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Bioeconomic modeling of a single species fishery with Von Bertalanffy law of growth

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This paper aims to study a single species fishery model with Von Bertalanffy law of growth. The dynamical and the bionomic steady states are determined and their natures are examined from the biological and the economical points of view. The optimal harvest policy is studied by taking the fishing effort as a dynamic control variable.

keywords: Single species fishery, Von Bertalanffy growth law, Pontryagin's maximum principle, Shadow price.

1 Introduction

One of the major renewable resources for the human community is fish and hence there is a real necessity to mathematically analyse the different aspects of the functioning of a fishery. Various form of equation have been proposed and used in fishery biology; for example Logistic model, Von Bertalanffy model and Gompertz model (Iwasa, 1978). Biologist M.B.Schaefer developed a fishery model, known as Schaefer model, obeying the Logistic law of growth and adopted the CPUE (catch-per-unit-effort) hypothesis to represent the catch-rate function (Schaefer, 1957; Clark, 1990). In spite of various limitations of the Logistic growth function, the Schaefer model is still being used for its simplicity in commercial marine fisheries. The model was also extended to non-selective harvesting of two ecologically independed species (Clark, 1990) and also to the case

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of combined harvesting of two competing species (Chaudhuri, 1986, 1988). Schaefer (1954) also discussed various aspects of the dynamics of populations important to the management of the commercial marine fisheries: this article was reprinted (Schaefer, 1991) a few years ago.

Here we develop a single species fishery model on the basis of the Von Bertaranphy model (Bertalanffy, 1957; Iwasa, 1978; Thompson and Cauley, 1979). The growth rate function of this model follows Medawar's 'law of growth' (Thompson and Cauley, 1979). The silent features of Medawar's laws are:

(i)size is a monotonically increasing function, that is, a fish always grows larger when the population size is small,

(ii) the process of growth is multiplicative and in general, will be described by equations of logarithmic form,

(iii) the specific acceleration of growth is always negative under conditions of growth.

The problem is first formulated on the basis of the CPUE hypothesis and then we discuss the dynamical behaviour, bionomic equilibrium and optimal harvest policy. We carry out dynamical optimization, in the sense that the harvesting effort $E(t)$ is taken as a dynamic variable, using Pontryagin's maximum principle (Pontryagin, Boltyanskii and Gamkrelidze). Silent features of the results are discussed in the relevant section of the paper.

Our motivation for introducing the Von Bertalanffy growth law primarily lies in the superiority of the Von Bertalanffy law over the Logistic law. Since ideal living conditions prevail in the initial stage, there should be very rapid growth initially. Thereafter, as the population grows, the limitations of resources forces the growth rate to decline and the population gradually approaches the saturation level. Compared to the logistic law, Von Bertalanffy law exhibits faster early growth, but a slower approach to the asymptote, with a longer period of linear growth about the point of inflexion.

When the fish population size becomes considerably large, it tends to maintain a stronger pressure on the newly produced biomass through cannibalism and also there is intra specific competition amongst the individuals in the population for the use of limited resources available in the habitat. These effects, coupled together, should retard the growth of the population to a large extent and as a result, the population size should approach its asymptote rapidly as in the case of logistic model. These retarding effects are, however, counterbalanced to some extend by group movement (Mostofsky, 1978) which is a special behavioural characteristic of a fish population. As a result, the approach of the population size to the asymptote is slowed down. This feature of faster early growth and slower approach to the asymptote is reflected in the Von Bertalanffy law of growth.

Moreover, implication of exploiting a biological population obeying the Von Bertalanffy law of growth have not so far been studied. This fact has motivated us to develop the present model.

2 Statement of the problem

Tsoularis (2001) proposed a modified Verhulst logistic equation which they termed the generalized logistic equation. This has the form

$$\frac{dx}{dt} = rx^{\alpha}\left[1 - \left(\frac{x}{k}\right)^{\beta}\right]^{\gamma}$$

where $x(t)$ denote the population density at time t and α, β, γ are positive real numbers. Here we consider a single species fishery. It is assumed that the density of fish population follows Von Bertalanffy growth function (VBGF) at a rate $qEx^{2/3}$ where E denote the harvesting effort and q is a constant, called catchability coefficient. In VBGF, growth is defined as the net result of the process of anabolism and catabolism which is the particular case of generalized logistic growth model when $\alpha = 2/3, \beta = 1/3, \gamma = 1$. Then growth equation becomes

$$\frac{dx}{dt} = rx^{2/3} - \frac{r}{k^{1/3}}x - qEx^{2/3} = F(x) - qEx^{2/3} \tag{1}$$

where r is the intrinsic growth rate and k is a constant, called the carrying capacity, such that $k = \lim_{t\to\infty} x(t)$ when $E = 0$.

