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# Asymptotic behavior for a nonlocal diffusion problem with Neumann boundary conditions and a reaction term

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**Abstract.** In this paper, we consider the following initial value problem

$$u_t(x,t) = \int_\Omega J(x-y)(u(y,t)-u(x,t))dy - \gamma u^p(x,t) \quad \text{in} \quad \overline{\Omega}\times(0,\infty),$$

$$u(x,0) = u_0(x) > 0 \quad \text{in} \quad \overline{\Omega},$$

where $\gamma \in \{-1, 1\}$ is a parameter, $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p > 1$, $J\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ is a kernel which is nonnegative, measurable, symmetric, bounded and $\int_{\mathbb{R}^N} J(z)dz = 1$, the initial datum $u_0 \in C^0(\overline{\Omega})$, $u_0(x) > 0$ in $\overline{\Omega}$. We show that, if $\gamma = 1$, then the solution $u$ of the above problem tends to zero as $t \to \infty$ uniformly in $x \in \overline{\Omega}$, and a description of its asymptotic behavior is given. We also prove that, if $\gamma = -1$, then the solution $u$ blows up in a finite time, and its blow-up time goes to that of the solution of a certain ODE as the $L^\infty$ norm of the initial datum goes to infinity.

**Keywords:** Nonlocal diffusion, asymptotic behavior, blow-up time

**MSC 2000 classification:** 35B40, 45A07, 35G10

## 1 Introduction

Consider the following initial value problem

$$u_t(x,t) = \int_\Omega J(x-y)(u(y,t)-u(x,t))dy - \gamma u^p(x,t) \quad \text{in} \quad \overline{\Omega}\times(0,\infty), \tag{1}$$

$$u(x,0) = u_0(x) > 0 \quad \text{in} \quad \overline{\Omega}, \tag{2}$$

where $\gamma \in \{-1, 1\}$ is a parameter, $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p > 1$, $J$: $\mathbb{R}^N \longrightarrow \mathbb{R}$ is a kernel which is nonnegative, measurable, symmetric, bounded and $\int_{\mathbb{R}^N} J(z)dz = 1$, the initial datum $u_0 \in C^0(\overline{\Omega})$, $u_0(x) > 0$ in $\overline{\Omega}$. Recently, nonlocal diffusion problems have been the subject of investigations of many authors (see, [1], [2], [4]–[7], [13]–[18], [20]–[22], [25]–[27], [31], and the references cited therein). Nonlocal evolution equations of the form

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)(u(y,t) - u(x,t))dy,$$

and variations of it, have been used by a lot of authors to model diffusion processes (see, [4]–[6], [13], [20], [21]). The solution $u(x,t)$ can be interpreted as the density of a single population at the point $x$, at the time $t$, and $J(x-y)$ as the probability distribution of jumping from location $y$ to location $x$. Then the convolution $(J * u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy$ is the rate at which individuals are arriving to position $x$ from all other places, and $-u(x,t) = -\int_{\mathbb{R}^N} J(x-y)u(x,t)dy$ is the rate at which they are leaving location $x$ to travel to any other side (see, [20]). Let us notice that for our equation, the term of the source $-\gamma u^p(x,t)$ can be rewritten as follows

$$-\gamma u^p(x,t) = \int_{\mathbb{R}^N} J(x-y)(-\gamma u^p(x,t))dy.$$

Therefore, in view of the above equality, the term of the source $-\gamma u^p(x,t)$ can be interpreted as a force which increases the rate of individuals that leave location $x$ to travel to any other site when $\gamma = 1$, and decreases this rate when $\gamma = -1$.
In this paper, we are interested in the asymptotic behavior of the solution of (1)–(2) when $\gamma = 1$, and the blow-up of the solution of (1)–(2) when $\gamma = -1$. For local diffusion problems, the asymptotic behavior of solutions has been the subject of investigations of several authors (see, [3], [9]–[12], [23], [24], and the references cited therein). For our problem, in the case where $\gamma = 1$ and $\Omega = \mathbb{R}^N$, Pazoto and Rossi studied in [26] the asymptotic behavior of solutions. On the other hand, when $\gamma = -1$, Perez-LLanos and Rossi proved that the solution $u$ of (1)–(2) blows up in a finite time and some results about blow-up rate and set have been given. In the same way, in [25], we considered the following initial-boundary value problem

$$u_t(x,t) = \varepsilon \int_{\Omega} J(x-y)(u(y,t) - u(x,t))dy + f(u(x,t)) \quad \text{in} \quad \Omega \times (0,T),$$

$$u(x,t) = 0 \quad \text{in} \quad (\mathbb{R}^N - \Omega) \times (0,T), \quad u(x,0) = u_0(x) \quad \text{in} \quad \overline{\Omega},$$

