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An interesting table related to Fibonacci numbers

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Abstract. In this paper, we show that sum of the row elements on the table formed by a given recurrence relation, and each element on this table can be obtained using two different Fibonacci numbers.

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1 Introduction

There are interesting recurrence relations in [2] and then, we take an infinite-dimensional matrix $A = (a_{i,j})$ formed by the following recurrence relation.

$$\begin{aligned} a_{1,j} &= F_j + (-1)^j F_1 \\ a_{2,j} &= F_{j+1} + (-1)^j F_2 \\ a_{i,j} + a_{i+1,j} &= a_{i+2,j}, \qquad i > 1 \end{aligned} \tag{1}$$

2 Results on the table

Using the recurrence relation (1), we have the following results.

1 Theorem.
$$a_{i,j} = F_{i+j-1} + (-1)^j F_i, \quad i \geq 1.$$

PROOF. By the recurrence relation (1), the result is true for $i = 1, 2$.
We now assume that it is true for all integer less than n, where $n > 2$:

$$a_{n-1,j} = F_{n+j-2} + (-1)^j F_{n-1}.$$

We then get

$$a_{n,j} = a_{n-2,j} + a_{n-1,j} = F_{n+j-3} + (-1)^j F_{n-2} + F_{n+j-2} + (-1)^j F_{n-1}$$
$$= F_{n+j-1} + (-1)^j F_n.$$

Thus, the result is true for all $n \geq 2$. QED

TABLE

i/j	1	2	3	4	5	6	7	8	9	10	...
1	0	2	1	4	4	9	12	22	33	56	
2	0	3	2	6	7	14	20	35	54	90	
3	0	5	3	10	11	23	32	57	87	146	
4	0	8	5	16	18	37	52	92	141	236	
5	0	13	8	26	29	60	84	149	228	382	
6	0	21	13	42	47	97	136	241	369	618	
7	0	34	21	68	76	157	220	390	597	1000	
8	0	55	34	110	123	254	356	631	966	1618	
9	0	89	55	178	199	411	576	1021	1563	2618	
10	0	144	89	288	322	665	932	1652	2529	4236	
$\vdots$											

2 Corollary.

$$a_{i,j} + a_{i,j+1} = F_{i+j+1}, \quad i, j \geq 1.$$

Proof. Since $a_{i,j} = F_{i+j-1} + (-1)^j F_i$, by Theorem 1,

$$a_{i,j} + a_{i,j+1} = F_{i+j-1} + (-1)^j F_i + F_{i+j} + (-1)^{j+1} F_i$$
$$= F_{i+j-1} + F_{i+j} = F_{i+j+1}.$$

QED

We now examine the sum of the first n elements on any row:

$$a_{i,1} = F_i + (-1)^1 F_i = F_i - F_i,$$
$$a_{i,2} = F_{i+1} + (-1)^2 F_i = F_{i+1} + F_i,$$
$$a_{i,3} = F_{i+2} + (-1)^3 F_i = F_{i+2} - F_i,$$
$$\vdots$$
$$a_{i,n} = F_{i+n-1} + (-1)^n F_i.$$

Here, it is important that n is even or odd in these equations. Therefore, we first assume that n is even. Adding these equations, we get

$$\sum_{j=1}^{n} a_{i,j} = F_{i+1} + F_{i+2} + \cdots + F_{i+n-1} + F_i$$
$$= \sum_{k=0}^{n-1} F_{i+k} = \sum_{r=1}^{i+n-1} F_r - \sum_{r=1}^{i-1} F_r = F_{i+n+1} - F_{i+1}.$$

Now let n be odd. Then, we get

$$\sum_{j=1}^{n} a_{i,j} = F_{i+1} + F_{i+2} + \cdots + F_{i+n-1}$$
$$= \sum_{k=1}^{n-1} F_{i+k} = \sum_{r=1}^{i+n-1} F_r - \sum_{r=1}^{i} F_r = F_{i+n+1} - F_{i+2}.$$

Thus,

3 Theorem.

$$\sum_{j=1}^{n} a_{i,j} = \begin{cases} F_{i+n+1} - F_{i+2} & \text{if } n \text{ is odd,} \\ F_{i+n+1} - F_{i+1} & \text{if } n \text{ is even.} \end{cases}$$

We now examine a determinant of a special matrix $A = (a_{i,j})_{2\times 2}$ taken from the above table. We now take the matrix

$$A = \begin{bmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{bmatrix}$$

and calculate the determinant of this matrix as follows.