Here we have taken the production function of the form

$$h = qE\phi(x), \phi(x) = x^{2/3} \tag{1a}$$

where

$$\phi'(x) = \frac{2}{3}x^{-1/3} > 0 \text{ and } \phi''(x) = -\frac{2}{9}x^{-4/3} < 0.$$

Therefore, the production function $h(x, E)$ exhibits decreasing marginal returns to the input factor x as a consequence of ultimate gear saturation (Clark, 1990).

2.1 Equilibrium Solution

In order to find out the equilibrium solution of (1),we set $dx/dt = 0$. Now,

$$\frac{dx}{dt} = 0 \text{ gives } x = \begin{cases} 0 = \bar{x}_1 \text{ (say)} \\ \frac{k}{r^3}(r - qE)^3 = \bar{x}_2\text{(say)} \end{cases} \tag{2}$$

We observe that the existence of the nontrivial steady state ($\bar{x}_2$) depends on the biotechnical productivity (BTP) $\frac{r}{q}$ (Clark, 1990). When the effort level (E) lies below the BTP of this fish species, there exists a non-trivial steady state ($\overline{x}_2$) with non-extinction of fish species. Here we assume that

$$E < \left(\frac{r}{q}\right) \tag{3}$$

We now prove the following theorems:

Theorem 1. The trivial steady state $\bar{x}_1$ is unstable and the non-trivial steady state $\bar{x}_2$ is stable.

Proof. We have

$$\frac{d}{dx}\left(\frac{dx}{dt}\right)\bigg|_{x=\epsilon} \to \infty \ as \ \epsilon \ \to 0+$$

and

$$\frac{d}{dx}\left(\frac{dx}{dt}\right)\bigg|_{x=\bar{x}_2} = -\frac{r}{3k^{1/3}} \ < \ 0.$$

Hence $\bar{x}_1$ is an unstable equilibrium and $\bar{x}_2 = (k/r^3)(r - qE)^3$ is a stable equilibrium.

Theorem 2. The non-trivial steady state $\bar{x}_2$ is globally asymptotically stable for $x(t) > 0$.

Proof. Now $x(t) > \bar{x}_2$

$$\Rightarrow (r - qE)\, x^{-1/3} - \frac{r}{k^{1/3}} \ < \ 0 \ \left(\text{since } E \ < \ \tfrac{r}{q}\right)$$

$$\Rightarrow \frac{dx}{dt} \ = \ x\left[(r - qE)\, x^{-1/3} - \frac{r}{k^{1/3}}\right] \ > \ 0, \ \text{provided } x(t) \ > \ 0.$$

Similarly $x(t) \ < \bar{x}_2 \Rightarrow \ dx/dt \ > \ 0$, provided $x(t) \ > \ 0$. Thus $x(t)$ increases for $0 \ < x \ < \bar{x}_2$ and decreases for $x(t) \ > \bar{x}_2$. Hence $x(t) = \bar{x}_2$ is a globally asymptotically stable equilibrium point for positive $x(t)$ in the sense that $\lim_{t\to\infty} x(t) = \bar{x}_2$ provided $x(0) > 0$.

Theorem 3. The non-trivial steady state $\bar{x}_2 >$ or $<$ $(8k/27)$ according to whether $E \ < r/3q$ or $> \ r/3q$ where r/q =(biotic-potential)/(catchability coefficient) is called the Biotechnical Productivity (Clark, 1990) of the fish species being harvested.

Proof. $F(x) = rx^{2/3} - \dfrac{r}{k^{1/3}}x$

$$F'(x) = r\left[\frac{2}{3}x^{-1/3} - \frac{1}{k^{1/3}}\right] \text{ and } F''(x) = -\frac{2r}{9}x^{-4/3} \ < \ 0.$$

$F'(x) = 0$ gives $x = \frac{8}{27}k$.

So at $x = 8k/27$, $F(x)$ has a maximum value.

Thus the biological productivity of the population is maximized at $x = 8k/27$. Now,

$$\bar{x}_2 = k\left[1 - \frac{qE}{r}\right]^3 \ > \ \frac{8}{27}k$$

$$\Rightarrow \left(\frac{1}{3} - \frac{qE}{r}\right)\left[\left(1 - \frac{qE}{r}\right)^2 + \frac{2}{3}\left(1 - \frac{qE}{r}\right) + \frac{4}{9}\right] > 0$$

$$\Rightarrow E \ < \ \frac{r}{3q} \text{ since } E \ < \frac{r}{q}$$

Similarly, $\bar{x}_2 \ < \dfrac{8}{27}k \Rightarrow E \ > \ \dfrac{r}{3q}$.