where $f : [0,\infty) \to [0,\infty)$ is $C^1$ nondecreasing function, and $\varepsilon$ is a positive parameter. Under some assumptions, we showed that the solution $u$ of the above problem blows up in a finite time, and its blow-up time goes to that of the solution of the following ODE $\alpha^{'}(t) = f(\alpha(t))$, $t > 0$, $\alpha(0) = \|u_0\|_\infty$, as $\varepsilon$ goes to zero. In the current paper, we obtain an analogous result when $\gamma = -1$ taking as parameter the $L^\infty$ norm of the initial datum. We also prove that when $\gamma = 1$, then the solution $u$ of (1)–(2) tends to zero as $t \to \infty$ uniformly in $x \in \overline{\Omega}$, and a complete description of its asymptotic behavior is exhibited. The remainder of the paper is written in the following manner. In the next section, we provide some material about the maximum principle for nonlocal problems. In the third section, we prove the local existence and uniqueness of solutions. In the fourth and fifth sections, we obtain the main results when $\gamma = 1$ and $\gamma = -1$, respectively.

## 2 Maximum principle

In this section, we give some insights about the maximum principle for nonlocal problems for our subsequent use.
The following lemma is a form of the maximum principle for nonlocal problems.

**Lemma 1.** *Let* $b \in C^0(\overline{\Omega} \times [0,\infty))$ *and let* $u \in C^{0,1}(\overline{\Omega} \times [0,\infty))$ *satisfy the following inequalities*

$$u_t(x,t) - \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + b(x,t)u(x,t) \geq 0 \quad in \quad \overline{\Omega} \times (0,\infty),$$

$$u(x,0) \geq 0 \quad in \quad \overline{\Omega}.$$

*Then we have* $u(x,t) \geq 0$ *in* $\overline{\Omega} \times (0,\infty)$.

*Proof.* Let $T_0$ be any positive quantity satisfying $T_0 < \infty$, and let $\lambda$ be such that $b(x,t) - \lambda > 0$ in $\overline{\Omega} \times [0,T_0]$. Introduce the function $z(x,t) = e^{\lambda t}u(x,t)$, and suppose $m = \min_{x\in\overline{\Omega}, t\in[0,T_0]} z(x,t)$. Then there exists $(x_0,t_0) \in \overline{\Omega} \times [0,T_0]$ such that $m = z(x_0,t_0)$. We get $z(x_0,t_0) \leq z(x_0,t)$ for $t \leq t_0$ and $z(x_0,t_0) \leq z(y,t_0)$ for $y \in \Omega$, which implies that

$$z_t(x_0,t_0) \leq 0, \tag{3}$$

and

$$\int_\Omega J(x_0-y)(z(y,t_0)-z(x_0,t_0))dy \geq 0. \tag{4}$$

Using the first inequality of the lemma, it is not hard to see that

$$z_t(x_0,t_0) - \int_\Omega J(x_0-y)(z(y,t_0)-z(x_0,t_0))dy + (b(x_0,t_0)-\lambda)z(x_0,t_0) \geq 0.$$

Making use of (3) and (4), we observe that the first two terms on the left hand side of the above inequality are nonpositive. We deduce that $(b(x_0,t_0)-\lambda)z(x_0,t_0) \geq 0$, which implies that $z(x_0,t_0) \geq 0$ because $b(x_0,t_0) - \lambda > 0$. Consequently, we have $u(x,t) \geq 0$ in $\overline{\Omega} \times [0,T_0]$, which leads us to the result. QED

Another version of the maximum principle for nonlocal problems is the following comparison lemma.

**Lemma 2.** *Let* $u, v \in C^{0,1}(\overline{\Omega} \times [0,\infty))$ *be such that*

$$\begin{aligned} u_t(x,t) &- \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + \gamma u^p(x,t) > \\ v_t(x,t) &- \int_\Omega J(x-y)(v(y,t)-v(x,t))dy + \gamma v^p(x,t) \quad in \quad \overline{\Omega} \times (0,\infty), \end{aligned}$$

$$u(x,0) > v(x,0) \quad in \quad \overline{\Omega}.$$

*Then we have* $u(x,t) > v(x,t)$ *in* $\overline{\Omega} \times (0,\infty)$.

