ \left(\frac{1}{\theta}-1\right)\sum_{i=1}^{n}\frac{(1+x_i^c)^{-2}e^{-\alpha x_i^{\beta}}x_i^c\ln(x_i)}{1-(1+x_i^c)^{-1}e^{-\alpha x_i^{\beta}}},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\ell}{\partial\alpha} &= (\delta-1)\sum_{i=1}^{n}\frac{x_i^{\beta}(1+x_i^c)^{-1}e^{-\alpha x_i^{\beta}}}{\ln(1-(1+x_i^c)^{-1}e^{-\alpha x_i^{\beta}})} + \sum_{i=1}^{n}\frac{\beta x_i^{\beta-1}}{(1+x_i^c)^{-1}cx_i^{c-1}+\alpha\beta x_i^{\beta-1}} \\
&- \sum_{i=1}^{n}x_i^{\beta} + \left(\frac{1}{\theta}-1\right)\sum_{i=1}^{n}\frac{(1+x_i^c)^{-1}e^{-\alpha x_i^{\beta}}x_i^{\beta}}{1-(1+x_i^c)^{-1}e^{-\alpha x_i^{\beta}}},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\ell}{\partial\beta} &= -(\delta-1)\sum_{i=1}^{n}\frac{\alpha x_i^{\beta}(1+x_i^c)^{-1}\ln(x_i)}{\ln(1-(1+x_i^c)^{-1}e^{-\alpha x_i^{\beta}})} + \sum_{i=1}^{n}\frac{\alpha x_i^{\beta-1}[1+\beta\ln(x_i)]}{(1+x_i^c)^{-1}cx_i^{c-1}+\alpha\beta x_i^{\beta-1}} \\
&- \alpha\sum_{i=1}^{n}x_i^{\beta} + \left(\frac{1}{\theta}-1\right)\sum_{i=1}^{n}\frac{(1+x_i^c)^{-1}e^{-\alpha x_i^{\beta}}\alpha x_i^{\beta}\ln(x_i)}{1-(1+x_i^c)^{-1}e^{-\alpha x_i^{\beta}}},
\end{aligned}$$

$$\frac{\partial\ell}{\partial\delta} = -\frac{n\Gamma(\delta)}{\Gamma(\delta)} + \sum_{i=1}^{n}\ln\left[-\ln\left(1-(1+x_i^c)^{-1}e^{-\alpha x_i^{\beta}}\right)\right] - n\ln(\theta),$$

and

$$\frac{\partial \ell}{\partial \theta} = -\frac{n\delta}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} \ln\left(1 - (1 + x_i^c)^{-1} e^{-\alpha x_i^{\beta}}\right).$$

The total log-likelihood function based on a random sample of n observations: $x_1, x_2, ..., x_n$ drawn from the GLLoGW distribution is given by $\ell = \ell(\mathbf{\Delta}) = \sum_{i=1}^{n} \ell_i(\mathbf{\Delta})$, where $\ell_i(\mathbf{\Delta})$, $i = 1, 2, ..., n$ is the log-likelihood of a single observation x of the random variable X. The equations obtained by setting the above partial derivatives to zero are not in closed form and the values of the parameters c, α, β, δ, and θ must be found by using iterative methods. The maximum likelihood estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell}{\partial c}, \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial \theta})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by $\mathbf{I}(\mathbf{\Delta}) = [\mathbf{I}_{\theta_i, \theta_j}]_{5X5} = E(-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j})$, $i, j = 1, 2, 3, 4, 5$ can be numerically obtained by MATLAB, SAS or R software. The total Fisher information matrix $n\mathbf{I}(\mathbf{\Delta})$ can be approximated by

$$\mathbf{J}_n(\hat{\mathbf{\Delta}}) \approx \left[-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j}\bigg|_{\mathbf{\Delta}=\hat{\mathbf{\Delta}}} \right]_{5X5}, \quad i, j = 1, 2, 3, 4, 5. \tag{33}$$

For a given set of observations, the matrix given in equation (33) is obtained after the convergence of the Newton-Raphson procedure. The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\Delta} = (\hat{c}, \hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\theta})$ be the maximum likelihood estimate of $\mathbf{\Delta} = (c, \alpha, \beta, \delta, \theta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Delta}} - \mathbf{\Delta}) \xrightarrow{d} N_5(\underline{0}, I^{-1}(\mathbf{\Delta}))$, where $I(\mathbf{\Delta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\mathbf{\Delta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Delta}}$, that is $J(\hat{\mathbf{\Delta}})$. The multivariate normal distribution $N_5(\underline{0}, J(\hat{\mathbf{\Delta}})^{-1})$, where the mean vector $\underline{\mathbf{0}} = (0, 0, 0, 0, 0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. That is, the approximate $100(1-\eta)\%$ two-sided confidence intervals for c, α, β, δ and θ are given by:

$$\widehat{c} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{cc}^{-1}(\widehat{\mathbf{\Delta}})}, \quad \widehat{\alpha} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\alpha\alpha}^{-1}(\widehat{\mathbf{\Delta}})}, \quad \widehat{\beta} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\beta\beta}^{-1}(\widehat{\mathbf{\Delta}})}, \quad \widehat{\delta} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\delta\delta}^{-1}(\widehat{\mathbf{\Delta}})},$$

and $\widehat{\theta} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\theta\theta}^{-1}(\widehat{\mathbf{\Delta}})}$, respectively, where $\mathbf{I}_{cc}^{-1}(\widehat{\mathbf{\Delta}})$, $\mathbf{I}_{\alpha\alpha}^{-1}(\widehat{\mathbf{\Delta}})$, $\mathbf{I}_{\beta\beta}^{-1}(\widehat{\mathbf{\Delta}})$, $\mathbf{I}_{\delta\delta}^{-1}(\widehat{\mathbf{\Delta}})$, and $\mathbf{I}_{\theta\theta}^{-1}(\widehat{\mathbf{\Delta}})$, are the diagonal elements of $\mathbf{I}_n^{-1}(\hat{\mathbf{\Delta}}) = (n\mathbf{I}(\hat{\mathbf{\Delta}}))^{-1}$, and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

We maximized the likelihood function using NLmixed in SAS as well as the function nlm in R (Team (2011)). These functions were applied and executed for wide range of initial values. This process often results or lead to more than one maximum, however, in these cases, we take the MLEs corresponding to the largest value of the maxima. In a few cases, no maximum was identified for the selected initial values. In these cases, a new initial value was tried in order to obtain a maximum.