2.2 General Solution

The Von Bertalanffy model with a given value of the effort E is shown in Figures 1 and 2.

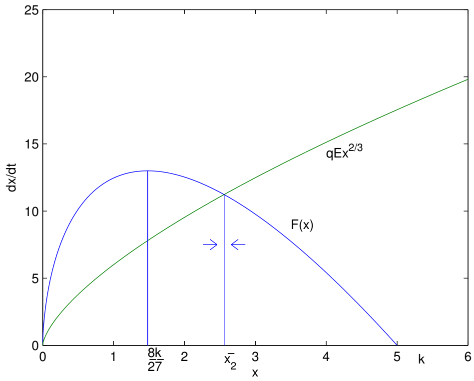

Figure 1: von Bertalanffy growth model with constant rate of effort E $(r = 30, k = 5, E = 60, q = .1)$

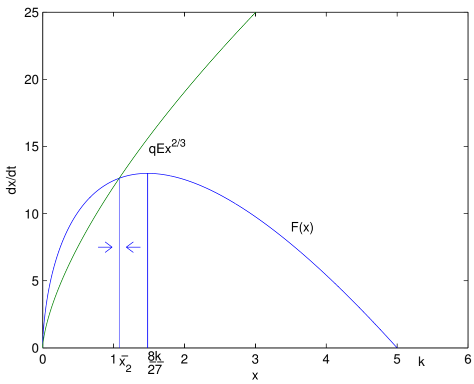

Figure 2: von Bertalanffy growth model with constant rate of effort E $(r = 30, k = 5, E = 120, q = .1)$

We now solve Equation (1). We have

$$\frac{dx}{dt} = x(r_1 x^{-1/3} - rk^{-1/3}), \text{ where } r_1 = r - qE > 0$$

or $$\frac{(kx)^{1/3}dx}{x(r_1 k^{1/3} - rx^{1/3})} = dt$$

integrating this relation, we have

$$\left|r_1k^{1/3} - rx^{1/3}\right| = C_1\, e^{-\frac{r}{3k^{1/3}}t}$$

as $t \to \infty$, $x(t) \to \bar{x}_2$. It is thus once more confirmed that the population level $x = \bar{x}_2$ is globally asymptotically stable.

Theorem 4. For the Von Bertalanffy law of population growth, the exploited population undergoes accelerated growth when $x(t) \in (0, 8\bar{x}_2/27)$ and retarded growth when $x(t) \in (8\bar{x}_2/27, \bar{x}_2)$. Also population decreases with positive acceleration when $x(t) > \bar{x}_2$.
Proof. From (1), we have

$$\frac{d^2x}{dt^2} = r^2x^{1/3}\left\{\left(\frac{x}{k}\right)^{1/3} - \left(1 - \frac{qE}{r}\right)\right\}\left\{\left(\frac{x}{k}\right)^{1/3} - \frac{2}{3}\left(1 - \frac{qE}{r}\right)\right\}$$

Therefore, $\frac{dx}{dt} > 0$ and $\frac{d^2x}{dt^2} > 0$ when $x(t) \in \left(0, \frac{8\bar{x}_2}{27}\right)$.

Also $\frac{dx}{dt} > 0$ and $\frac{d^2x}{dt^2} < 0$ when $x(t) \in \left(\frac{8}{27}\bar{x}_2, \bar{x}_2\right)$, since $\bar{x}_2 = k\left(1 - \frac{qE}{r}\right)^3 > 0$ as $E < \frac{r}{q}$.
Similarly $\frac{dx}{dt} < 0$ and $\frac{d^2x}{dt^2} > 0$ when $x(t) > \bar{x}_2$.
The solution curves are shown in Figure 3.

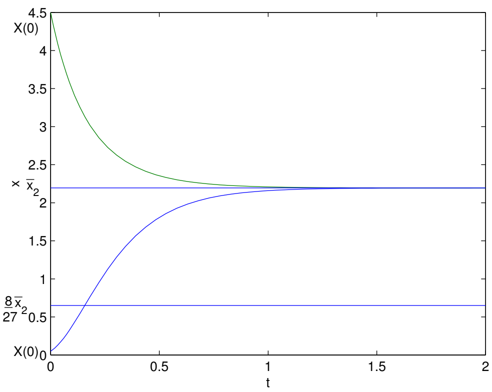

Figure 3: Solution curves $(r = 25, k = 5, E = 60, q = .1)$

2.3 Maximum Sustainable Yield

The Maximum Sustainable Yield (MSY) of a biological resource population is the maximum rate at which it can be harvested even after maintaining the population at a constant level.