*Proof.* Let $w = u - v$ in $\overline{\Omega} \times [0, \infty)$. A straightforward computation reveals that

$$w_t(x,t) - \int_\Omega J(x-y)(w(y,t) - w(x,t))dy + b(x,t)w(x,t) > 0 \text{ in } \overline{\Omega} \times (0,\infty),$$

$$w(x,0) > 0 \quad \text{in} \quad \overline{\Omega},$$

where $b(x,t) = p\gamma \int_0^1 (su(x,t) + (1-s)v(x,t))^{p-1} ds$. We know from Lemma 1 that $u(x,t) \geq 0$ in $\overline{\Omega} \times (0,\infty)$. Assume that $w(x_0,t_0) = 0$ for a certain $(x_0,t_0) \in \overline{\Omega} \times (0,\infty)$. This implies that $w_t(x_0,t_0) \leq 0$ and $\int_\Omega J(x-y)(w(y,t_0) - w(x_0,t_0))dy \geq 0$. We infer that

$$w_t(x_0,t_0) - \int_\Omega J(x-y)(w(y,t_0) - w(x_0,t_0))dy + b(x_0,t_0)w(x_0,t_0) \leq 0,$$

which is a contradiction. This finishes the proof. QED

# 3 Local existence

In this section, we shall establish the existence and uniqueness of nonnegative solutions of (1)–(2) in $\overline{\Omega} \times (0,T)$ for small $T$. We shall also point out that global existence occurs when $\gamma = 1$.

Without loss of generality, we may replace the reaction term $-\gamma u^p(x,t)$ by the term $-\gamma |u(x,t)|^{p-1} u(x,t)$. Indeed, if one shows the existence and uniqueness of nonnegative solutions of (1)–(2) in $\overline{\Omega} \times (0,T)$ taking this last reaction term, then the nonnegativity of solutions implies the existence and uniqueness of solutions of (1)–(2) in $\overline{\Omega} \times (0,T)$ for the first reaction term. Let $t_0 > 0$ be fixed and define the function space

$$Y_{t_0} = \{u;\ u \in C([0,t_0], C(\overline{\Omega}))\}$$

equipped with the norm defined by $\|u\|_{Y_{t_0}} = \max_{0 \leq t \leq t_0} \|u(\cdot,t)\|_\infty$ for $u \in Y_{t_0}$. It is easy to see that $Y_{t_0}$ is a Banach space. Introduce the set

$$X_{t_0} = \{u;\ u \in Y_{t_0}, \|u\|_{Y_{t_0}} \leq b_0\},$$

where $b_0 = 2\|u_0\|_\infty + 1$. We observe that $X_{t_0}$ is a nonempty bounded closed convex subset of $Y_{t_0}$. Define the map $R$ as follows

$$R : X_{t_0} \longrightarrow X_{t_0},$$

$$R(v)(x,t) = u_0(x) + \int_0^t \int_\Omega J(x-y)(v(y,s) - v(x,s))dyds - \gamma \int_0^t |v(x,s)|^{p-1} v(x,s)ds.$$

**Theorem 1.** *Assume that $u_0 \in C^0(\overline{\Omega})$. Then $R$ maps $X_{t_0}$ into $X_{t_0}$, and $R$ is strictly contractive if $t_0$ is appropriately small relative to $\|u_0\|_\infty$.*

*Proof.* We get

$$|R(v)(x,t) - u_0(x)| \leq 2\|J\|_\infty |\Omega| \|v\|_{Y_{t_0}} t + \|v\|^p_{Y_{t_0}} t,$$

which implies that $\|R(v)\|_{Y_{t_0}} \leq \|u_0\|_\infty + 2\|J\|_\infty b_0 |\Omega| t_0 + b_0^p t_0$.

Consequently, if

$$t_0 \leq \frac{b_0 - \|u_0\|_\infty}{2|\Omega| \|J\|_\infty b_0 + b_0^p}, \tag{5}$$

then $\|R(v)\|_{Y_{t_0}} \leq b_0$. Therefore if (6) holds, then $R$ maps $X_{t_0}$ into $X_{t_0}$. Now we are going to prove that the map $R$ is strictly contractive. Let $v, z \in X_{t_0}$. Setting $\alpha = v - z$, we discover that

$$|(R(v) - R(z))(x,t)| \leq |\int_0^t \int_\Omega J(x-y)(\alpha(y,s) - \alpha(x,s))dyds|$$
$$+|\int_0^t (|v(x,s)|^{p-1}v(x,s) - |z(x,s)|^{p-1}z(x,s))ds|.$$

Use Taylor's expansion to obtain

$$|(R(v) - R(z))(x,t)| \leq 2\|J\|_\infty |\Omega| \|\alpha\|_{Y_{t_0}} t + t\|v - z\|_{Y_{t_0}} p\|\beta\|_{Y_{t_0}}^{p-1},$$

where $\beta$ is a function which is localized between $v$ and $z$. We deduce that

$$\|R(v) - R(z)\|_{Y_{t_0}} \leq 2\|J\|_\infty |\Omega| \|\alpha\|_{Y_{t_0}} t_0 + t_0\|v - z\|_{Y_{t_0}} p\|\beta\|_{Y_{t_0}}^{p-1},$$

which implies that $\|R(v) - R(z)\|_{Y_{t_0}} \leq (2\|J\|_\infty|\Omega|t_0 + t_0 p b_0^{p-1})\|v - z\|_{Y_{t_0}}$. If

$$t_0 \leq \frac{1}{4\|J\|_\infty|\Omega| + 2pb_0^{p-1}},$$

then $\|R(v) - R(z)\|_{Y_{t_0}} \leq \frac{1}{2}\|v - z\|_{Y_{t_0}}$. Hence, we see that $R(v)$ is a strict contraction in $Y_{t_0}$, and the proof is complete. $\boxed{QED}$

