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Some companions of Ostrowski type inequalities for twice differentiable functions

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Abstract. The main aim of this paper is to establish some companions of Ostrowski type integral inequalities for functions whose second derivatives are bounded. Moreover, some Ostrowski type inequalities are given for mappings whose first derivatives are of bounded variation. Some applications for special means and quadrature formulae are also given.

Keywords: Function of bounded variation, Ostrowski type inequalities, Riemann-Stieltjes integral.

MSC 2000 classification: primary 26D15, 26A45 secondary 26D10, 41A55

1 Introduction

In 1938, Ostrowski [27] established a following useful inequality:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\left\| f' \right\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible.

Inequality (1.1) is referred to, in the literature, as the Ostrowski inequality. Numerous studies were devoted to extensions and generalizations of this inequality in both the integral and discrete case. For some examples, please refer to ([10], [11], [17]-[26], [28]-[35])

Definition 1. Let $P : a = x_0 < x_1 < ... < x_n = b$ be any partition of $[a,b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$, then f is said to be of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta f(x_i)|$$

is bounded for all such partitions.

Definition 2. Let f be of bounded variation on $[a,b]$, and $\sum \Delta f(P)$ denotes the sum $\sum\limits_{i=1}^{n} |\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number

$$\bigvee_a^b (f) := \sup\left\{ \sum \Delta f(P) : P \in P([a,b]) \right\},$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [12], Dragomir proved the following Ostrowski type inequality for functions of bounded variation:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left| \int_a^b f(t)dt - (b-a) f(x) \right| \le \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f) \tag{1.2}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{2}$ is the best possible.

A great many of authors worked on Ostrowski type inequality for functions of bounded variation (or derivatives of bounded variation), for some of them please see ([1]-[9], [13]-[16])

The main purpose of this paper is to obtain some companions of Ostrowski type inequalities for function whose second derivatives are bounded. Moreover, some inequalities for derivatives of bounded variation and some applications are also given. This paper is divided into the following four sections. In Section 2, the first part of main result is presented. We establish an identity for twice differantiable functions and using this identity we obtain an Ostrowski type integral inequality for mappings whose second derivatives are bounded. In Section 3, some integral inequalities for function whose first derivatives are of bounded variation and some corollaries for special cases are given. In section 4, we give some applications for special means using the inequality obtained in Section 2. Finally, in Section 5, we presented an application for quadrature formula via Ostrowski type ineqaulity for derivatives of bounded variation given in Section 3.

2 Inequalities for Functions Whose Second Derivatives are Bounded

Before we start our main results, we state and prove the following lemma:

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a twice differantiable function on (a,b). Then we have the following identity*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \qquad (2.1)\\
&+\frac{(b-a)}{36}\left[f'(b)-f'(a)\right]\\
=\;& \frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}}\left(t-\frac{2a+b}{3}\right)\left(t-\frac{5a+b}{6}\right)f''(t)dt\right.\\
&\left.+\int_{\frac{a+b}{2}}^b\left(t-\frac{a+2b}{3}\right)\left(t-\frac{a+5b}{6}\right)f''(t)dt\right].
\end{aligned}$$

Proof. Using the integration by parts, we have

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}}\left(t-\frac{2a+b}{3}\right)\left(t-\frac{5a+b}{6}\right)f''(t)dt \qquad (2.2)\\
=\;&\left(t-\frac{2a+b}{3}\right)\left(t-\frac{5a+b}{6}\right)f'(t)\Big|_a^{\frac{a+b}{2}}-2\int_a^{\frac{a+b}{2}}\left(t-\frac{3a+b}{4}\right)f'(t)dt\\
=\;&\frac{(b-a)^2}{18}f'\left(\frac{a+b}{2}\right)-\frac{(b-a)^2}{18}f'(a)-2\left(t-\frac{3a+b}{4}\right)f(t)\Big|_a^{\frac{a+b}{2}}+2\int_a^{\frac{a+b}{2}}f(t)dt\\
=\;&\frac{(b-a)^2}{18}\left[f'\left(\frac{a+b}{2}\right)-f'(a)\right]-\frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right)+f(a)\right]+2\int_a^{\frac{a+b}{2}}f(t)dt
\end{aligned}$$