The issues of existence and uniqueness of the MLEs are theoretical interest and has been studied by several authors for different distributions including Seregin (2010), Santos Silva and Tenreyro (2010), Zhou (2009), and Xia et al. (2009). At this point we are not able to address the theoretical aspects (existence, uniqueness) of the MLE of the parameters of the GLLoGW distribution.

The maximum likelihood estimates (MLEs) of the GLLoGW parameters c, α, β, δ, and θ are computed by maximizing the objective function via the subroutine NLmixed in SAS and the function nlm in R. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2\ln(L)$, Bayesian Information Criterion, $BIC = p\ln(n) - 2\ln(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\boldsymbol{\Delta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented. In order to compare the models, we use the criteria stated above. Note that for the value of the log-likelihood function at its maximum ($\ell = \ln(L)$), larger value is good and preferred, and for AIC, AICC and BIC, smaller values are preferred. The GLLoGW distribution is fitted to the data sets and these fits are compared to the fits of the GLLoGE, GLLoGR, gamma-Dagum (Oluyede et al. (2014)), exponentiated Kumaraswamy Dagum (EKD) (Huang and Oluyede (2014)), beta modified Weibull (BMW) (Silva et al. (2010)) and beta Weibull-Poisson (BWP) (Percontini et al. (2013)) distributions.

We can use the likelihood ratio (LR) test to compare the fit of the GLLoGW distribution with its sub-models for a given data set. For example, to test $\theta = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{c}, \hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\theta})) - \ln(L(\tilde{c}, \tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, 1))]$, where $\hat{c}$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\delta}$ and $\hat{\theta}$, are the unrestricted estimates, and $\tilde{c}$, $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\delta}$, are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_\epsilon$, where χ^2_ϵ denote the upper $100\epsilon\%$ point of the χ^2 distribution with 1 degree of freedom.

5.2 Ordinary Least Squares

In this subsection, we discuss the method of ordinary least squares. Let $y_{1:n} < y_{2:n} < ... < y_{n:n}$ denote the order statistics based on a random sample of size n from the distribution with cdf $G(y)$, then

$$E[G(y_{i:n})] = \frac{i}{n+1} \quad \text{and} \quad Var[G(y_{i:n})] = \frac{i(n-i+1)}{(n+1)^2(n+2)}. \tag{34}$$

The ordinary least-square (OLS) estimates of the GLLoGW parameters $\boldsymbol{\Delta} = (c, \alpha, \beta, \delta, \theta)^T$, say, $(\hat{c}_{OLS}, \hat{\alpha}_{OLS}, \hat{\beta}_{OLS}, \hat{\delta}_{OLS}, \hat{\theta}_{OLS})^T$ are obtained by minimizing the function

$$Q(\boldsymbol{\Delta}|\mathbf{y}) = \sum_{i=1}^{n} \left(G(y_{i:n}|\boldsymbol{\Delta}) - \frac{i}{n+1} \right)^2. \tag{35}$$

The OLS estimates of the parameters, denoted by $\hat{\boldsymbol{\Delta}}_{\mathbf{OLS}}$ is obtained by solving the nonlinear equations

$$\left(\frac{\partial Q(\boldsymbol{\Delta}|\mathbf{y})}{\partial c}, \frac{\partial Q(\boldsymbol{\Delta}|\mathbf{y})}{\partial \delta}, \frac{\partial Q(\boldsymbol{\Delta}|\mathbf{y})}{\partial \alpha}, \frac{\partial Q(\boldsymbol{\Delta}|\mathbf{y})}{\partial \beta}, \frac{\partial Q(\boldsymbol{\Delta}|\mathbf{y})}{\partial \theta} \right)^T = \mathbf{0},$$

using numerical methods.

5.3 Weighted Least Squares

The weighted least square (WLS) estimates of the GLLoGW parameters $\boldsymbol{\Delta} = (c, \alpha, \beta, \delta, \theta)^T$, say, $(\hat{c}_{WLS}, \hat{\alpha}_{WLS}, \hat{\beta}_{WLS}, \hat{\delta}_{WLS}, \hat{\theta}_{WLS})^T$ are obtained by minimizing the function

$$\begin{aligned} W(\boldsymbol{\Delta}|\mathbf{y}) &= \sum_{i=1}^{n} \frac{1}{Var[G(y_{i:n})]} \left(G(y_{i:n}|\boldsymbol{\Delta}) - \frac{i}{n+1} \right)^2 \\ &= \sum_{i=1}^{n} w_i \left(G(y_{i:n}|\boldsymbol{\Delta}) - \frac{i}{n+1} \right)^2, \end{aligned} \tag{36}$$

where $w_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}$. The WLS estimates of the parameters, denoted by $\hat{\boldsymbol{\Delta}}_{\mathbf{WLS}}$ is obtained by solving the nonlinear equations

$$\left(\frac{\partial W(\boldsymbol{\Delta}|\mathbf{y})}{\partial c}, \frac{\partial W(\boldsymbol{\Delta}|\mathbf{y})}{\partial \delta}, \frac{\partial W(\boldsymbol{\Delta}|\mathbf{y})}{\partial \alpha}, \frac{\partial W(\boldsymbol{\Delta}|\mathbf{y})}{\partial \beta}, \frac{\partial W(\boldsymbol{\Delta}|\mathbf{y})}{\partial \theta} \right)^T = \mathbf{0},$$

using a numerical method.