It follows from the contraction mapping principle that for appropriately chosen $t_0 \in (0,1)$, $R$ has a unique fixed point $u \in X_{t_0}$ which is a solution of (1)–(2). Making use of Lemma 1, one easily sees that this solution is nonnegative in $\overline{\Omega} \times [0, t_0]$.

Now, let us reveal that the solution $u$ of (1)–(2) is global when $\gamma = 1$. In order to prove this assertion, we need to show an a priori estimate. More precisely, we shall exhibit that

$$\|u(\cdot,t)\|_\infty \leq \|u_0\|_\infty \quad \text{for} \quad t > 0.$$

To demonstrate this estimate, we proceed in the following manner. Multiply both sides of (1) by $(u(x,t) - \|u_0\|_\infty)_+$ and integrate over $\Omega$ to obtain

$$\frac{d}{dt}\int_\Omega \frac{(u(x,t) - \|u_0\|_\infty)_+^2}{2} dx$$

$$= \int_\Omega \int_\Omega J(x-y)(u(y,t) - u(x,t))(u(x,t) - \|u_0\|_\infty)_+ dxdy$$

$$- \int_\Omega |u(x,t)|^{p-1}u(x,t)(u(x,t) - \|u_0\|_\infty)_+ dx,$$

where $(x)_+$ denotes $\max(x, 0)$. It is not hard to check that

$$(A - B)(A_+ - B_+) \geq (A_+ - B_+)^2.$$

On the other hand, according to the fact that the kernel $J$ is symmetric, we realize that

$$\int_\Omega \int_\Omega J(x-y)(\varphi(y) - \varphi(x))\psi(x)dxdy$$

$$= -\frac{1}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))(\psi(y)-\psi(x))dxdy.$$

Use the above relations to arrive at

$$\frac{d}{dt}\int_\Omega \frac{(u(x,t)-\|u_0\|_\infty)_+^2}{2}dx$$

$$\leq -\frac{1}{2}\int_\Omega\int_\Omega J(x-y)|(u(y,t)-\|u_0\|_\infty)_+ - (u(x,t)-\|u_0\|_\infty)_+|^2 dxdy$$

$$-\int_\Omega |u(x,t)|^{p-1}u(x,t)(u(x,t)-\|u_0\|_\infty)_+ dx \leq 0,$$

which implies that $\int_\Omega \frac{(u(x,t)-\|u_0\|_\infty)_+^2}{2}dx = 0$. We infer that

$$\|u(\cdot,t)\|_\infty \leq \|u_0\|_\infty \quad \text{for} \quad t>0,$$

and our estimate is proved. Now, let us show that, if $\gamma = 1$, then there exists a unique nonnegative global solution of (1)-(2). We know that for $t_0 > 0$ small enough, the problem (1)-(2) admits a unique bounded solution $u$ in $\overline{\Omega}\times[0,t_0]$. Taking as initial datum $u(\cdot,t_0) \in C(\overline{\Omega})$ and arguing as before, it is possible to extend the solution up to some interval $[0,t_1)$ for certain $t_1 > t_0$. Repeating this procedure, we easily prove the existence of a unique solution of (1)-(2) in $\overline{\Omega}\times[0,\infty)$. Making use of Lemma 1, one easily sees that the above solution is nonnegative in $\overline{\Omega}\times[0,\infty)$

# 4 Asymptotic behavior of solutions

In this section, we show that if $\gamma = 1$, then the solution $u$ of (1)-(2) tends to zero as $t$ approaches infinity uniformly in $x \in \overline{\Omega}$. We also give its asymptotic behavior as $t \to \infty$.

Introduce the function $\mu(x)$ defined by

$$\mu(x) = (C_0+x)^p - \lambda(C_0+x) \quad \text{for} \quad x \in [-C_0,\infty),$$

where $C_0 = (\frac{1}{p-1})^{\frac{1}{p-1}}$ and $\lambda = \frac{1}{p-1}$, which is crucial for the asymptotic behavior of solutions. We have $\mu(0) = 0$ and $\mu'(0) = 1$. We deduce that $\mu(x) > 0$ for any $x \geq 0$ and $\mu(x) < 0$ for any $x \in (-C_0, 0)$.

The lemma below shows that the solution $u$ of the problem (1)-(2) tends to zero as $t$ approaches infinity uniformly in $x \in \overline{\Omega}$ when $\gamma = 1$.