and

$$\int_{\frac{a+b}{2}}^b\left(t-\frac{a+2b}{3}\right)\left(t-\frac{a+5b}{6}\right)f''(t)dt \qquad (2.3)$$

$$= \left(t-\frac{a+2b}{3}\right)\left(t-\frac{a+5b}{6}\right)f'(t)\Big|_{\frac{a+b}{2}}^{b} - 2\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+3b}{4}\right)f'(t)dt$$

$$= \frac{(b-a)^2}{18}f'(b) - \frac{(b-a)^2}{18}f'\left(\frac{a+b}{2}\right) - 2\left(t-\frac{a+3b}{4}\right)f(t)\Big|_{\frac{a+b}{2}}^{b} + 2\int_{\frac{a+b}{2}}^{b} f(t)dt$$

$$= \frac{(b-a)^2}{18}\left[f'(b) - f'\left(\frac{a+b}{2}\right)\right] - \frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right)+f(b)\right] + 2\int_{\frac{a+b}{2}}^{b} f(t)dt.$$

If we add the equality (2.2) and (2.3) and divide by $2(b-a)$, we obtain required identity. $\boxed{QED}$

Now using the above Lemma, we state and prove the following inequality:

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a twice differantiable function on (a,b), whose second derivative $f'':(a,b)\to\mathbb{R}$ is bounded on (a,b). Then we have the inequality inequalities*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right. \\ &\left. + \frac{(b-a)}{36}\left[f'(b)-f'(a)\right]\right| \\ \leq\ & \frac{11}{6^4}(b-a)^2\left\|f''(t)\right\|_\infty . \end{aligned} \tag{2.4}$$

Proof. Taking the modulus identity (2.1), we have

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right. \\ &\left. + \frac{(b-a)}{36}\left[f'(b)-f'(a)\right]\right| \\ \leq\ & \frac{1}{2(b-a)}\left[\left|\int_a^{\frac{a+b}{2}}\left(t-\frac{2a+b}{3}\right)\left(t-\frac{5a+b}{6}\right)f''(t)dt\right|\right. \\ & \left. + \left|\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+2b}{3}\right)\left(t-\frac{a+5b}{6}\right)f''(t)dt\right|\right] \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
&\leq \frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}}\left|\left(t-\frac{2a+b}{3}\right)\left(t-\frac{5a+b}{6}\right)\right|\left|f''(t)\right|dt\right. \\
&\left.+\int_{\frac{a+b}{2}}^{b}\left|\left(t-\frac{a+2b}{3}\right)\left(t-\frac{a+5b}{6}\right)\right|\left|f''(t)\right|dt\right].
\end{aligned}
$$

Since f'' is bounded on (a,b) we have

$$
\int_a^{\frac{a+b}{2}}\left|\left(t-\frac{2a+b}{3}\right)\left(t-\frac{5a+b}{6}\right)\right|\left|f''(t)\right|dt \tag{2.6}
$$

$$
\begin{aligned}
&\leq \left\|f''(t)\right\|_{\left[a,\frac{a+b}{2}\right],\infty}\int_a^{\frac{a+b}{2}}\left|\left(t-\frac{2a+b}{3}\right)\left(t-\frac{5a+b}{6}\right)\right|dt \\
&= \frac{11}{6^4}(b-a)^3\left\|f''(t)\right\|_{\left[a,\frac{a+b}{2}\right],\infty}
\end{aligned}
$$

and

$$
\int_{\frac{a+b}{2}}^{b}\left|\left(t-\frac{a+2b}{3}\right)\left(t-\frac{a+5b}{6}\right)\right|\left|f''(t)\right|dt \tag{2.7}
$$

$$
\begin{aligned}
&\leq \left\|f''(t)\right\|_{\left[\frac{a+b}{2},b\right],\infty}\int_{\frac{a+b}{2}}^{b}\left|\left(t-\frac{a+2b}{3}\right)\left(t-\frac{a+5b}{6}\right)\right|dt \\
&= \frac{11}{6^4}(b-a)^3\left\|f''(t)\right\|_{\left[\frac{a+b}{2},b\right],\infty}.
\end{aligned}
$$