5.4 Minimum Distance Methods

The estimates of the GLLoGW parameters can be obtained via the minimization of the well known Anderson-Darling and Cramér-von Mises goodness-of-fit statistics. This class of goodness-of-fit statistics is based on the difference between the estimates of the GLLoGW cdf and the corresponding empirical distribution function.

5.4.1 Anderson-Darling Method

The Anderson-Darling (AD) estimates of the GLLoGW model parameters $\boldsymbol{\Delta}_{AD}$, say $\hat{\boldsymbol{\Delta}}_{AD}$ are obtained by minimizing the function

$$AD(\boldsymbol{\Delta}|\mathbf{y}) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \log \left(G(y_{i:n}|\boldsymbol{\Delta})[1 - G(y_{n+1-i:n}|\boldsymbol{\Delta})] \right). \tag{37}$$

with respect to parameters. The AD estimates of the parameters, denoted by $\hat{\boldsymbol{\Delta}}_{\mathbf{AD}}$ is obtained by solving the nonlinear equations

$$\left(\frac{\partial AD(\boldsymbol{\Delta}|\mathbf{y})}{\partial c}, \frac{\partial AD(\boldsymbol{\Delta}|\mathbf{y})}{\partial \delta}, \frac{\partial AD(\boldsymbol{\Delta}|\mathbf{y})}{\partial \alpha}, \frac{\partial AD(\boldsymbol{\Delta}|\mathbf{y})}{\partial \beta}, \frac{\partial AD(\boldsymbol{\Delta}|\mathbf{y})}{\partial \theta} \right)^T = \mathbf{0},$$

using a numerical method.

5.4.2 Cramér-von Mises Method

The Cramér-von Mises (CVM) estimates of the GLLoGW parameters $\mathbf{\Delta}_{CVM}$, say $\hat{\mathbf{\Delta}}_{CVM}$ are obtained by minimizing the function

$$CVM(\mathbf{\Delta}|\mathbf{y}) = \frac{1}{12n} + \sum_{i=1}^{n}\left(G(y_{i:n}|\mathbf{\Delta}) - \frac{2i-1}{2n}\right)^2, \tag{38}$$

with respect to the parameters $\mathbf{\Delta} = (c, \alpha, \beta, \delta, \theta)^T$.

5.5 Maximum Product of Spacings Method

The $(n+1)$ uniform spacings of the first order of the sample are given by $D_1 = G(y_{1:n}|\mathbf{\Delta})$, $D_{n+1} = 1 - G(y_{n:n}|\mathbf{\Delta})$ and $D_i = G(y_{i:n}|\mathbf{\Delta}) - G(y_{(i-1):n}|\mathbf{\Delta})$, $i = 1, 2, ..., n$. The maximum product of spacings (MPS) method consist of finding the values of $\mathbf{\Delta}$ which maximizes the geometric mean of the spacings given by

$$GM(\mathbf{\Delta}|\mathbf{y}) = \left(\prod_{i=1}^{n} D_i\right)^{\frac{1}{(n+1)}}, \tag{39}$$

or equivalently

$$\log\left(GM(\mathbf{\Delta}|\mathbf{y})\right) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log(D_i), \tag{40}$$

by taking $0 = G(y_{0:n}|\mathbf{\Delta}) < G(y_{1:n}|\mathbf{\Delta}) < ... < G(y_{n:n}|\mathbf{\Delta}) < G(y_{n+1:n}|\mathbf{\Delta}) = 1$. The MPS estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}_{\mathbf{MPS}}$ is obtained by solving the nonlinear equations

$$\left(\frac{\partial GM(\mathbf{\Delta}|\mathbf{y})}{\partial c}, \frac{\partial GM(\mathbf{\Delta}|\mathbf{y})}{\partial \delta}, \frac{\partial GM(\mathbf{\Delta}|\mathbf{y})}{\partial \alpha}, \frac{\partial GM(\mathbf{\Delta}|\mathbf{y})}{\partial \beta}, \frac{GM(\mathbf{\Delta}|\mathbf{y})}{\partial \theta}\right)^T = \mathbf{0},$$

using a numerical method. See Chen and Amin (1983) for additional details.

6 Simulation Study

In this section, we examine the performance of the GLLoGW distribution by conducting various simulations for different sizes (n=25, 50, 100, 200, 400, 800) via the R package. We simulate 1000 samples for the true parameters values given in the Table 3. The table lists the mean MLEs of the five model parameters along with the respective root mean squared errors (RMSEs). The bias and RMSE for the estimated parameter $\hat{\theta}$, say, are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{N}\hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