**Lemma 3.** *Let $u$ be the solution of (1)–(2). If $\gamma = 1$, then we have*

$$u(x,t) > 0 \quad \textit{in} \quad \overline{\Omega}\times[0,\infty) \quad \textit{and} \quad \lim_{t\to\infty} u(x,t) = 0,$$

*uniformly in $x \in \overline{\Omega}$.*

*Proof.* Introduce the function $\alpha(t)$ defined as follows

$$\alpha(t) = ((u_{0\min})^{1-p} + (p-1)t)^{1/(1-p)} \quad \text{for} \quad t \in [0,\infty),$$

where $u_{0\min} = \min_{x\in\overline{\Omega}} u_0(x) > 0$. It is not hard to see that $\alpha'(t) = -\alpha^p(t)$, $t > 0$, $\alpha(0) = u_{0\min}$. Setting $e(x,t) = u(x,t) - \alpha(t)$, an application of the mean value theorem leads us to

$$e_t(x,t) - \int_\Omega J(x-y)(e(y,t)-e(x,t))dy + \beta(x,t)e(x,t) = 0 \quad \text{in} \quad \overline{\Omega}\times(0,\infty),$$

$$e(x,0) \geq 0 \quad \text{in} \quad \overline{\Omega},$$

where $\beta(x,t) = p\int_0^1 (su(x,t)+(1-s)\alpha(t))^{p-1}ds$. It follows readily from Lemma 1 that $e(x,t) \geq 0$ in $\overline{\Omega}\times(0,\infty)$, that is $u(x,t) \geq \alpha(t) > 0$ in $\overline{\Omega}\times(0,\infty)$. Thus, the first part of the lemma is proved. In order to demonstrate the second one, we proceed as follows. Let $z(x,t) = C_0 t^{-\lambda}$ in $\overline{\Omega}\times[1,\infty)$, where $\lambda = \frac{1}{p-1}$ and $C_0 = (\frac{1}{p-1})^{\frac{1}{p-1}}$. A straightforward computation reveals that

$$z_t(x,t) - \int_\Omega J(x-y)(z(y,t)-z(x,t))dy + z^p(x,t) = 0 \quad \text{in} \quad \overline{\Omega}\times(1,\infty),$$

$$z(x,1) = C_0 \quad \text{in} \quad \overline{\Omega}.$$

Let $k > 1$ be so large that $kz(x,1) = kC_0 > u(x,1)$ in $\overline{\Omega}$. Obviously $kz^p(x,t) < (kz)^p(x,t)$, which implies that

$$(kz)_t(x,t) - \int_\Omega J(x-y)(kz(y,t)-kz(x,t))dy + (kz)^p(x,t) > 0 \quad \text{in} \quad \overline{\Omega}\times(1,\infty),$$

$$kz(x,1) > u(x,1) \quad \text{in} \quad \overline{\Omega}.$$

An application of Lemma 2 renders $0 \leq u(x,t) < kz(x,t)$ in $\overline{\Omega}\times(1,\infty)$, or equivalently $0 \leq u(x,t) < kC_0 t^{-\lambda}$ in $\overline{\Omega}\times(1,\infty)$. We deduce that $\lim_{t\to\infty} u(x,t) = 0$, uniformly in $x\in\overline{\Omega}$, and the proof is complete. QED

Now, let us give the asymptotic behavior of the solution $u$ when $\gamma = 1$. We have the following result.

**Theorem 2.** *Let $u$ be the solution of (1)-(2). If $\gamma = 1$, then we have*

$$u(x,t) = C_0 t^{-\lambda}(1+o(1)) \quad as \quad t\to\infty,$$

*uniformly in $x\in\overline{\Omega}$, where $C_0 = (\frac{1}{p-1})^{\frac{1}{p-1}}$ and $\lambda = \frac{1}{p-1}$.*

The proof of the above theorem is based on the following lemmas.

**Lemma 4.** *Let $u$ be the solution of (1)-(2). If $\gamma = 1$, then for any $\varepsilon > 0$ small enough, there exist two times $\tau \geq T \geq 1$ such that*

$$u(x,t+\tau) \leq (C_0+\varepsilon)(t+T)^{-\lambda} + (t+T)^{-\lambda-1} \quad in \quad \overline{\Omega}\times(0,\infty).$$

*Proof.* Introduce the function $w(x,t)$ defined as follows

$$w(x,t) = (C_0+\varepsilon)t^{-\lambda} + t^{-\lambda-1} \quad \text{in} \quad \overline{\Omega}\times[1,\infty).$$

A direct calculation yields

$$w_t(x,t) - \int_\Omega J(x-y)(w(y,t)-w(x,t))dy + w^p(x,t)$$

$$= t^{-\lambda-1}\left(-\lambda(C_0+\varepsilon) - (\lambda+1)t^{-1}\right) + t^{-\lambda p}\left(C_0+\varepsilon+t^{-1}\right)^p \quad \text{in} \quad \overline{\Omega}\times(1,\infty).$$