If we substitute the inequalities (2.6) and (2.7) in (2.5), then we get

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]+\frac{(b-a)}{36}\left[f'(b)-f'(a)\right]\right| \\
&\leq \frac{1}{2(b-a)}\left[\frac{11}{6^4}(b-a)^3\left\|f''(t)\right\|_{\left[a,\frac{a+b}{2}\right],\infty}+\frac{11}{6^4}(b-a)^3\left\|f''(t)\right\|_{\left[\frac{a+b}{2},b\right],\infty}\right] \\
&\leq \frac{11}{6^4}(b-a)^2\left\|f''(t)\right\|_{\infty}
\end{aligned}
$$

which completes the proof. $\boxed{QED}$

Corollary 1. *If we choose $f'(b) = f'(a)$, then the following Bullen type inequality holds*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \leq \frac{11}{6^4}(b-a)^2 \left\|f''(t)\right\|_\infty .$$

3 Inequalities for Functions Whose First Derivatives are of Bounded Variation

For functions whose first derivatives are of bounded variation, the following theorem holds:

Theorem 4. *Let : $f : [a,b] \to \mathbb{R}$ be a twice differantiable function on I° and $[a,b] \subset I^\circ$. If the first derivative f' is of bounded variation on $[a,b]$, then*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right. \\ &\left.+\frac{(b-a)}{36}\left[f'(b) - f'(a)\right]\right| \\ \leq\ &\frac{b-a}{36}\bigvee_a^b(f'). \end{aligned} \tag{3.1}$$

Proof. Using the integration by parts for Riemann-Stieltjes, we have the equality

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] \\ &+\frac{(b-a)}{36}\left[f'(b) - f'(a)\right] \\ =\ &\frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}}\left(t-\frac{2a+b}{3}\right)\left(t-\frac{5a+b}{6}\right)df'(t)\right. \\ &\left.+\int_{\frac{a+b}{2}}^b\left(t-\frac{a+2b}{3}\right)\left(t-\frac{a+5b}{6}\right)df'(t)\right]. \end{aligned} \tag{3.2}$$

Taking the madulus in (3.2), we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right. \qquad (3.3)\\
&\left.+\frac{(b-a)}{36}\left[f'(b)-f'(a)\right]\right|\\
=\;&\frac{1}{2(b-a)}\left[\left|\int_a^{\frac{a+b}{2}}\left(t-\frac{2a+b}{3}\right)\left(t-\frac{5a+b}{6}\right)df'(t)\right|\right.\\
&\left.+\left|\int_{\frac{a+b}{2}}^b\left(t-\frac{a+2b}{3}\right)\left(t-\frac{a+5b}{6}\right)df'(t)\right|\right].
\end{aligned}$$

It is well known that if $g,f:[a,b]\to\mathbb{R}$ are such that g is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, then $\int_a^b g(t)df(t)$ exists and

$$\left|\int_a^b g(t)df(t)\right|\leq\sup_{t\in[a,b]}|g(t)|\bigvee_a^b(f). \tag{3.4}$$

Since f' is of bounded variation on $[a,b]$, appliying the inequality (3.4), we get

$$\begin{aligned}
&\left|\int_a^{\frac{a+b}{2}}\left(t-\frac{2a+b}{3}\right)\left(t-\frac{5a+b}{6}\right)df'(t)\right| \qquad (3.5)\\
\leq\;&\sup_{t\in\left[a,\frac{a+b}{2}\right]}\left|\left(t-\frac{2a+b}{3}\right)\left(t-\frac{5a+b}{6}\right)\right|\bigvee_a^{\frac{a+b}{2}}(f')\\
=\;&\frac{(b-a)^2}{18}\bigvee_a^{\frac{a+b}{2}}(f'),
\end{aligned}$$

and similarly

$$\left|\int_{\frac{a+b}{2}}^b\left(t-\frac{a+2b}{3}\right)\left(t-\frac{a+5b}{6}\right)df'(t)\right| \tag{3.6}$$

$$\leq \sup_{t\in\left[\frac{a+b}{2},b\right]} \left| \left(t-\frac{a+2b}{3}\right)\left(t-\frac{a+5b}{6}\right)\right| \bigvee_{\frac{a+b}{2}}^{b}(f')$$

$$\leq \frac{(b-a)^2}{18} \bigvee_{\frac{a+b}{2}}^{b}(f').$$

If we substitute the inequalities (3.5) and (3.6) in (3.3), then we obtain required result. $\boxed{QED}$

Under assumption of of Theorem 3, we have the following corollaries:

Corollary 2. *Let* $f \in C^2[a,b]$. *Then we have the inequality*

$$\left| \frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] + \frac{(b-a)}{36}\left[f'(b)-f'(a)\right]\right| \leq \frac{b-a}{36}\left\|f''\right\|_{[a,b],1} \tag{3.7}$$

where $\|.\|_{[a,b],1}$ *is the* L_1-*norm, namely*

$$\left\|f''\right\|_{[a,b],1} = \int_a^b \left|f''(t)\right| dt.$$

Corollary 3. *Let* $f' : [a,b] \to \mathbb{R}$ *be a Lipschitzian mapping with the constants* $L>0$. *Then, we have the inequality*

$$\left| \frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] + \frac{(b-a)}{36}\left[f'(b)-f'(a)\right]\right| \leq \frac{L(b-a)^3}{36}. \tag{3.8}$$

Proof. As f' is L-Lipschitzian on $[a,b]$, it is also of bounded variation. If $P([a,b])$

denotes the family of divisions on $[a,b]$, then

$$\begin{aligned}\bigvee_a^b(f') &= \sup_{P\in P([a,b])}\sum_{i=0}^{n-1}\left|f'(x_{i+1})-f'(x_i)\right| \\ &\leq L\sup_{P\in P([a,b])}\sum_{i=0}^{n-1}|x_{i+1}-x_i| \\ &= L\,(b-a)\end{aligned}$$

and the required result (3.8) is proved. $\boxed{QED}$

4 Some applications for special means

Let us recall the following special means of the two positive number u, v:
(1) Arithmetic mean,

$$A(u,v)=\frac{u+v}{2},\ u,v\in\mathbb{R}$$

(2) Geometric mean,

$$G(u,v)=\sqrt{u.v},\ u,v>0$$

(3) Harmonic mean

$$H(u,v)=\frac{2uv}{u+v},$$

(4) Logarithmic mean,

$$L(u,v)=\begin{cases} u & \text{if } u=v \\ \frac{u-v}{\ln u-\ln v} & \text{if } u\neq v\end{cases},\ u,v>0$$

(5) Generalized $log-$mean

$$L_p(u,v)=\begin{cases} u & \text{if } u=v \\ \left[\frac{u^{p+1}-v^{p+1}}{(p+1)(u-v)}\right]^{\frac{1}{p}} & \text{if } u\neq v\end{cases},\ u,v>0,\ p\neq -1,0$$

(6) Identric mean

$$I(u,v)=\begin{cases} u & \text{if } u=v \\ \frac{1}{e}\left(\frac{v^v}{u^u}\right)^{\frac{1}{v-u}} & \text{if } u\neq v\end{cases},\ u,v>0$$

Proposition 1. *Let $a,b\in\mathbb{R}$, $a<b$, and $n\in\mathbb{Z}\backslash\{-1,0\}$. Then the following inequality holds:*

$$\begin{aligned}&\left|L_n^n(a,b)-\frac{A^n(a,b)+A(a^n,b^n)}{2}+\frac{n(n-1)(b-a)^2}{36}L_{n-1}^{n-1}(a,b)\right|\\ \leq\;&\frac{11}{6^4}(b-a)^2\delta_n(a,b).\end{aligned}$$

Proof. Let us reconsider the inequality (2.4):

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right.\\ &\left.+\frac{(b-a)}{36}\left[f'(b)-f'(a)\right]\right|\\ \leq\;&\frac{11}{6^4}(b-a)^2\left\|f''(t)\right\|_\infty.\end{aligned}\tag{4.1}$$

Consider the mapping $f:(0,\infty)\to\mathbb{R}$, $f(x)=x^n$, $n\in\mathbb{Z}\backslash\{-1,0\}$. Then, $0<a<b$, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)dt&=L_n^n(a,b),\\ f\left(\frac{a+b}{2}\right)&=A^n(a,b),\ \frac{f(a)+f(b)}{2}=A(a^n,b^n)\\ f'(b)-f'(a)&=n\left[b^{n-1}-a^{n-1}\right]=n(n-1)(b-a)L_{n-1}^{n-1}(a,b)\end{aligned}$$

and

$$\left\|f''(t)\right\|_\infty=\begin{cases}|n(n-1)|\,b^{n-2}, & n>2\\ |n(n-1)|\,b^{n-2}, & n\in(-\infty,2)\setminus\{-1,0\}\end{cases}$$