Table 3: Monte Carlo Simulation Results

		(0.5,0.5,0.5,1.0,0.5)			(0.5,0.5,1.5,0.5,0.5)			(1.5,0.5,0.5,0.5,0.5)			(1.5,0.5,1.5,0.5,1.0)		
Parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
c	25	0.50349	0.51090	0.00349	0.92682	1.05883	0.42682	1.78882	1.39574	0.28882	1.40911	1.39782	-0.09089
	50	0.52445	0.48357	0.02445	0.78218	0.85076	0.28218	1.70189	1.07199	0.20189	1.46830	1.34564	-0.03170
	100	0.54416	0.44591	0.04416	0.73391	0.78469	0.23391	1.87113	1.00403	0.37113	1.45139	1.10828	-0.04861
	200	0.51418	0.33501	0.01418	0.55625	0.46202	0.05625	1.82026	0.79130	0.32026	1.46159	0.96153	-0.03841
	400	0.51691	0.26688	0.01691	0.50126	0.29768	0.00126	1.83186	0.59378	0.33186	1.42311	0.77963	-0.07689
	800	0.55989	0.23020	0.05989	0.48629	0.22182	-0.01371	1.59715	0.20322	0.09715	1.44362	0.68520	-0.05638
α	25	0.56005	0.79062	0.06005	0.30875	0.50502	-0.19125	0.34020	0.74532	-0.15980	0.19361	0.56112	-0.30639
	50	0.45829	0.73237	-0.04171	0.36056	0.46231	-0.13944	0.36912	0.71901	-0.13088	0.20699	0.55552	-0.29301
	100	0.42565	0.66603	-0.07435	0.39052	0.41766	-0.10948	0.39880	0.70204	-0.10120	0.21957	0.54936	-0.28043
	200	0.29558	0.51160	-0.20442	0.42225	0.33127	-0.07775	0.35497	0.48715	-0.14503	0.23529	0.48556	-0.26471
	400	0.24028	0.49085	-0.25972	0.44076	0.27483	-0.05924	0.34636	0.39525	-0.15364	0.30822	0.44752	-0.19178
	800	0.14952	0.42474	-0.35048	0.47165	0.22976	-0.02835	0.28681	0.34900	-0.21319	0.32108	0.39046	-0.17892
β	25	1.06880	1.06842	0.56880	2.38343	1.45351	0.88343	1.34617	1.29404	0.84617	2.98822	2.13240	1.48822
	50	1.00065	0.92213	0.50065	1.99872	1.02567	0.49872	1.22533	1.07191	0.72533	2.62163	1.76029	1.12163
	100	0.93050	0.76593	0.43050	1.76285	0.70397	0.26285	1.10686	0.90064	0.60686	2.35043	1.36861	0.85043
	200	0.87566	0.62675	0.37566	1.52383	0.38036	0.02383	1.07418	0.78407	0.57418	2.07283	1.02657	0.57283
	400	0.86731	0.53572	0.36731	1.45947	0.24041	-0.04053	0.91203	0.54163	0.41203	1.81593	0.70815	0.31593
	800	0.85775	0.45953	0.35775	1.44554	0.18939	-0.05446	0.93860	0.52222	0.43860	1.69349	0.50992	0.19349
δ	25	3.05901	3.11895	2.05901	1.37248	1.75419	0.87248	0.99606	1.23210	0.49606	2.76975	3.79555	2.26975
	50	2.85361	2.95947	1.85361	1.23828	1.43481	0.73828	0.93836	1.06923	0.43836	2.40413	2.89916	1.90413
	100	2.48184	2.45608	1.48184	1.13163	1.20276	0.63163	0.72740	0.78416	0.22740	2.12614	2.50945	1.62614
	200	2.34926	2.15511	1.34926	1.09980	1.06594	0.59980	0.54779	0.52618	0.04779	1.69375	1.93107	1.19375
	400	2.11730	1.84130	1.11730	0.98631	0.88001	0.48631	0.44366	0.31246	-0.05634	1.35132	1.47893	0.85132
	800	1.82457	1.40759	0.82457	0.86807	0.73852	0.36807	0.45602	0.14973	-0.04398	1.08573	1.08238	0.58573
θ	25	0.61381	1.17146	0.11381	0.73200	1.09051	0.23200	1.12433	1.48809	0.62433	0.94222	1.59218	-0.05778
	50	0.61968	1.00959	0.11968	0.68143	0.88259	0.18143	1.05462	1.27864	0.55462	0.99730	1.57407	-0.00270
	100	0.63558	0.83042	0.13558	0.65848	0.86922	0.15848	1.10153	1.12593	0.60153	0.94431	1.20821	-0.05569
	200	0.55018	0.57044	0.05018	0.51652	0.55278	0.01652	1.05392	0.89702	0.55392	0.95035	0.96200	-0.04965
	400	0.53647	0.44394	0.03647	0.46985	0.35564	-0.03015	1.04568	0.73329	0.54568	0.89945	0.72044	-0.10055
	800	0.58488	0.37554	0.08488	0.47908	0.29190	-0.02092	0.85576	0.40709	0.35576	0.93725	0.62316	-0.06275

From the results, we can readily verify that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero.

7 Application

In this section, we present an example to illustrate the flexibility and usefulness of the GLLoGW distribution and its sub-models for data modeling. We also compare the GLLoGW distribution with the gamma-Dagum (GD) (Oluyede et al. (2014)), exponentiated Kumaraswamy Dagum (EKD) (Huang and Oluyede (2014)), beta modified Weibull (BMW) (Silva et al. (2010)) and beta Weibull-Poisson (BWP)(Percontini et al. (2013)) distributions.