Due to the fact that $p\lambda = \lambda+1$, we arrive at

$$w_t(x,t) - \int_\Omega J(x-y)(w(y,t)-w(x,t))dy + w^p(x,t)$$

$$= t^{-\lambda-1}\left(-\lambda(C_0+\varepsilon)-(\lambda+1)t^{-1}+(C_0+\varepsilon+t^{-1})^p\right) \quad \text{in} \quad \overline{\Omega}\times(1,\infty).$$

Applying Taylor's expansion, we get $(C_0+\varepsilon+t^{-1})^p = (C_0+\varepsilon)^p + M(t)t^{-1}$ for $t \geq 1$, where $M(t)$ is a bounded function for $t \geq 1$. Hence, we find that

$$\begin{aligned} w_t(x,t) &- \int_\Omega J(x-y)(w(y,t)-w(x,t))dy + w^p(x,t) \\ &= t^{-\lambda-1}(\mu(\varepsilon)-(\lambda+1)t^{-1}+M(t)t^{-1}) \quad \text{in} \quad \overline{\Omega}\times(1,\infty). \end{aligned}$$

Having in mind that $\varepsilon > 0$ is small enough, we discover that $\mu(\varepsilon) > 0$. Therefore, there exists a time $T \geq 1$ such that

$$w_t(x,t) - \int_\Omega J(x-y)(w(y,t)-w(x,t))dy + w^p(x,t) > 0 \quad \text{in} \quad \overline{\Omega}\times(T,\infty).$$

Since $u$ goes to zero as $t$ approaches infinity uniformly in $x \in \overline{\Omega}$ owing to Lemma 3, there exists $\tau \geq T$ such that $u(x,\tau) < w(x,T)$ in $\overline{\Omega}$. Setting $z(x,t) = u(x,t+\tau-T)$, we easily see that

$$z_t(x,t) - \int_\Omega J(x-y)(z(y,t)-z(x,t))dy + z^p(x,t) = 0 \quad \text{in} \quad \overline{\Omega}\times(T,\infty),$$

$$z(x,T) = u(x,\tau) < w(x,T) \quad \text{in} \quad \overline{\Omega}.$$

Comparison Lemma 2 implies that $z(x,t) \leq w(x,t)$ in $\overline{\Omega}\times(T,\infty)$, or equivalently $u(x,t+\tau-T) \leq (C_0+\varepsilon)t^{-\lambda} + t^{-\lambda-1}$ in $\overline{\Omega}\times(T,\infty)$. We deduce that

$$u(x,t+\tau) \leq (C_0+\varepsilon)(t+T)^{-\lambda} + (t+T)^{-\lambda-1} \quad \text{in} \quad \overline{\Omega}\times(0,\infty),$$

and the proof is complete. QED

**Lemma 5.** *Let $u$ be the solution of (1)-(2). If $\gamma = 1$, then for any $\varepsilon > 0$ small enough, there exists a time $\tau \geq 1$ such that*

$$u(x,t+1) \geq (C_0-\varepsilon)(t+\tau)^{-\lambda} + (t+\tau)^{-\lambda-1} \quad \textit{in} \quad \overline{\Omega}\times(0,\infty).$$

*Proof.* Introduce the function $w(x,t)$ defined by

$$w(x,t) = (C_0-\varepsilon)t^{-\lambda} + t^{-\lambda-1} \quad \text{in} \quad \overline{\Omega}\times[1,\infty).$$

As in the proof of Lemma 4, we find that

$$\begin{aligned} w_t(x,t) &- \int_\Omega J(x-y)(w(y,t)-w(x,t))dy + w^p(x,t) \\ &= t^{-\lambda-1}(\mu(-\varepsilon)-(\lambda+1)t^{-1}+M(t)t^{-1}) \quad \text{in} \quad \overline{\Omega}\times(1,\infty), \end{aligned}$$

where $M(t)$ is a bounded function for $t \geq 1$. Since $\varepsilon > 0$ is small enough, we discover that $\mu(-\varepsilon) < 0$. Consequently, there exists a time $T \geq 1$ such that

$$w_t(x,t) - \int_\Omega J(x-y)(w(y,t)-w(x,t))dy + w^p(x,t) < 0 \quad \text{in} \quad \overline{\Omega}\times(T,\infty).$$

