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Space-like hypersurfaces with vanishing conformal forms in the conformal space

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Abstract. We study the space-like hypersurfaces with vanishing conformal form in the conformal geometry, and classify the the Einstein space-like hypersurfaces in the conformal space.

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MSC 2000 classification: 53A30; 53C21; 53C40

Introduction

We define the pseudo-Euclidean inner product $\langle \cdot, \cdot \rangle_s$ in R^{n+p} as

$$\langle X, Y\rangle_s = -\sum_{i=1}^{s} x_i y_i + \sum_{i=s+1}^{n+p} x_i y_i, X = (x_i), Y = (y_i) \in R^{n+p},$$

Let RP^{n+2} be a real projective space. $< \cdot, \cdot >_2$ is the pseudo-Euclidean inner product in R^{n+3}. The quadratic surface

$$Q_1^{n+1} = \{[\xi] \in RP^{n+2} | \langle \xi, \xi \rangle_2 = 0\}$$

in RP^{n+2} is called the conformal space.

Suppose that $x : M^n \to Q_1^{n+1}$ is a space-like hypersurface in the conformal space Q_1^{n+1}, $\{e_i\}$ is a local orthonormal frame of M^n for the standard metric $I = dx \cdot dx$ with dual basis $\{\theta_i\}$. Then we define the first fundamental form I, the second fundamental form II and the mean curvature of x as

$$I = \langle dx, dx\rangle_1 = \sum_i \theta_i \otimes \theta_i; \;\; II = \sum_{ij} h_{ij} \theta_i \otimes \theta_j; \;\; H = \frac{1}{n} \sum_i h_{ii}.$$

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In [5], Nie and Wu classified the hypersurfaces with parallel conformal second fundamental form.They obtained

Theorem 1. [5] *Let $x : M^n \to Q_1^{n+1}$ be a space-like hypersurface with parallel conformal second fundamental form, then M^n is conformally equivalent to an open part of one of the following hypersurfaces in Q_1^{n+1}:*

1) $S^k(a) \times H^{n-k}(b) \subset S_1^{n+1}$

2) $H^k(a) \times H^{n-k}(b) \subset H_1^{n+1}$;

3) $H^k(a) \times R^{n-k} \subset R_1^{n+1}$;

4) $WP(p,q,a) \subset R_1^{n+1}$, $WP(p,q,a)$ is the warped product embedding u : $S^p(a) \times H^q(b) \times R^+ \times R^{n-p-q-1} \subset R_1^{n+2} \to R_1^{n+1}, a > 1, b = \sqrt{a^2-1}$, which is given by

$$u = (tu_1, tu_2, tu_3), u_1 \in S^p(a), u_2 \in H^q(b), u_3 \in R^{n-p-q-1}, t \in R^+.$$

In this paper, we consider the space-like hypersurfaces M^n with conformal form $C = 0$, which also have harmonic curvature. Here is our main theorem

Theorem 2. *Let $x : M^n \to Q_1^{n+1}$ be a space-like hypersurface in Q_1^{n+1} without umbilics. If its conformal form $C = 0$ and its curvature tensor is harmonic, then its conformal second fundamental form is parallel.*

Every manifold with parallel Ricci tensor has harmonic curvature. This applies, for instance, to Einstein manifolds. Consequently, we have the following corollary

Corollary 1. *Let $x : M^n \to Q_1^{n+1}$ be a space-like hypersurface in Q_1^{n+1} without umbilics. If its conformal form $C = 0$ and M^n is Einstein hypersurface with respect to conformal metric g, then M^n is conformally equivalent to an open part of one of the following hypersurfaces in Q_1^{n+1}:*

1) $H^k(a) \times H^{n-k}(b) \subset H_1^{n+1}$;

2) $H^1(a) \times R^{n-1} \subset R_1^{n+1}$.