The pdf of EKD distribution is given by

$$\begin{aligned} g_{EKD}(x) &= \alpha\lambda\delta\phi\theta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\alpha-1}[1-(1+\lambda x^{-\delta})^{-\alpha}]^{\phi-1} \\ &\times \{1-[1-(1+\lambda x^{-\delta})^{-\alpha}]^{\phi}\}^{\theta-1}, \end{aligned} \tag{41}$$

for $\alpha, \lambda, \delta, \phi, \theta > 0$, and $x > 0$. The BMW and BWP pdfs are given by

$$g_{BMW}(x) = \frac{\alpha x^{\gamma-1}(\gamma + \lambda x)\exp(\lambda x)}{B(a,b)} e^{-b\alpha x^{\gamma}\exp(\lambda x)}(1 - e^{-\alpha x^{\gamma}\exp(\lambda x)})^{a-1}, \quad x > 0,$$

and

$$g_{BWP}(x) = \frac{\alpha\beta\lambda x^{\alpha-1}e^{\lambda e^{-\beta x^{\alpha}}-\lambda-\beta x^{\alpha}}(e^{\lambda}-1)^{2-a-b}(e^{\lambda}-e^{\lambda e^{-\beta x^{\alpha}}})^{a-1}(e^{\lambda e^{-\beta x^{\alpha}}}-1)^{b-1}}{B(a,b)(1-e^{-\lambda})} \tag{42}$$

for $a, b, \alpha, \beta, \lambda > 0$, and $x > 0$, respectively. The GD pdf is given by

$$\begin{aligned} g_{GD}(x) &= \frac{\lambda\beta\delta x^{-\delta-1}}{\Gamma(\alpha)\theta^{\alpha}}(1+\lambda x^{-\delta})^{-\beta-1}\left(-\log[1-(1+\lambda x^{-\delta})^{-\beta}]\right)^{\alpha-1} \\ &\times \ [1-(1+\lambda x^{-\delta})^{-\beta}]^{(1/\theta)-1}. \end{aligned} \tag{43}$$

Plots of the fitted densities, the histogram of the data and the probability plots (Chambers et al. (1983)) are given in Figure 16. For the probability plot, we plotted $G_{GLLoGW}(x_{(j)}; \hat{c}, \hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\theta})$ against $\dfrac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS = \sum_{j=1}^{n}\left[G_{GLLoGW}(x_{(j)}) - \left(\frac{j-0.375}{n+0.25}\right)\right]^2.$$

The goodness-of-fit statistics W^* and A^*, Chen and Balakrishnan (1995) are also presented in the tables. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit.

7.1 Strengths of Glass Fibers Data

The data set ($n = 63$) is on the strengths of 1.5 cm glass fibers measured at the National Physical Laboratory, England, and is obtained from Smith and Naylor (1987). Estimates of the parameters of GLLoGW distribution and its related sub-models (standard error in parentheses), AIC, BIC, W*, A* and SS are give in Table 4. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 16. The estimated variance-covariance matrix for the GLLoGW distribution is given by:

$$\begin{pmatrix} 0.2089 & -0.0070 & 0.3476 & -0.3580 & 0.0215 \\ -0.0070 & 0.0013 & -0.0283 & -0.0126 & 0.0002 \\ 0.3476 & -0.0283 & 1.2877 & -0.9017 & 0.0597 \\ -0.3580 & -0.0126 & -0.9017 & 2.3355 & -0.1239 \\ 0.0215 & 0.0002 & 0.0597 & -0.1239 & 0.0070 \end{pmatrix},$$

and the 95% confidence intervals for the model parameters are given by $c \in (0.8317 \pm 1.96 \times 0.4571), \alpha \in (0.0498 \pm 1.96 \times 0.0361), \beta \in (5.5639 \pm 1.96 \times 1.1348), \delta \in (1.9867 \pm 1.96 \times 1.5282)$ and $\theta \in (0.1519 \pm 1.96 \times 0.0836)$, respectively.

Table 4: Estimates of Models for Strengths of Glass Fibers Data

	Estimates					Statistics								
Model	c	α	β	δ	θ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
GLLoGW	0.8317	0.0498	5.5639	1.9867	0.1519	25.36	35.36	36.41	46.07	0.0981	0.5793	0.0979	0.5815	0.0879
	(0.4571)	(0.0361)	(1.1348)	(1.5282)	(0.0836)									
GLLoGE or Sub1	0.0002	13.4591	1	0.0432	14.8535	194.22	202.22	203.27	210.79	0.5330	2.9269	0.4273	0.0000	4.1096
	(0.0007)	(2.5222)	-	(0.0089)	(0.1773)									
GLLoGR or Sub2	0.0447	0.2702	2	11.2749	0.0283	32.58	40.58	41.63	49.15	0.2753	1.5165	0.1664	0.0610	0.2717
	(0.2658)	(0.0510)	-	(0.0010)	(0.0028)									
GLLoG or Sub3	1.1524	0	0	17.6581	0.0288	58.01	64.01	65.06	70.44	0.7319	3.9987	0.2359	0.0018	0.7288
	(0.0947)	-	-	(0.0028)	(0.0011)									
$GLLoGW(1,\alpha,\beta,\delta,1)$ or Sub5	1	0.0000	0.5229	0.9528	1	229.03	235.03	236.09	241.46	0.7120	3.8920	0.3916	0.0000	3.8275
	-	(0.0000)	(0.0323)	(0.0948)	-									
$GLLoGW(1,\alpha,1,\delta,1)$ or Sub6	1	3.8108	1	0.1586	1	174.75	178.75	179.80	183.03	0.5995	3.2852	0.4101	0.0000	3.6782
	-	(2.4279)	-	(0.0913)	-									
$GLLoGW(1,\alpha,2,\delta,1)$ or Sub7	1	5.9406	2	0.0689	1	100.00	104.00	105.05	108.28	0.4796	2.6302	0.3345	0.0000	2.3030
	-	(1.6606)	-	(0.0205)	-									
$GLLoGW(c,0,0,1,1)$ or Sub10	3.4506	0	0	1	1	136.94	138.94	140.00	141.09	0.8045	4.4035	0.5346	0.0000	6.5223
	(0.3414)	-	-	-	-									
$GLLoGW(1,\alpha,\beta,1,1)$ or Sub11	1	0.0106	8.2434	1	1	91.16	95.16	96.21	99.45	0.2279	1.4489	0.4379	0.0000	4.7748
	-	(0.0064)	(0.9305)	-	-									
$GLLoGW(1,\alpha,1,1,1)$ or Sub12	1	0.2620	1	1	1	215.57	217.57	218.63	219.72	0.6417	3.5137	0.5345	0.0000	6.4159
	-	(0.0833)	-	-	-									
$GLLoGW(1,\alpha,2,1,1)$ or Sub13	1	0.2811	2	1	1	170.81	172.81	173.86	174.95	0.4528	2.4832	0.5674	0.0000	7.5302
	-	(0.0526)	-	-	-									
GD	λ	β	δ	α	θ									
	3.9073	1.3029	1.2928	10.1331	0.0235	50.06	60.06	61.11	70.78	0.6025	3.3054	0.2232	0.0038	0.5981
	(1.8210)	(0.6723)	(0.5411)	(0.1002)	(0.0231)									
BetaMW	a	b	α	γ	λ									
	1.3236	0.8588	0.0159	1.3255	2.3344	28.52	38.52	39.57	49.23	0.1734	0.9705	0.1373	0.1859	0.1676
	(1.6534)	(1.3511)	(0.0215)	(6.1595)	(2.1702)									
EKD	α	λ	δ	ϕ	θ									
	2.4980	94.2472	7.0993	7.1672	0.2876	28.09	38.09	39.15	48.81	0.1927	1.0609	0.1543	0.0996	0.2098
	(1.3516)	(106.792)	(2.5412)	(8.7799)	(0.1918)									
BetaWP	a	b	α	β	λ									
	1.7697	4.2396	3.8950	0.0064	10.1766	34.83	44.83	45.88	55.55	0.3363	1.8427	0.1833	0.0290	0.3457
	(1.8377)	(0.7316)	(2.4905)	(0.0124)	(0.6390)									