We know from Lemma 3 that $u(x,1) > 0$ in $\overline{\Omega}$. Since $\lim_{t\to\infty} w(x,t) = 0$, uniformly in $x \in \overline{\Omega}$, there exists a time $\tau \geq T$ such that $w(x,\tau) < u(x,1)$ in $\overline{\Omega}$. Setting $z(x,t) = w(x,t+\tau-1)$, it is not difficult to see that

$$z_t(x,t) - \int_\Omega J(x-y)(z(y,t)-z(x,t))dy + z^p(x,t) < 0 \quad \text{in} \quad \overline{\Omega}\times(1,\infty),$$

$$z(x,1) = w(x,\tau) < u(x,1) \quad \text{in} \quad \overline{\Omega}.$$

It follows from Comparison Lemma 2 that $z(x,t) < u(x,t)$ in $\overline{\Omega} \times (1,\infty)$, which implies that $u(x,t) \geq (C_0 - \varepsilon)(t+\tau-1)^{-\lambda} + (t+\tau-1)^{-\lambda-1}$ in $\overline{\Omega} \times (1,\infty)$. We deduce that

$$u(x,t+1) \geq (C_0 - \varepsilon)(t+\tau)^{-\lambda} + (t+\tau)^{-\lambda-1} \quad \text{in} \quad \overline{\Omega} \times (0,\infty),$$

and the proof is complete. $\boxed{QED}$

Now, we are in a position to prove the main result of this section.

*Proof of Theorem 2.* It follows from Lemmas 4 and 5 that

$$C_0 - \varepsilon \leq \lim_{t\to\infty} \inf(\frac{u(x,t)}{t^{-\lambda}}) \leq \lim_{t\to\infty} \sup(\frac{u(x,t)}{t^{-\lambda}}) \leq C_0 + \varepsilon,$$

which gives the desired result. $\boxed{QED}$

# 5 Blow-up solutions

It is well known (see, [27]) that the solution $u$ of (1)–(2) blows up in a finite time when $\gamma = -1$. In this section, we prove that, if $\gamma = -1$ and the $L^\infty$ norm of the initial datum is bigger than one, then the solution $u$ of (1)–(2) blows up in a finite time $T$, and its blow-up time goes to that of the solution of a certain ODE when the $L^\infty$ norm of the initial datum goes to infinity. These results are stated in the theorem below.

**Theorem 3.** *Let $u$ be the solution of (1)–(2), and assume that the initial datum satisfies $\|u_0\|_\infty > 1$. If $\gamma = -1$, then the solution $u$ blows up in a finite time $T$, and the following estimates hold*

$$0 \leq T - T_{u_0} \leq T_{u_0}\|u_0\|_\infty^{1-p} + o(T_{u_0}\|u_0\|_\infty^{1-p}) \quad \text{as} \quad \|u_0\|_\infty \to \infty, \tag{6}$$

*where $T_{u_0} = \frac{\|u_0\|_\infty^{1-p}}{p-1}$ is the blow-up time of the solution $\alpha(t)$ of the ODE defined below*

$$\alpha^{'}(t) = \alpha^p(t), \quad t > 0, \quad \alpha(0) = \|u_0\|_\infty.$$

*Proof.* Let $(0,T)$ be the maximal time interval of existence of the solution $u$. Our aim is to show that $T$ is finite and satisfies the above estimates. Due to the fact that the initial value is positive in $\overline{\Omega}$, it is clear that the solution $u$ is nonnegative in $\overline{\Omega} \times (0,T)$ because of Lemma 1. We note that $\int_\Omega J(x-y) \leq \int_{\mathbb{R}^N} J(x-y)dy = 1$, which implies that

$$u_t(x,t) \geq -u(x,t) + u^p(x,t) \quad \text{in} \quad \overline{\Omega} \times (0,T), \tag{7}$$

because $J(z) \geq 0$ for $z \in \mathbb{R}^N$. The estimate (8) can be rewritten as follows

$$u_t(x,t) \geq u^p(x,t)(1 - u^{1-p}(x,t)) \quad \text{in} \quad \overline{\Omega} \times (0,T). \tag{8}$$

Introduce the function $U(t)$ defined as follows

$$U(t) = \|u(\cdot,t)\|_\infty \quad \text{for} \quad t \in [0,T).$$

Let $t_1, t_2 \in [0,T)$. Then there exist $x_1, x_2 \in \overline{\Omega}$ such that $U(t_1) = u(x_1,t_1)$ and $U(t_2) = u(x_2,t_2)$. Making use of Taylor's expansion, it is easy to observe that

$$U(t_2) - U(t_1) \geq u(x_1,t_2) - u(x_1,t_1) = (t_2 - t_1)u_t(x_1,t_1) + o(t_2 - t_1),$$

and

$$U(t_2) - U(t_1) \le u(x_2, t_2) - u(x_2, t_1) = (t_2 - t_1)u_t(x_2, t_2) + o(t_2 - t_1),$$

which implies that $U(t)$ is Lipschitz continuous. Further, if $t_2 > t_1$, then

$$\frac{U(t_2) - U(t_1)}{t_2 - t_1} \ge u_t(x_1, t_1) + o(1) \ge u^p(x_1, t_1)(1 - u^{1-p}(x_1, t_1)) + o(1).$$

