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The residually weakly primitive geometries of J_2

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Abstract. We announce the end of the classification of all firm and residually connected geometries satisfying the conditions $(IP)_2$ and $(2T)_1$ and on which the Hall-Janko group J_2 acts flag-transitively and residually weakly primitively. We state some facts regarding the results. The complete list of geometries is available as a supplement to this paper [9].

Keywords: Janko group J_2; flag-transitive automorphism group; coset geometry.

MSC 2000 classification: 51E24.

Introduction

In [6], Dehon and the author described two algorithms to classify all geometries Γ of a given group G such that Γ is a firm and residually connected geometry and G acts flag-transitively and residually weakly primitively on Γ. We stated in that paper that these programs were able to classify all geometries with Borel subgroup not equal to the identity of G when G is the Hall-Janko group J_2. In the meantime, we have succeeded in classifying all geometries with Borel subgroup the identity of G for J_2 as we describe in Section 2.

In the present paper, we announce the results we obtained using our set of programs. These programs were written in Magma [1].

The paper is organised as follows. In section 1, we recall the basic definitions needed to understand this paper. In section 2, we explain how we dealt with the case where the Borel subgroup is reduced to the identity for the Hall-Janko group J_2. In section 3, we state the results obtained and state some facts concerning these results. Finally, in section 4, we describe what can be found in the supplement to this paper [9].

1 Definitions and notations

The basic concepts about geometries constructed from a group and some of its subgroups are due to Tits [13] (see also [3], chapter 3).

Let $I = \{1, \ldots, n\}$ and let $\Gamma(G; (G_i)_{i \in I})$ be a coset geometry constructed from a group G and some subgroups G_i of G (where $i \in I$). In the present paper, we follow the set of axioms described in [4], i.e. we want Γ to be

- *flag-transitive* (FT): G acts transitively on all chambers of Γ, hence also on all flags of any given type J, where J is a subset of I; this property also implies that every flag of Γ is contained in a chamber and hence that Γ is a *geometry* in the sense of Buekenhout [2];
- *firm* (F): every flag of rank $\mid I \mid -1$ is contained in at least two chambers;
- *residually connected* (RC): the incidence graph of each residue of rank ≥ 2 is a connected graph;
- *residually weakly primitive* (RWPRI): the stabilizer G_F of the residue Γ_F of each flag F is primitive on the set of elements of type i of Γ_F for at least one $i \in t(\Gamma_F)$;
- $(IP)_2$: every rank 2 residue of Γ is either a partial linear space or a generalized digon [3];
- $(2T)_1$: the stabilizer G_F of any given flag F of Γ of corank 1 acts two-transitively on the residue Γ_F.

As in [7], for any $\emptyset \neq J \subset I$ we set $G_J := \bigcap_{j \in J} G_j$, $B := G_I$ and $G_\emptyset := G$. Then we call $\mathcal{L}(\Gamma) := \{G_J \mid J \subset I\}$ the *sublattice* (of the subgroup lattice of G) *spanned* by the collection $(G_i)_{i \in I}$. The subgroup B is called the *Borel subgroup* of Γ.

The group of type-preserving automorphisms of Γ is denoted by $Aut(\Gamma)$ and the automorphism group of Γ is denoted by $Cor(\Gamma)$.

As to notation for groups, we follow the conventions of the Atlas [5].

2 Geometries with $B = 1$

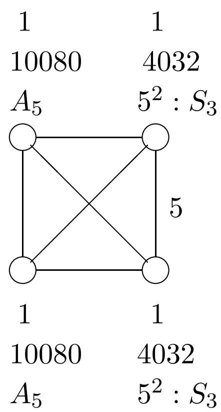
In the case $B = 1$, the programs described in [6] were unable to complete the classification with the computer we had. One gigabyte of memory was not enough. The RWPRI condition implies that at least one subgroup G_i forming the geometry $\Gamma(G; (G_i)_{i \in I})$ must be maximal in G. So we decided to build another series of programs that construct all geometries of each maximal subgroup M of G and then tries to extend all these geometries to a geometry of G. These programs took a long time to execute but after months of computation, the computer ruled out all extensions of geometries of rank four of the maximal subgroups of J_2 and found two rank three geometries of maximal subgroups of

J_2 that could be extended to rank four geometries of J_2. These two geometries are mentioned in Section 3 as number 1 and 2.

3 The geometries of J_2

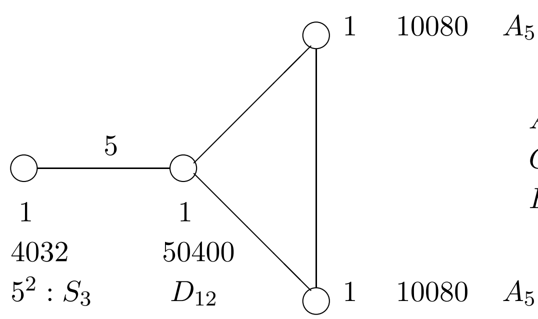
Using our set of programs, we obtained 8, 19, 3, 0 geometries of rank 2, 3, 4 and ≥ 5 satisfying the set of axioms described in Section 1. We only mention here the diagrams of the three rank four geometries obtained. Two of them have a Borel subgroup reduced to the identity, the third one has a dihedral group of order 8 as Borel subgroup. For the complete list of geometries obtained, we refer to [9].

1 (2)



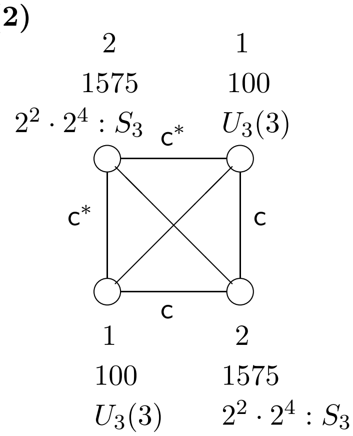
$Aut(\Gamma) = J_2$
$Cor(\Gamma) = J_2 \times 2^2$
$B = 1$

2 (2)



$Aut(\Gamma) = J_2$
$Cor(\Gamma) = J_2 \times 2$
$B = 1$

3 (2)



$Aut(\Gamma) = J_2$
$Cor(\Gamma) = J_2 \times 2$
$B = D_8$

For a geometric construction of geometry number 3, we refer to [8]. The first two geometries are very nice in the sense that they are thin geometries. Up to now, these are the only thin geometries satisfying our set of axioms that we obtained for the sporadic groups investigated so far (i.e. M_{11}, M_{12}, J_1 and J_2). Geometry number 1 may be obtained from geometry number 2 using a construction described in [10] and commonly called "doubling".

4 What is available in the supplement [9]?

For every geometry Γ, we give its diagram, the automorphism and correlation groups, the Borel subgroups, the kernels. We mention when a geometry is a truncation of a higher rank geometry. We also say when a geometry can be constructed from another using a construction described in Theorem 4.1 of [10] and that Pasini calls doubling [12]. At the end of the supplement, the full subgroup pattern of J_2 is given. This subgroup pattern was first computed by Pahlings [11].

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