The LR test statistic for testing H_0: GLLoGE against H_a: GLLoGW and H_0: GLLoG against H_a: GLLoGW are 168.866 (p-value < 0.0001) and 32.6528 (p-value < 0.0001). We conclude that there are significant differences between the GLLoGE and the GLLoGW distributions, as well as between the LLoG and the GLLoGW distributions, respectively based on the LR tests. The GLLoGW distribution is significantly better than any of the sub-models considered above. The values of the statistics: AIC, AICC, and BIC are smallest for the GLLoGW distribution. Also, the goodness-of-fit statistics W^* and A^* are the smallest and definitely points to the GLLoGW distribution as the "best"fit for the glass fibers data when compared to the corresponding values for the sub-models. The goodness-of-fit statistics W^* and A^* are also better for the GLLoGW distribution when compared to the values for the non-nested GD, BMW, EKD and BWP distributions. The values of SS from the probability plot is also smallest ($SS = 0.0879$) for the GLLoGW distribution. Thus, there is indeed convincing evidence that the GLLoGW distribution is the "best"fit for the glass fibers data.

8 Concluding Remarks

A new generalized distribution called the gamma log-logistic Weibull (GLLoGW) distribution is presented. The GLLoGW distribution has several new and known distributions

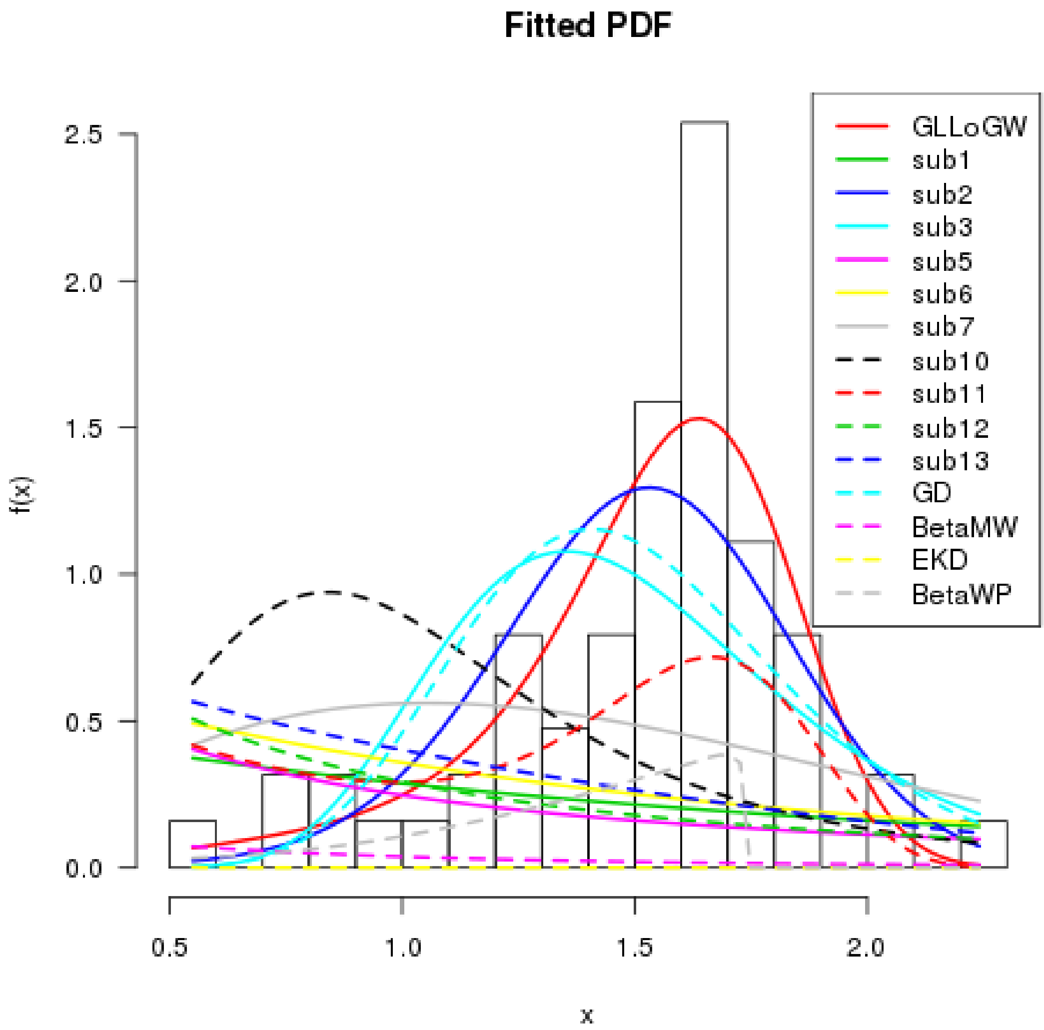

Figure 15: Fitted Densities for Strengths of Glass Fibers Data