Letting $t_1 \to t_2$, we obtain

$$\frac{dU(t)}{dt} \ge U^p(t)(1 - U^{1-p}(t)) \quad \text{for} \quad \text{a.e.} \quad t \in (0, T). \tag{9}$$

We claim that $U(t) > \|u_0\|_\infty$ for $t \in (0, T)$. Indeed, using estimate (10), one gets

$$U(t) - \|u_0\|_\infty \ge \int_0^t U^p(s)(1 - U^{1-p}(s))ds, \tag{10}$$

for any $t \in [0, T]$. Since $U$ is continuous and $U^p(0)(1 - U^{1-p}(0)) > 0$, there exists $\delta > 0$, such that $U^p(s)(1 - U^{1-p}(s)) > 0$ for any $s \in (0, \delta)$. Hence, $U(t) > \|u_0\|_\infty$ for $t \in (0, \delta)$. Suppose by contradiction that $U(t) - \|u_0\|_\infty$ is not everywhere positive in $(0, T)$. Then, there would exist $t_0 \in (0, T)$ such that $U(t) > \|u_0\|_\infty$ for any $t \in (0, t_0)$ and $U(t_0) = \|u_0\|_\infty$. Since $\|u_0\|_\infty > 1$, by assumptions, $U^p(s)(1 - U^{1-p}(s)) > 0$ for any $s \in (0, t_0)$. Inequality (11) then would show that $U(t_0) - \|u_0\|_\infty > 0$: a contradiction. In view of the claim and (10), we discover that $U^{'}(t) \ge (1 - \|u_0\|_\infty^{1-p})U^p(t)$ for a.e. $t \in (0, T)$, or equivalently $\frac{dU}{U^p} \ge (1 - \|u_0\|_\infty^{1-p})dt$ for a.e. $t \in (0, T)$. Integrate the above estimate over $(0, T)$ to obtain

$$T \le \frac{\|u_0\|_\infty^{1-p}}{(p-1)(1 - \|u_0\|_\infty^{1-p})}. \tag{11}$$

We infer that the solution $u$ of (1)–(2) blows up in a finite time because the quantity on the right hand side of the above inequality is finite. Now, let us show that the estimates in (7) hold. Apply Taylor's expansion to obtain

$$\frac{1}{1 - \|u_0\|_\infty^{1-p}} = 1 + \|u_0\|_\infty^{1-p} + o(\|u_0\|_\infty^{1-p}) \quad \text{as} \quad \|u_0\|_\infty \to \infty.$$

Exploiting the above relation and (12), we find that

$$T - T_{u_0} \le T_{u_0}\|u_0\|_\infty^{1-p} + o(T_{u_0}\|u_0\|_\infty^{1-p}) \quad \text{as} \quad \|u_0\|_\infty \to \infty,$$

and the second estimate of the theorem is demonstrated. In order to prove the first one, we proceed in the following manner. Introduce the function $z(x, t)$ defined as follows

$$z(x, t) = \alpha(t) - u(x, t) \quad \text{in} \quad \overline{\Omega} \times [0, T_*),$$

where $T_* = \min\{T, T_{u_0}\}$. Invoking the mean value theorem, it is not hard to see that

$$z_t(x, t) = \int_\Omega J(x - y)(z(y, t) - z(x, t))dy + p\xi(x, t)z(x, t) \quad \text{in} \quad \overline{\Omega} \times (0, T_*),$$

$$z(x, 0) \ge 0 \quad \text{in} \quad \overline{\Omega},$$

where $\xi(x, t) = \int_0^1 (\sigma\alpha(t) + (1 - \sigma)u(x, t))^{p-1}d\sigma$. It follows from Lemma 1 that

$$z(x, t) = \alpha(t) - u(x, t) \ge 0 \quad \text{in} \quad \overline{\Omega} \times (0, T_*). \tag{12}$$

We claim that $T \ge T_{u_0}$. To prove this assertion, we argue by contradiction. Assume that $T < T_{u_0}$. In view of (13), we note that $\|u(\cdot, T)\|_\infty \le \alpha(T) < \infty$, which contradicts the fact that $(0, T)$ is the maximal time interval of existence of the solution $u$. Consequently, the claim is demonstrated, and the proof is complete. $\boxed{QED}$

**Remark 1.** Let us notice that the estimates in (7) can be rewritten as follows

$$0 \leq \frac{T}{T_{u_0}} - 1 \leq \|u_0\|_\infty^{1-p} + o(\|u_0\|_\infty^{1-p}) \quad \text{as} \quad \|u_0\|_\infty \to \infty.$$

We infer that $\lim_{\|u_0\|_\infty \to \infty} \frac{T}{T_{u_0}} = 1$.

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