The harvesting effort and the population level corresponding to MSY are denoted by E_{MSY} and x_{MSY} respectively.

Theorem 5. The maximum Sustainable Yield (MSY)= $4rk^{2/3}/27$ occurs at the effort level $E_{MSY} = r/3q$ and the population level $x_{MSY} = 8k/27$.

Proof. For a given effort E, the sustainable yield $Y(E)$ is given by

$$Y(E) = qE(\bar{x}_2)^{2/3} = qEk^{2/3}\left(\frac{r-qE}{r}\right)^2 \text{ by (2)}$$

so $\dfrac{dY}{dE} = qk^{2/3}\left(1-\dfrac{qE}{r}\right)\left(1-\dfrac{3qE}{r}\right)$ and $\dfrac{dY}{dE} = 0 \Rightarrow E = \dfrac{r}{3q}$ since $E < \frac{r}{q}$.

Again $\dfrac{d^2Y}{dE^2} = -\dfrac{q^2k^{2/3}}{r^2}\{(r-3qE)+3(r-qE)\}$

so $\left.\dfrac{d^2Y}{dE^2}\right|_{E=\frac{r}{3q}} = -\dfrac{2q^2k^{2/3}}{r} < 0.$

Therefore, $Y(E)$ is maximum when $E = \dfrac{r}{3q}(= E_{MSY})$.

Hence, $MSY = Y(E)|_{E=\frac{r}{3q}} = \dfrac{4}{27}rk^{2/3}$

and $x_{MSY} = k\left.\left(\dfrac{r-qE}{r}\right)^3\right|_{E=\frac{r}{3q}} = \dfrac{8k}{27}.$

Thus the MSY $= 4rk^{2/3}/27$ occurs at the effort level $E_{MSY} = r/3q = (BTP)/3$ and the corresponding population level is $x_{MSY} = 8k/27$. Any value of $E > E_{MSY}, r/3q < E < r/q$. Then the sustainable yield $Y(E)$ monotonically decreases with E towards zero. Biologists call it a case of biological overexploitation (Clark, 1990; Pradhan and Chaudhuri, 1998) whenever the effort level exceeds its MSY level.

Observe that the population level $x = x_{MSY}$ at which productivity of the biological resource (fish) is maximized, is not the natural equilibrium level k, it is only 8/27 times that level. It is also noted that, in the case of the Schaefer model (Schaefer, 1957) the MSY population level is $x_{MSY} = k/2 > 8k/27$. Thus the Schaefer model prescribes a MSY population level higher that in the present model.

2.4 Yield-Effort Curve

The yield obtainable from exploitation of a biological resource depends on the effort rate at which it is harvested. Mathematically it is expressed as $Y = Y(E)$ where 'Y' stands for yield. A pictorial representation of this relation in the $Y - E$ plane is known as the yield-effort curve.
The yield-effort curve is shown in Figure 4.

3 Bionomic Equilibrium

The term bionomic equilibrium is an amalgamation of the concept of biological equilibrium and economic equilibrium. Biological equilibrium is given by $dx/dt = 0$ and economic equilibrium is said to be achieved when TR (total revenue obtained by selling the harvested biomass) equals TC (total cost for the effort devoted to harvesting).

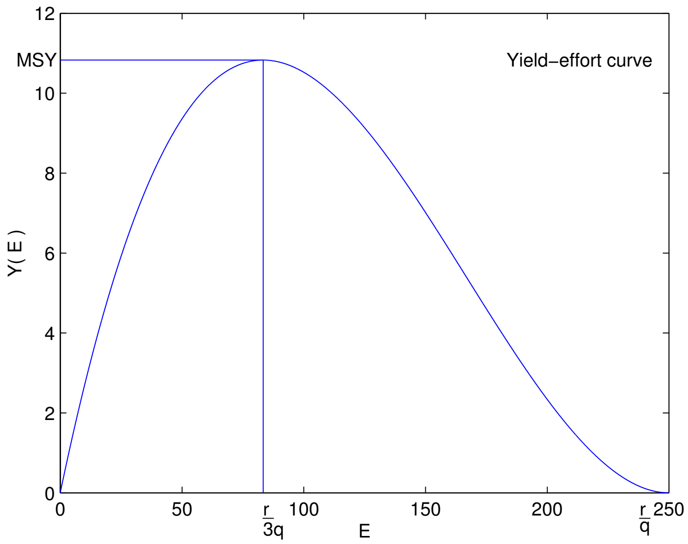

Figure 4: Yield-effort curve $(r = 25, k = 5, q = .1)$

The bionomic equilibrium (x_∞, E_∞) is given as a solution of the system of equations