4 Theorem.

$$\begin{vmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} = (-1)^{i+j+1}[F_{j+1} - 1], \quad i, j \geq 1.$$

Proof. Using the Theorem 1, we get

$$\begin{vmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} = F_{i+j-1}F_{i+j+1} + (-1)^{j+1}F_{i+j-1}F_{i+1} +$$
$$(-1)^j F_{i+j+1}F_i - F_{i+j}^2 - (-1)^j F_{i+j}F_{i+1} - (-1)^{j+1}F_{i+j}F_i$$
$$= F_{i+j-1}F_{i+j+1} - F_{i+j}^2 + (-1)^{j+1}(F_{i+j-1}F_{i+1} - F_{i+j}F_i) +$$
$$(-1)^j(F_{i+j+1}F_i - F_{i+j}F_{i+1}).$$

By the well-known identities about the Fibonacci numbers [2, p.74, 87]

$$F_{r-1}F_{r+1} - F_r^2 = (-1)^r, F_mF_n - F_{m+k}F_{n-k} = (-1)^r F_{m+k-n}F_k,$$

we obtain

$$\begin{aligned}\begin{vmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} &= (-1)^{j+i} + (-1)^{i+j+1}F_{j-1} + (-1)^{i+2j}F_{1-j} \\ &= (-1)^{j+i} + (-1)^{i+j+1}F_{j-1} + (-1)^{i+3j+1}F_j \\ &= (-1)^{i+j+1}[F_{j-1} + F_j - 1] = (-1)^{i+j+1}[F_{j+1} - 1].\end{aligned}$$

QED

We can generalize Theorem 4, as the next theorem shows.

5 Theorem.

$$\begin{vmatrix} a_{i,j} & a_{i,k} \\ a_{i+1,j+1} & a_{i+1,k} \end{vmatrix} = (-1)^{i+j+1}F_{k-j} + (-1)^{i+j+k}[F_{1-j} - F_{1-k}], \quad i, j \geq 1, k > j.$$

Proof. From the definition of determinant of a matrix, we obtain

$$\begin{aligned}\begin{vmatrix} a_{i,j} & a_{i,k} \\ a_{i+1,j+1} & a_{i+1,k} \end{vmatrix} &= a_{i,j}a_{i+1,k} - a_{i,k}a_{i+1,j+1} \\ &= (F_{i+j-1}+(-1)^jF_i)(F_{i+k}+(-1)^kF_{i+1})-(F_{i+k-1}+(-1)^kF_i)(F_{i+j}+(-1)^jF_{i+1}) \\ &= F_{i+j-1}F_{i+k} + (-1)^kF_{i+j-1}F_{i+1} + (-1)^jF_iF_{i+k} + (-1)^{j+k}F_iF_{i+1} \\ &\quad - F_{i+j}F_{i+k-1} - (-1)^kF_{i+j}F_i - (-1)^jF_{i+1}F_{i+k-1} - (-1)^{j+k}F_{i+1}F_i \\ &= F_{i+j-1}F_{i+k} + (-1)^k(F_{i+j-1}F_{i+1} - F_{i+j}F_i) + (-1)^j(F_iF_{i+k} - F_{i+1}F_{i+k-1})\end{aligned}$$

Since by [1]

$$F_{n+h}F_{n+k} - F_nF_{n+h+k} = (-1)^nF_hF_k,$$

we get

$$\begin{aligned}\begin{vmatrix} a_{i,j} & a_{i,k} \\ a_{i+1,j+1} & a_{i+1,k} \end{vmatrix} &= (-1)^{i+j}F_{k-j} + (-1)^k(-1)^{i+j}F_{1-j} + (-1)^j(-1)^{i+k}F_{1-k} \\ &= (-1)^{i+j}F_{k-j} + (-1)^{i+j+k}(F_{1-j} + F_{1-k}),\end{aligned}$$

as desired. QED

References

[1] Everman et al: *Problem E 1396*, The American Mathematical Monthly, **67**, (1960), 697.

[2] T. Koshy: Fibonacci and Lucas Numbers in Applications, A Wiley-Interscience Publication, New York, 2001.