as special cases or sub-models. The density of this new class of distributions can be expressed as a linear combination of Burr XII-Weibull density functions. The GLLoGW distribution possesses hazard function with flexible behavior. We also obtain closed form expressions for the moments, mean and median deviations, distribution of order statistics and entropy. Maximum likelihood estimation technique is used to estimate the model parameters. The performance of the GLLoGW distribution was examined by conducting various simulations for different sizes. Finally, the GLLoGW distribution is fitted to a real dataset to illustrate the applicability and usefulness of the distribution.

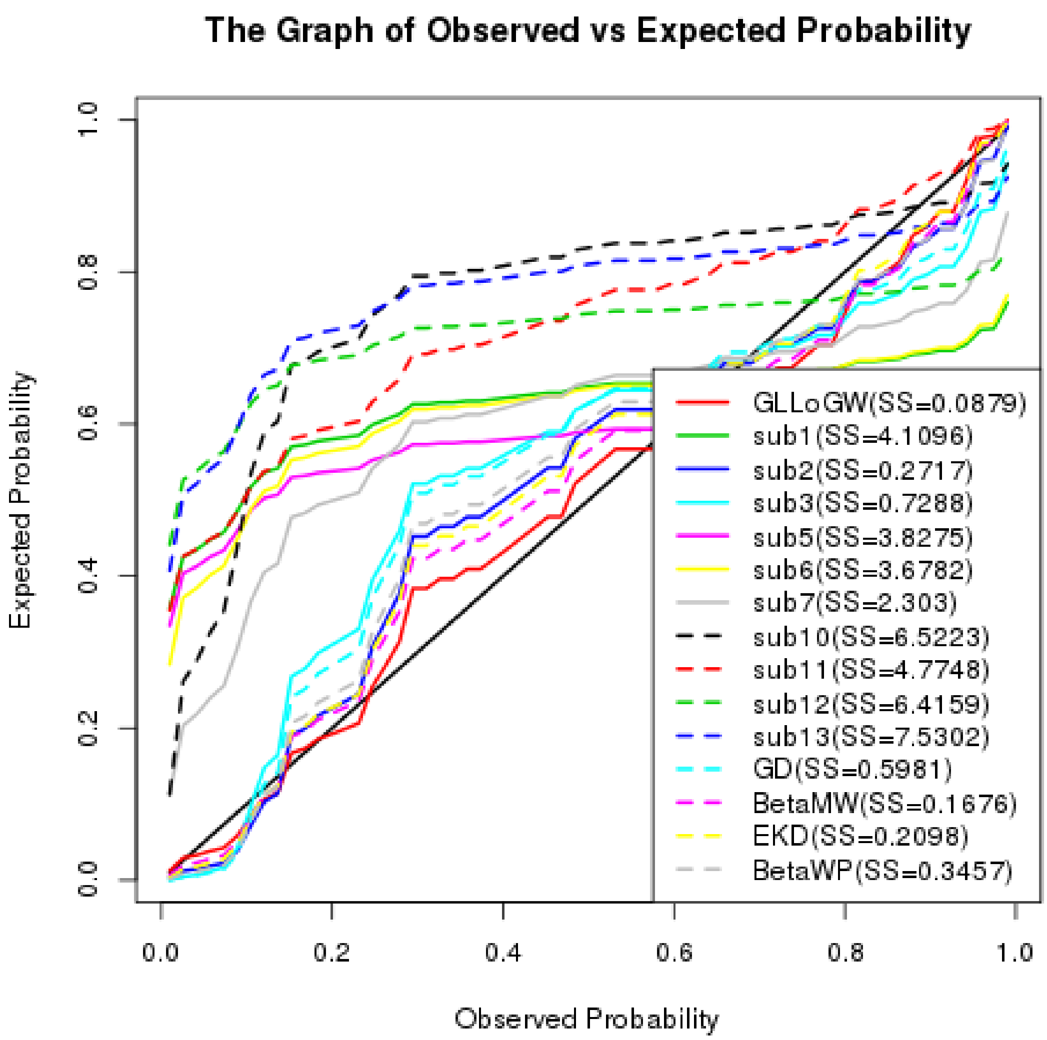

Figure 16: Probability Plots for Strengths of Glass Fibers Data

Acknowledgement

The authors are very grateful to the editor and referees for some useful comments on an earlier version of this manuscript which led to this improved version.

References

Chambers, J., Cleveland, W., Kleiner, B., and Tukey, P. (1983). *Graphical Methods of Data Analysis*. Chapman and Hall.

Chen, G. and Balakrishnan, N. (1995). A general purpose approximate goodness-of-fit test. *Journal of Quality Technology*, 27:154–161.

Chen, R. and Amin, N. A. K. (1983). Estimating parameters in a continuous distribution with a shifted origin. *Journal of Royal Statistical Society B*, 45(3):394–403.

Gradshteyn, I. and Ryzhik, I. M. (2000). *Table of Integrals, Series and Products*. Academic Press, San Diego.