1 Conformal invariants for space-like hypersurfaces in Q_1^{n+1}

Let $x : M^n \to Q_1^{n+1}$ be a space-like hypersurface in the conformal space Q_1^{n+1}. The cone of light in R^{n+3} is given by

$$C^{n+2} = \{\xi \in R^{n+3} | \langle \xi, \xi \rangle_2 = 0, \xi \neq 0\}.$$

Then there exists a unique lift $Y : M^n \to C^{n+2}$ of x such that $g = \langle dY, dY \rangle$ up to a sign, Y is called the canonical lift of x. Then we have

Theorem 3. [6] *Two space-like hypersurfaces* $x, \tilde{x} : M^n \to Q_1^{n+1}$ *are conformally equivalent if and only if there exists a pseudo-orthogonal transformation* $T \in O(n, 2)$ *in* R_2^{n+3} *such that* $Y = \tilde{Y}T$.

It follows immediately from Theorem 3 that $g = \langle dY, dY\rangle = e^{2\tau} dx \cdot dx$, $e^{2\tau} = \dfrac{n}{n-1}(\sum_{ij}(h_{ij})^2 - nH^2)$ is a conformal invariant, which is called the conformal metric of $x : M^n \to Q_1^{n+1}$.

Let Δ be the Laplacian operator with respect to g. We define

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle \Delta Y, \Delta Y\rangle Y. \tag{1}$$

It is easy to see that

$$\langle \Delta Y, Y\rangle_2 = -n, \langle Y, dY\rangle_2 = 0, \tag{2}$$

$$\langle Y, Y\rangle_2 = \langle N, N\rangle_2 = 0, \langle Y, N\rangle_2 = 1. \tag{3}$$

Let $\{E_i := e^{-\tau} e_i\}$ be a local orthonormal basis for the conformal metric g with dual basis $\{\omega_i = e^{\tau}\theta_i\}$. Writing $\{Y_i = E_i(Y)\}$, we have

$$\langle Y_i, Y_j\rangle_2 = \delta_{ij}, \langle Y_i, Y\rangle_2 = \langle Y_i, N\rangle_2 = 0, \quad 1 \le i, j \le n. \tag{4}$$

If we denote by V is the orthogonal complement space of the subspace $span\{Y, N, Y_1, \cdots, Y_n\}$ in R_2^{n+3}, then we have

$$R_2^{n+3} = span\{Y, N\} \oplus span\{Y_1, \cdots, Y_n\} \oplus V,$$

here V is called the conformal normal bundle of $x : M^n \to S^{n+1}$. We define the local orthonormal basis of V by

$$E = E_{n+1} := (H, Hx + e_{n+1}), \tag{5}$$

then $\{Y, N, Y_1, \cdots, Y_n, E\}$ is a moving frame of R_2^{n+3} along M^n, we can write the structure equations as

$$dY = \sum_i Y_i \omega_i, \tag{6}$$

$$dN = \sum_i \psi_i Y_i + CE, \tag{7}$$

$$dY_i = -\psi_i Y - \omega_i N + \sum_j \omega_{ij} Y_j + \omega_{i,n+1} E, \tag{8}$$

$$dE = CY + \sum_i \omega_{i,n+1} Y_i. \tag{9}$$

The tensors $A = \sum_{i,j} A_{ij}\omega_i\omega_j$, $B = \sum_{i,j} B_{ij}\omega_i\omega_j$, $C = \sum_i C_i\omega_i$ are called the Blaschke tensor, the conformal second fundamental form and the conformal form respectively. All of them are conformal invariants. The relations between conformal invariants and Euclidean invariants of x are given by

$$B_{ij} = e^{-\tau}(h_{ij} - H\delta_{ij}), \tag{10}$$

$$C_i = e^{-2\tau}(H\tau_i - \sum_j h_{ij}\tau_j - H_i), \tag{11}$$

$$A_{ij} = e^{-2\tau}[\tau_i\tau_j - \tau_{i,j} - Hh_{ij} + \frac{1}{2}(H^2 - \sum_k(\tau_k)^2 + \epsilon)\delta_{ij}]. \tag{12}$$

Here $\tau_i = e_i(\tau), H_i = e_i(H)$. $\tau_{i,j}$ and ∇ are called the Hessian-matrix and the gradient with respect to $I = dx \cdot dx$ respectively.