$$\frac{dx}{dt} = (r - qE)x^{2/3} - \frac{r}{k^{1/3}}x = 0 \tag{4}$$

and

$$\Pi = TR - TC = pqEx^{2/3} - cE = 0 \tag{5}$$

where c= constant fishing cost per unit effort and p= constant price per unit biomass. From (5),

$$x_\infty = \left(\frac{c}{pq}\right)^{3/2} \text{ or } E_\infty = 0. \tag{6}$$

Substituting $E_\infty = 0$ in (4) we get $x_\infty = 0$ or k and

$$E_\infty = \frac{r}{q}\left[1 - \frac{1}{k^{1/3}}\left(\frac{c}{pq}\right)^{1/2}\right] \text{ when } x_\infty = \left(\frac{c}{pq}\right)^{\frac{3}{2}}. \tag{7}$$

To examine the nature of the bionomic equilibrium level of effort E_∞, we analyse the following cases :

(i) If $c > k^{2/3}pq$ then no positive bionomic equilibrium exists. So $E_\infty = 0$ and fishery remains unexploited. Hence we conclude that if the fishing cost is sufficiently high relative to the price of fish as in curve TC_1 in Figure 5 ,then the fisherman's choice should be to stop fishing. Because the market value of fish, even though it may be positive, does not offset the expense of catching the fish.

(ii) If $c < k^{2/3}pq$ then unique positive bionomic equilibrium point exist.

Two cases arise here.

(a) $4k^{2/3}/9 < c/pq < k^{2/3}$. This implies $x_{MSY} = 8k/27 < x_\infty < k$ and $E_{MSY} = r/3q > E_\infty$, the bionomic equilibrium is established at a level such as E_∞^2 in Figure 5. At this stage the effort is still below the level of maximum sustainable biological yield E_{MSY} and the fishery becomes profitable. The bionomic equilibrium ensures that there is no biological overfishing and fisherman should, therefore, operate at the bionomic equilibrium (x_∞, E_∞). The MSY level (x_{MSY}, E_{MSY}) can be the fisherman's choice only if there is no consideration of economic factors. If effort can somehow reduced then TR and TC both are decrease, but positive economic rent achieved. Due to this, fisherman applies excessive effort and ultimately producing zero rent i.e. economic overfishing occures.

(b) If $0 < c/pq < 4k^{2/3}/9$, i.e., $x_\infty < x_{MSY} = 8k/27$, then $E_\infty > E_{MSY}$. Therefore, the bionomic equilibrium population level is lower than its MSY level and the bionomic equilibrium level of effort exceeds its MSY level. This situation also indicates that if the cost-price ratio becomes sufficient low, equilibrium is established at a level $E_\infty > E_{MSY}$ and biological over-fishing occurs as in curve TC_3 in Figure 5. In this case the fisherman should not operate at the bionomic equilibrium. Here also economic overfishing occures.

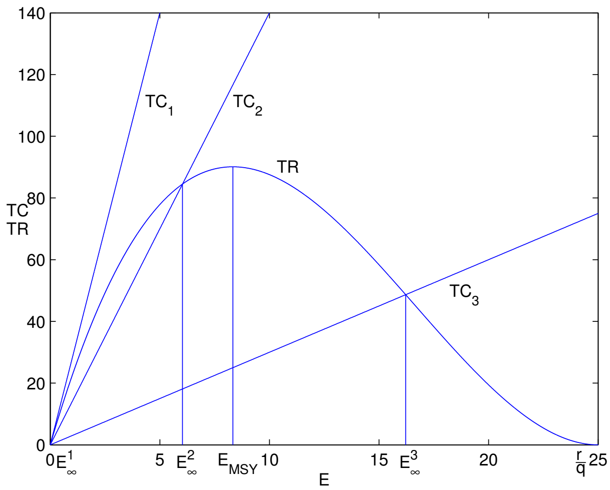

Figure 5: Bionomic equilibrium levels ($r = 2.5, k = 15, q = .1, p = 40, c_1 = 28, c_2 = 14, c_3 = 3$)

4 Optimal Harvest Policy

The net economic revenue at any time is given by

$$\Pi = \{pqx^{2/3}(t) - c\}E(t) \tag{8}$$

Our purpose is to maximize the present value J of a continuous time-stream of revenues given by (Clark, 1990)

$$J = \int_0^\infty e^{-\delta t}\{pqx^{2/3}(t) - c\}E(t)dt \tag{9}$$

where $\delta(> 0)$ is the instantaneous annual rate of discount. If one unit of money be invested at an annual interest rate δ(per unit of money), continuously compounded, then the amount in the account at some time t will be $e^{\delta t}$. Hence one has to multiply any future revenue by the discounting factor $e^{-\delta t}$ in order to convert it to the present value.