Then, we obtain

$$\begin{aligned}&\left|L_n^n(a,b)-\frac{A^n(a,b)+A(a^n,b^n)}{2}+\frac{n(n-1)(b-a)^2}{36}L_{n-1}^{n-1}(a,b)\right|\\ \leq\;&\frac{11}{6^4}(b-a)^2\delta_n(a,b)\end{aligned}$$

where

$$\delta_n(a,b)=\begin{cases}|n(n-1)|\,b^{n-2}, & n>2\\ |n(n-1)|\,b^{n-2}, & n\in(-\infty,2)\setminus\{-1,0\}.\end{cases}$$

This completes the proof. $\boxed{QED}$

Proposition 2. *Let $a, b \in \mathbb{R}$, $a < b$. Then the following inequality holds:*

$$\left| L^{-1}(a,b) - \frac{A^{-1}(a,b) + H^{-1}(a,b)}{2} + \frac{(b-a)^2}{18ab} H^{-1}(a,b) \right| \leq \frac{22}{6^4} \cdot \frac{(b-a)^2}{a^3}.$$

Proof. The proof is obvious from Theorem 3 applied to the function $f(x) = \frac{1}{x}$. $\boxed{QED}$

Proposition 3. *Let $a, b \in \mathbb{R}$, $a < b$. Then the following inequality holds:*

$$\left| \ln \left[\frac{I(a,b)}{\sqrt{A(a,b)G(a,b)}} \right] + \frac{b-a}{18} H^{-1}(a,b) \right| \leq \frac{11}{6^4} \cdot \frac{(b-a)^2}{a^2}.$$

Proof. Consider the mapping $f : (0, \infty) \to \mathbb{R}$, $f(x) = lnx$, and $0 < a < b$.

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(t)dt &= \ln I(a,b), \; f\left(\frac{a+b}{2}\right) = \ln A(a,b), \; \frac{f(a) + f(b)}{2} = \ln G(a,b) \\ f'(b) - f'(a) &= 2H^{-1}(a,b), \text{ and } \left\| f''(t) \right\|_\infty = \frac{1}{a^2} \end{aligned}$$

and then ,by (4.1), we obtain the desired inequality. $\boxed{QED}$

5 Application to quadrature formula

Our obtained inequalities for function of bounded variation have many applications but in this paper, we apply our result only for efficient quadrature rule.

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$ with $h_i := x_{i+1} - x_i$ and $v(h) := \max\{h_i |\ i = 0, ..., n-1\}$. Then the following Theorem holds:

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be such that f' is a continuous function of bounded variation on $[a,b]$. Then we have the quadrature formula:*

$$\begin{aligned} & \int_a^b f(t)dt \\ = & \sum_{i=0}^{n-1} \left[\frac{1}{2} \left[f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{f(x_i) + f(x_{i+1})}{2} \right] + \frac{h_i}{36} \left[f'(x_{i+1}) - f'(x_i) \right] \right] h_i \\ & + R(I_n, f). \end{aligned}$$

The remainder term $R(I_n, f)$ satisfies

$$|R(I_n, f)| \leq \frac{1}{36}(v(h))^2 \bigvee_a^b (f')$$

Proof. Applying Theorem 3 to interval $[x_i, x_{i+1}]$, we have

$$\left| \int_{x_i}^{x_{i+1}} f(t)dt - \left[\frac{1}{2} \left[f\left(\frac{x_i + x_{i+1}}{2} \right) + \frac{f(x_i) + f(x_{i+1})}{2} \right] \right. \right. \tag{5.1}$$

$$\left. \left. + \frac{h_i}{36} \left[f'(x_{i+1}) - f'(x_i) \right] \right] h_i \right|$$

$$\leq \frac{h_i^2}{36} \bigvee_{x_i}^{x_{i+1}} (f').$$

Summing the inequality (5.1) over i from 0 to $n-1$, then we have

$$|R(I_n, f, \xi)| \leq \frac{1}{36} \sum_{i=0}^{n-1} \frac{h_i^2}{36} \bigvee_{x_i}^{x_{i+1}} (f') \leq \frac{1}{36}(v(h))^2 \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f') \leq \frac{1}{36}(v(h))^2 \bigvee_a^b (f').$$

This completes the proof of the Theorem. $\boxed{QED}$

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