We define the covariant derivatives of C_i, A_{ij}, B_{ij} as follows

$$\sum_j C_{i,j}\omega_j = dC_i - \sum_j C_j\omega_{ji}, \tag{13}$$

$$\sum_k A_{ij,k}\omega_k = dA_{ij} - \sum_k A_{ik}\omega_{kj} - \sum_k A_{kj}\omega_{ki}, \tag{14}$$

$$\sum_k B_{ij,k}\omega_k = dB_{ij} - \sum_k B_{ik}\omega_{kj} - \sum_k B_{kj}\omega_{ki}. \tag{15}$$

Then the structure equations (6)–(9) are equivalent to

$$A_{ij,k} - A_{ik,j} = B_{ik}C_i - B_{ij}C_k, \tag{16}$$

$$C_{i,j} - C_{j,i} = \sum_k(B_{ik}A_{kj} - B_{kj}A_{ki}), \tag{17}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{18}$$

$$R_{ijkl} = \delta_{ik}A_{jl} - \delta_{il}A_{jk} + \delta_{jl}A_{ik} - \delta_{jk}A_{il} - (B_{ik}B_{jl} - B_{il}B_{jk}), \tag{19}$$

$$R_{ij} := \sum_k R_{ikjk} = \sum_k B_{ik}B_{jk} + (trA)\delta_{ij} + (n-2)A_{ij}, \tag{20}$$

$$\sum_i B_{ii} = 0, \ \sum_{i,j}(B_{ij})^2 = \frac{n-1}{n}, \ trA = \sum_i A_{ii} = \frac{1}{2n}(n^2\kappa - 1). \tag{21}$$

Here R_{ijkl} is the curvature tensor of g, $Q = \sum_{i,j} R_{ij}\omega_i \otimes \omega_j$ is the conformal Ricci curvature and $\kappa = \frac{1}{n(n-1)}\sum_i R_{ii}$ is the normalized conformal scalar curvature of $x : M^n \to S^{n+1}$.

Theorem 4. [6] *Two space-like hypersurfaces $x : M^n \to Q_1^{n+1}$ and $\tilde{x} : \tilde{M}^n \to Q_1^{n+1} (n \geq 3)$ are conformally equivalent if and only if there exists a diffeomorphism $\sigma : M^n \to \tilde{M}^n$ which preserves the conformal metric g and conformal second fundamental form B.*

2 The proof of main results

Proof of Theorem 2. Let $x : M^n \to Q_1^{n+1}$ be a space-like hypersurface in Q_1^{n+1} with conformal form $C = 0$. From (17), we choose a local orthonormal frame $\{e_i\}$ with respect to g such that A, B are diagonalizable at the same time, i.e.,

$$B_{ij} = b_i \delta_{ij}, \;\; A_{ij} = a_i \delta_{ij}, \;\; 1 \le i, j \le n. \tag{22}$$

We are assuming that x has harmonic conformal curvature, i.e., $\sum_i R_{ijkl,i} = 0$. This happens if and only if the Ricci tensor is Codazzi tensor, i.e., $R_{ij,k} = R_{ik,j}$. Thus the scalar curvature of x with respect to g is constant and $tr(A)$ is constant too. From (20), we have

$$R_{ij,k} = \sum_l B_{il,k} B_{lj} + \sum_l B_{il} B_{lj,k} + (n-2) A_{ij,k}, \tag{23}$$

$$R_{ik,j} = \sum_l B_{il,j} B_{lk} + \sum_l B_{il} B_{lk,j} + (n-2) A_{ik,j}, \tag{24}$$

Since $C = 0$, form(17), (18), we have $B_{ij,k} = B_{ik,j}, A_{ij,k} = A_{ik,j}$. Thus from (22) and (23), we get

$$\sum_l B_{il,k} B_{lj} = \sum_l B_{il,j} B_{lk}.$$

By using (22), for any indices of i, j, k, we get

$$B_{ij,k} b_j = B_{ik,j} b_k, \tag{25}$$

If $b_j \neq b_k$, then we have

$$B_{ij,k} = 0. \tag{26}$$

If $b_j = b_k$, since

$$\begin{aligned} \sum_l B_{jk,l}\omega_l &= dB_{jk} + \sum_l B_{jl}\omega_{lk} + \sum_l B_{lj}\omega_{li} \\ &= dB_{jk} + (b_j - b_k)\omega_{jk} \end{aligned} \tag{27}$$

It is easy to see from (22) that

$$B_{ij,k} = 0. \tag{28}$$

From (26), (28) and $\sum_j B_{ij,j} = 0$, for any indices of i, j, k, we get

$$B_{ij,k} = 0. \tag{29}$$

Therefore we obtain our main Theorem 2.