We intend to maximize (9) subject to the state equation (1) by invoking Pontryagin's Maximum Principle (Pontryagin, Boltyanskii and Gamkrelidze). The control variable $E(t)$ is subject to restriction $0 \leq E(t) \leq E_{max}$, E_{max} stands for a feasible upper limit of the harvesting effort and it may be a constant or a function of x and t. Here $V = [0, E_{max}]$ is the control set.

Our purpose is to determine $E(t)$ and the corresponding response $x(t)$ which maximize (9) subject to the conditions stated above. We now solve this optimal control problem by taking the control variable $E(t)$ as a dynamic (i.e. time dependent) variable.

In all the following conditions, the variables x and E are to be taken as functions of t irrespective of whether it is mentioned explicitly or not. Let us construct the Hamiltonian :

$$\begin{aligned} H &= e^{-\delta t}(pqx^{2/3} - c)E(t) + \lambda(t)\left[\{r - qE(t)\}x^{2/3} - \frac{r}{k^{1/3}}x\right] \\ &= \{e^{-\delta t}(pqx^{2/3} - c) - \lambda qx^{2/3}\}E(t) + \lambda(t)r\left\{x^{2/3} - \frac{x}{k^{1/3}}\right\} \\ &= \sigma(t)E(t) + \lambda(t)r\left\{x^{2/3} - \frac{x}{k^{1/3}}\right\} \end{aligned} \tag{10}$$

where

$$\sigma(t) = e^{-\delta t}(pqx^{2/3} - c) - \lambda qx^{2/3} \tag{11}$$

Considering that $E(t)$ is an optimal control and x is the corresponding response, the maximal principle ensures the existence of adjoint variable $\lambda(t)$ for all $t \geq 0$ such that

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial x} = -(e^{-\delta t}pq - \lambda q)\frac{2}{3}x^{-1/3}E(t) - r\lambda(t)\left\{\frac{2}{3}x^{-1/3} - \frac{1}{k^{1/3}}\right\}. \tag{12}$$

From (10), we have

$$H_E = \frac{\partial H}{\partial E} = \sigma(t) \tag{13}$$

According to the maximum principle, the optimal control $E(t)$ must maximize the Hamiltonian H. Clearly the optimal control $E(t)$ must satisfy the conditions :

$$E(t) = \begin{cases} E_{max} & \text{when } \sigma(t) > 0 \\ 0 & \text{when } \sigma(t) < 0 \end{cases}$$

Since $\sigma(t)$ causes $E(t)$ to switch between 0 and E_{max} , $\sigma(t)$ is called the switching function. Hence $E(t)$ is a bang-bang control switching from one extreme level (zero) to the other one (E_{max}) or vice-versa.

Now, $H_E >$ or < 0 according as $p - cx^{-2/3}/q >$ or $< \lambda e^{\delta t}$. Since a unit harvest ($h\Delta t = 1, h = qEx^{2/3}$) incurs a cost equal to $cE\Delta t = cx^{-2/3}h\Delta t/q = cx^{-2/3}/q$ (where$h = qEx^{2/3}$is the catch rate), we see that $cx^{-2/3}/q$ is equal to the unit harvesting cost when the population level is x. Therefore, $p\ - (cx^{-2/3}/q)$ equals to the net economic revenue on the unit harvest. Hence we conclude that $E = E_{max}$ or 0 according to whether the net economic revenue on a unit harvest is greater or less than the shadow price ($\lambda e^{\delta t}$) (Clark, 1990) of the population respectively (since $H_E = \sigma(t)$) by (13). When the net economic revenue on a unit harvest equals this shadow price, we have $H_E = \sigma(t) = 0$. Once $\sigma(t)$ vanishes, the Hamiltonian function H becomes independent of the control variable $E(t)$ and its optimal value cannot be determined by the above procedure. It is then called a singular control $E^*(t), 0 < E^*(t) < E_{max}$.