Huang, S. and Oluyede, B. (2014). Exponentiated kumaraswamy-dagum distribution with applications to income and lifetime data. *Journal of Statistical Distributions and Applications*, 1(8).

Oluyede, B., Foya, S., Warahena-Liyanage, G., and Huang, S. (2016). The log-logistic weibull distribution with applications to lifetime data. *Austrian Journal of Statistics*, 45:43–69.

Oluyede, B., Huang, S., and Pararai, M. (2014). A new class of generalized dagum distribution with applications to income and lifetime data. *Journal of Statistical and Econometric Methods*, 3(2):125–151.

Oluyede, B. O. (1999). On inequalities and selection of experiments for length biased distributrions. *Probability in the Engineering and Informational Sciences*, 13(2):129–149.

Percontini, A., Blas, B., and Cordeiro, G. (2013). The beta weibull poisson distribution. *Chilean Journal of Statistics*, 4(2):3–26.

Pinho, L., Cordeiro, G., and Nobre, J. S. (2012). The gamma exponentiated weibull distribution. *Journal of Statistical Theory and Applications*, 11(4):379–395.

Rényi, A. (1960). On measures of entropy and information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, volume 1, pages 547 – 561.

Ristic, M. and Balakrishnan, N. (2012). The gamma exponentiated exponential distributions. *Journal of Statistical Computation and Simulation*, 82(8):1191–1206.

Santos Silva, J. M. and Tenreyro, S. (2010). On the existence of maximum likelihood estimates in poisson regression. *Economics Letters*, 107:310–312.

Seregin, A. (2010). Uniqueness of the maximum likelihood estimator for k-monotone densities. *Proceedings of the American Mathematical Society*, 138:4511–4515.

Shaked, M. and Shanthikumar, J. (1994). *Stochastic Orders and Their Applications*. New York, Academic Press.

Silva, G., Ortega, E. M., and Cordeiro, G. (2010). The beta modified weibull distribution. *Lifetime Data Analysis*, 16:409–430.

Smith, R. L. and Naylor, J. (1987). A comparison of maximum likelihood and bayesian estimators for the three-parameter weibull distribution. *Journal of Applied Statistics*, pages 358–369.

Team, R. D. C. (2011). *A Language and Environment for Statistical Computing.* R Foundation for Statistical Computing, Vienna, Austria.

Xia, J., Mi, J., and Zhou, Y. Y. (2009). On the existence and uniqueness of the maximum likelihood estimators of normal and log-normal population parameters with grouped data. *Journal of Probability and Statistics*, Article id 310575:16 pages.

Zhou, C. (2009). Existence and consistency of the maximum likelihood estimator for the extreme index. *Journal of Multivariate Analysis*, 100:794–815.

Zografos, K. and Balakrishnan, N. (2009). On families of beta and generalized gamma-generated distribution and associated inference. *Statistical Methodology*, 6:344–362.

Appendix

R Algorithms

9 R Code

```
 ### define GLLoGW cdf
  GLLoGW_cdf=function(c,alpha,beta,delta,theta,x){
  u=ifelse(x==Inf,0,-(theta)^(-1)*log(1-(1+x^c)^(-1)
  *(exp(-alpha*x^beta))))
  y=1-pgamma(u,delta)
  return(y)
}

### define GLLoGW pdf
GLLoGW_pdf=function(c,alpha,beta,delta,theta,x){
  u1=(1+x^c)^(-1)
  u2=exp(-alpha*x^beta)
  y=1/(gamma(delta)*theta^delta)*(u1*u2)
  *(u1*c*x^(c-1)+alpha*beta*x^(beta-1))
  *((-log(1-u1*u2))^(delta-1))*((1-u1*u2)^(1/theta-1))
  y[!is.finite(y)]=0
  return(y)
}

### define GLLoGW hazard function
GLLoGW_hazard=function(c,alpha,beta,delta,theta,x){
  f=GLLoGW_pdf(c,alpha,beta,delta,theta,x)
  F=GLLoGW_cdf(c,alpha,beta,delta,theta,x)
  y=f/(1-F)
```

```
  return(y)
}

### define GLLoGW quantile function
GLLoGW_quantile=function(c,alpha,beta,delta,theta,u){
  f=function(x){
    GLLoGW_cdf(c,alpha,beta,delta,theta,x)-u
  }
  x=min(uniroot.all(f,lower=0,upper=100,tol=0.0001))
  return(x)
}

### define GLLoGW moments function
GLLoGW_moments=function(c,alpha,beta,delta,theta,k){
  f=function(x){
    (x^k)*(GLLoGW_pdf(c,alpha,beta,delta,theta,x))
  }
  y=integrate(f,lower=0,upper=Inf,subdivisions = 10000000)
  return(y$value$)
}

### GLLoGW fit
mysample_GLLoGW<-mle2(GLLoGW_neglogl,
start=list(c=0.1,alpha=0.1,beta=0.1,delta=0.1,theta=0.1),
method="L-BFGS-B",data=list(x=mysample),
lower=c(c=0,alpha=0,beta=0,delta=0,theta=0),
upper=c(c=Inf,alpha=Inf,beta=Inf,
delta=Inf,theta=Inf),use.ginv=TRUE)
mysample_GLLoGW_goodness=goodness.fit(pdf=function(par,x)
{GLLoGW_pdf(c=par[1],alpha=par[2],beta=par[3],
delta=par[4],theta=par[5],x)},
cdf=function(par,x){GLLoGW_cdf(c=par[1],alpha=par[2],
beta=par[3],delta=par[4],theta=par[5],x)},
data=mysample,method="L-BFGS-B",
mle=coef(mysample_GLLoGW)
[c('c','alpha','beta','delta','theta')])
```