Now let t be the number of the distinct eigenvalues of A, and $a_1, a_2, \cdots, a_t$ be all of distinct eigenvalues. Taking a suitably local orthonormal frame field $\{E_1, E_2, \cdots, E_n\}$ such that the matrix (A_{ij}) can be written as

$$(A_{ij}) = Diag(\underbrace{a_1, \cdots, a_1}_{k_1}, \underbrace{a_2, \cdots, a_2}_{k_2}, \cdots, \underbrace{a_t, \cdots, a_t}_{k_t}),$$

that is

$$A_1 = \cdots = A_{k_1} = a_1, \cdots, A_{n-k_t+1} = \cdots = A_n = a_t,$$

here $a_1, \cdots, a_t$ are not necessarily different from each other.

Similarly, under the same orthonormal frame field, the matrix (B_{ij}) can be written as

$$(B_{ij}) = Diag(\underbrace{b_1, \cdots, b_1}_{k_1}, \underbrace{b_2, \cdots, b_2}_{k_2}, \cdots, \underbrace{b_t, \cdots, b_t}_{k_t}),$$

or equivalently

$$B_1 = \cdots = B_{k_1} = b_1, \cdots, B_{n-k_t+1} = \cdots = B_n = b_t,$$

and $b_1, \cdots, b_t$ are not necessarily different from each other.

Proposition 1. *If the number of the distinct eigenvalues is $t \geq 3$, then $t = 3$.*

Proof. If $t > 3$, then there exist at least four indices i_1, i_2, i_3, i_4, such that $A_{i_1}, A_{i_2}, A_{i_3}, A_{i_4}$ are distinct from each other.

Making the convention on the ranges of indices as follows

$$1 \leq i_1, j_1 \leq k_1, \quad k_1 + 1 \leq i_2, j_2 \leq k_1 + k_2, \cdots, k_1 + k_2 + 1 \leq i_t, j_t \leq n.$$

From (15) and $B_{ij,k} = 0$, we have $\omega_{i_m i_n} = 0 (m \neq n, 1 \leq m, n \leq t)$. Using Gauss equation and from (19), we obtain

$$B_{i_1} B_{i_2} + A_{i_1} + A_{i_2} = 0, \quad B_{i_3} B_{i_4} + A_{i_3} + A_{i_4} = 0,$$

$$B_{i_1} B_{i_3} + A_{i_1} + A_{i_3} = 0, \quad B_{i_2} B_{i_4} + A_{i_2} + A_{i_4} = 0.$$

Consequently, $(A_{i_1} - A_{i_4})(A_{i_2} - A_{i_3}) = 0$, it contradicts with the assumption that $A_{i_1}, A_{i_2}, A_{i_3}, A_{i_4}$ are distinct from each other.

Proposition 2. *The Einstein space-like hypersurfaces with vanishing conformal form in conformal space have at most two different conformal principal curvatures.*

Proof. If that doesn't happen, $t=3$. Taking a local orthonormal frame field $\{E_1, E_2, \cdots, E_n\}$ such that

$$(B_{ij}) = Diag(\underbrace{b_1, \cdots, b_1}_{k_1}, \underbrace{b_2, \cdots, b_2}_{k_2}, \underbrace{b_3, \cdots, b_3}_{k_3})$$

with the multiplicity are k_1, k_2, k_3 respectively, and $k_1+k_2+k_3=n$.

$$(A_{ij}) = Diag(\underbrace{a_1, \cdots, a_1}_{k_1}, \underbrace{a_2, \cdots, a_2}_{k_2}, \underbrace{a_3, \cdots, a_3}_{k_3}).$$

Since B is parallel, By Using Gauss equation and from (15), (19), we obtain

$$\begin{cases} b_1b_2 + a_1 + a_2 = 0, \\ b_1b_3 + a_1 + a_3 = 0, \\ b_2b_3 + a_2 + a_3 = 0. \end{cases} \tag{30}$$