Therefore , the optimal harvest policy is $E = \begin{cases} E_{max} & \text{if } H_E > 0 \\ 0 & \text{if } H_E < 0 \\ E^*(t) & \text{if } H_E = 0 \end{cases}$

Using (13), the necessary condition for a singular extremal $H_E = 0$ yields

$$\lambda(t) = \frac{e^{-\delta t}(pqx^{2/3} - c)}{qx^{2/3}} \tag{14}$$

The other condition to be satisfied along a singular extremal are (Goh, 1980):
$DH_E = 0, D^2H_E = 0, (D^2H_E)_E > 0,$
where $D \equiv \frac{d}{dt}$ and the subscripts denote partial differentiation. Using (13) and (1), we get

$$DH_E = e^{-\delta t}pq\left[\frac{2}{3}rx^{1/3} - \left(\frac{2r}{3k^{1/3}} + \delta\right)x^{2/3}\right] - \frac{\lambda qr}{3k^{1/3}}x^{2/3} + c\delta e^{-\delta t} \tag{15}$$

Thus $DH_E = 0$ gives

$$\lambda = pe^{-\delta t}\left[2\left(\frac{k}{x}\right)^{1/3} - \left(2 + \frac{3\delta k^{1/3}}{r}\right) + \frac{3c\delta k^{1/3}}{pqrx^{2/3}}\right] \tag{16}$$

Again differentiating (15) with respect to t and after some simplifications, we have
$D^2H_E =$

$$e^{-\delta t}pq\left[\frac{2}{3}\delta x^{1/3}\left\{qE + 2r\left(\left(\frac{x}{k}\right)^{1/3} - 1\right)\right\} + \delta^2 x^{2/3} + \frac{2r^2}{9}\left\{1 - 2\left(\frac{x}{k}\right)^{1/3}\right\}\right.$$
$$\left.\left\{1 - \left(\frac{x}{k}\right)^{1/3}\right\} + \frac{2}{9}rqE\left\{3\left(\frac{x}{k}\right)^{1/3} - 1\right\}\right] - \frac{qr^2\lambda}{9}\left(\frac{x}{k}\right)^{2/3} - c\delta^2 e^{-\delta t} \tag{17}$$

Therefore,

$$(D^2H_E)_E = e^{-\delta t}pq^2\frac{2x^{1/3}}{3}\left\{\delta + r\left(\frac{1}{k^{1/3}} - \frac{1}{3x^{1/3}}\right)\right\} \tag{18}$$

For the existence of an optimal singular control, it is required to satisfy the generalized Legendre condition (Goh, 1980):

$$(D^2H_E)_E > 0 \Rightarrow x^{1/3} > \frac{rk^{1/3}}{3(r+\delta K^{1/3})} \tag{19}$$

Now $D^2H_E = 0$ gives (by using (14)):

$$E = \frac{\left(9\delta^2 - \frac{r^2}{k^{2/3}}\right)(pqx^{2/3} - c) - 12pq\delta r x^{1/3}\left\{1 - \left(\frac{x}{k}\right)^{1/3}\right\}}{2pq^2\left\{r - 3\delta x^{1/3} - 3r\left(\frac{x}{k}\right)^{1/3}\right\}} + \frac{2r^2pq\left\{1 - 3\left(\frac{x}{k}\right)^{1/3} + 2\left(\frac{x}{k}\right)^{2/3}\right\}}{2pq^2\left\{r - 3\delta x^{1/3} - 3r\left(\frac{x}{k}\right)^{1/3}\right\}} \tag{20}$$

Equation (20) yields the optimal singular control $E^*(t)$ in terms of the optimal population levels $x^*(t)$ where $x^*(t)$ is determined as follows:
From (14) and (16), we have

$$\frac{e^{-\delta t}(pqx^{2/3} - c)}{qx^{2/3}} = pe^{-\delta t}\left[2\left(\frac{k}{x}\right)^{1/3} - \left(2 + \frac{3\delta k^{1/3}}{r}\right) + \frac{3c\delta k^{1/3}}{pqrx^{2/3}}\right]$$

$$\Rightarrow \{x^*(t)\}^{1/3} = \frac{rk^{1/3} + \sqrt{r^2k^{2/3} + \frac{3c}{pq}(r+\delta k^{1/3})(r+3\delta k^{1/3})}}{3(r+\delta k^{1/3})} \tag{21}$$

since $x^*(t) > 0$. Which satisfies the Legendre condition (19). The necessary condition $H_E = 0$ for a singular extremal control $E^*(t)$, where $0 < E^*(t) < E_{max}$, implies

$$e^{-\delta t}(pqx^{2/3} - c) = \lambda q x^{2/3}$$

$$\Rightarrow e^{-\delta t}\frac{\partial \Pi}{\partial E} = \frac{\partial}{\partial E}(\lambda h) \text{ (by(8))} \tag{22}$$