Obviously, we have

$$b_3(b_1 - b_2) = -(a_1 - a_2). \tag{31}$$

Since M is Einstein manifold, i.e., the Ricci curvature $R_{ij} = \frac{r}{n}\delta_{ij} = (n-1)\kappa\delta_{ij} (n \geq 3)$, so its conformal scalar curvature κ is constant. From (20) we have

$$\begin{cases} (n-1)\kappa = tr(A) + (n-2)a_1 + b_1^2, \\ (n-1)\kappa = tr(A) + (n-2)a_2 + b_2^2, \\ (n-1)\kappa = tr(A) + (n-2)a_3 + b_3^2. \end{cases} \tag{32}$$

Subtracting the second formula from the the first formula, we get

$$(b_1 + b_2)(b_1 - b_2) = -(n-2)(a_1 - a_2). \tag{33}$$

Substituting (31) into (33), we obtain

$$b_1 + b_2 = -(n-2)b_3.$$

Similarly, we have

$$b_1 + b_3 = -(n-2)b_2.$$

Making subtraction in above two formulas and obtain $b_2 = b_3$, it contradicts with the assumption, so we complete the proof of Proposition 2.

Next, we give the proof of Corollary 1.

Proof of Corollary 1. Suppose that the number of different principle curvatures of the Einstein space-like hypersurfaces is $t=2$, by using the first two formulas of (21) to calculate b_1, b_2, we obtain

$$b_1 = \frac{1}{n}\sqrt{\frac{(n-k)(n-1)}{k}}, \quad b_2 = -\frac{1}{n}\sqrt{\frac{(n-1)k}{n-k}}.$$

$$a_1 + a_2 = b_1 \cdot b_2 = -\frac{n-1}{n^2} < 0.$$

Since$(M, g) = (M_1, g_1) \times (M_2, g_2)$, $dimM_1 = k$, $dimM_2 = n-k$. From (19), (M_1, g_1) and (M_2, g_2) have constant curvature R_1 and R_2. By direct calculation, we get

$$R_1 = 2a_1 - b_1^2, R_2 = 2a_2 - b_2^2.$$

$$R_1 + R_2 = -b_1^2 - b_2^2 + 2(a_1 + a_2) = -(b_1 - b_2)^2 < 0.$$

Then at least one of R_1, R_2 is negative, without generality, we let $R_1 < 0$, i.e., $2a_1 - b_1^2 < 0$.

Since M is Einstein manifold, from (20), we have

$$(n-1)\kappa\delta_{ij} = R_{ij} = tr(A)\delta_{ij} + (n-2)A_{ij} + \sum_k B_{ik}B_{jk}.$$

Furthermore, we have

$$\begin{cases} (n-1)\kappa = tr(A) + (n-2)a_1 + b_1^2, \\ (n-1)\kappa = tr(A) + (n-2)a_2 + b_2^2. \end{cases} \tag{34}$$

Adding the above two formulas, we get

$$2(n-1)\kappa = 2tr(A) + (n-2)(a_1 + a_2) + b_1^2 + b_2^2.$$

From (21), we obtain

$$\kappa = \frac{(1-k)(n-k-1)}{k(n-k)(n-2)}.$$

$$\begin{aligned} a_2 &= \frac{1}{n-2}[(n-1)\kappa - tr(A) - b_2^2] \\ &= \frac{1}{n-2}[\frac{(1-k)(n-k-1)}{2k(n-k)} + \frac{1}{2n} - \frac{(n-1)k}{n^2(n-k)}]. \end{aligned}$$

By direct calculation, we have

$$\begin{aligned} R_2 &= 2a_2 - b_2^2 \\ &= \frac{2}{n-2}[\frac{(1-k)(n-k-1)}{2k(n-k)} + \frac{1}{2n} - \frac{(n-1)k}{n^2(n-k)}] - \frac{(n-1)k}{n^2(n-k)} \\ &= \frac{(1-k)(n-1)}{k(n-2)(n-k)} \leq 0. \end{aligned}$$

Here we get " $=$ " if and only if $k = 1$. So we complete the proof of the Corollary 1.

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