where $h = qEx^{2/3}$ is the catch rate. In economics, the rate of change of the revenue Π with respect to the effort $E(= \partial\Pi/\partial E)$, is called the marginal revenue of effort. When this is multiplied by the discounting factor $e^{-\delta t}$,i.e. $e^{-\delta t}(\partial\Pi/\partial E)$, we get the present value of the marginal revenue of effort. The adjoint variable λ is interpreted as the marginal user cost along the optimal trajectory $x^*(t)$. To understand this interpretation of λ, one has to study the economic interpretation of the Maximum Principle (Clark,

1990). With this interpretation of $\lambda, \lambda h$ is the total user cost or simply user cost. The right hand side of (22) may be interpreted as the marginal user cost of effort. Therefore the optimal equilibrium solution is obtained when the present value of the marginal revenue of effort equals the marginal user cost of effort.
We now discuss the following cases :

(i) If $\delta = 0$, we have from (6) and (21)

$$\frac{x^{*1/3}}{x_{MSY}^{1/3}} = \frac{1}{2}\left\{1 + \sqrt{1 + \left(\frac{x_\infty}{k}\right)^{2/3}}\right\} > 1 \Rightarrow x^* > x_{MSY} \text{ since } x_{MSY} = 8k/27.$$

Therefore, for zero interest rate,the optimal population level (x^*) is always greater than the MSY population level (x_{MSY}). This result is valid only if there is no discounting of money, i.e. if the money-value does not change with time.

(ii) $x^* \to x_\infty$ as $\delta \to \infty$ by (6) and (21). Thus positive discount rate leads to progressively decreasing optimal population level x^* that tends to x_∞ as $\delta \to \infty$. In this case $\Pi \to 0$ from (5).

This implies that infinite interest rate leads to complete dissipation of the economic revenue(i.e. the total revenue equals to the total cost). This type of situation arises in an open-access fishery (Clark, 1990) where the fishermen are forced to set a zero value on future revenues because access to the fishery is open to anybody at all times. Mathematically, the discounting factor $e^{-\delta t} \to 0$ as the interest rate $\delta \to \infty$ and hence the present value of a future revenue is reduced to zero. Economically, a high interest rate implies a high inflation rate. Since the real value of the resource declines rapidly under a high rate of inflation, the management (or the owner) of the resource stock prefer to exploit it at a no-profit no-loss basis.

(iii) If c (fishing cost per unit effort)=0, we have $x_\infty = \left(\frac{c}{pq}\right)^{3/2} = 0$ and then

$$x^* = \frac{8k}{27\left(1 + \frac{\delta}{r}k^{1/3}\right)^3} < \frac{8k}{27} = x_{MSY}.$$

Therefore in this case $x_\infty = 0$ and $x^* \neq 0$.

Thus, when cost of fishing is zero (or negligible), the optimal population level (x^*) is always less than its MSY level (x_{MSY}).

Now, $\frac{\delta}{r} \to \infty \Rightarrow x^* \to 0$. The optimal harvest policy in this situation implies the most rapid possible extinction of the resource population. Therefore, if the growth rate (δ) of the economy being much higher than the growth rate (r) of the resource, the owner of the fishery makes no delay in cashing out the fishery and in investing the earning in the economy.

5 Conclusions

Present paper deals with a problem of single species fishery model in which growth of the fish population follows Von Bertalanffy growth function. It is seen that for this model

$x_{MSY} = 8k/27$ which is lower than that of Schaefer model ($x_{MSY} = k/2$) . This implies that the resource population can be safely exploited without causing over exploitation, up to a level much lower than the MSY population level in the Schaefer model. From the view point of conservation of resources, we conclude that there is no need of taking a conservative view to maintain the population level at an unnecessarily higher level in order to avoid over exploitation. When cost-price ratio becomes sufficiently low, then both biological and economic overfishing occures. If non negative bionomic equilibrium is less than E_{MSY} then economic overfishing occures but biological overfishing does not occure. So it is necessary to control the fishery by stopping fishing effort or forced to impose ever shorter fishing seasons by fishery management for long run. Using Pontryagin,s Maximum Principle, optimal harvest policy has been discussed using effort as a control variable. It has been found that the present value of the marginal revenue of effort equals the marginal user cost of effort. It has also been noted that infinite discount rate yields zero economic rent i.e. the fishery should remain unexploited and when cost of fishing is negligble, instantaneous annual rate of discount is verry high then resource population is extinct.

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