Note di Matematica
Note Mat. 1(2008), suppl. n. 1, 233-275
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v28n1supplp233
http://siba2.unile.it/notemat

Bi-harmonic maps and bi-Yang-Mills fields

Toshiyuki Ichiyama
Faculty of Economics, Asia University,
Sakai 5-24-10, Musashino, Tokyo, 180-8624, Japan
ichiyama@asia-u.ac.jp

Jun-ichi Inoguchi
Department of Mathematics, Faculty of Education, Utsunomiya University,
Utsunomiya, 321-8505, Japan
inoguchi@cc.utsunomiya-u.ac.jp

Hajime Urakawa
Division of Mathematics, Graduate School of Information Sciences, Tohoku University,
Aoba 6-3-09, Sendai, 980-8579, Japan
urakawa@math.is.tohoku.ac.jp

Received: 22/8/07; accepted: 5/10/07.

Abstract. In this paper, we show the first and second variational formulas of biharmonic maps and bi-Yang-Mills fields, and show the first variation formula of k-harmonic maps, and also give an overview of our recent results in [12], i.e., classifications of all biharmonic isoparametric hypersurfaces in the unit sphere, and all biharmonic homogeneous real hypersurfaces in the complex or quaternionic projective spaces, answers in case of bounded geometry to Chen's conjecture or Caddeo, Montaldo and Piu's one on biharmonic maps into a space of non positive curvature and the isolation phenomena of bi-Yang-Mills fields.

Keywords: harmonic maps, biharmonic maps, second variation formula

MSC 2000 classification: primary 58E20, secondary 53C43

Introduction

This is an expository and research paper. Harmonic maps play a central roll in variational problems, which are by definition for smooth maps between Riemannian manifolds $\varphi : M \to N$, critical maps of the energy functional $E(\varphi) = \frac{1}{2}\int_M \|d\varphi\|^2 \, v_g$. By extending the notion of harmonic maps, in 1983, J. Eells and L. Lemaire [8] proposed the problem to consider the k-harmonic maps which are critical maps of the functional

$$E_k(\varphi) = \frac{1}{2}\int_M \|(d+\delta)^k \varphi\|^2 \, v_g, \quad (k = 1, 2, \cdots).$$

After G.Y. Jiang [15] studied the first and second variation formulas of E_2 for $k = 2$, whose critical maps are called biharmonic maps, there have been extensive studies in this area (for instance, see [4], [18], [19], [23], [21], [13], [14], [25], etc.). Harmonic maps are always biharmonic maps by definition. One of main central problems is to classify the biharmonic maps, or to ask whether or not the converse to the above is true when the target Riemannian manifold (N, h) is non positive curvature (B. Y. Chen's conjecture [5] or Caddeo, Montaldo and Piu's one [4]). In this paper, we announce our results in [12], indeed we classify (1) all biharmonic hypersurfaces isoparametric hypersurfaces in the unit sphere, i.e., whose principal curvatures are constant, and (2) all biharmonic homogeneous real hypersurfaces in the complex or quaternionic projective spaces. Next, we show our answers to Chen's conjecture and Caddeo, Montaldo and Piu's one. Indeed, our result is that all biharmonic maps or biharmonic submanifolds of bounded geometry into the target space which is non positive curvature, must be harmonic ([12]). Here, that biharmonic maps are of bounded geometry means that the curvature of the domain manifold is bounded, and the norms of the tension field and its covariant derivative are L^2.

Recently, the notion of gauge field analogue of biharmonic maps, i.e., bi-Yang-Mills fields was proposed ([1]). In this paper, we also show the first and second variation formulas of bi-Yang-Mills fields, and the isolation phenomena of bi-Yang-Mills fields like the one for Yang-Mills fields (cf. Bourguignon-Lawson [3]), i.e., all bi-Yang-Mills fields over compact Riemanian manifolds whose Ricci curvature is bounded below by a positive constant k, and whose pointwise norm of curvature tensor is bounded above by $k/2$, must be Yang-Mills fields ([12]). We also show results of the L^2-isolation phenomena which are similar as Min-Oo's result ([20]) for Yang-Mills fields ([12]). These interesting phenomena suggest existence of a unified field theory between the the biharmonic maps and bi-Yang-Mills fields.

We thank to the referee who read carefully our paper and gave many suggestions to improve it.

1 Preliminaries

In this section, we prepare materials for the first and second variation formulas for the bi-energy functional and bi-harmonic maps. Let us recall the definition of a harmonic map $\varphi : (M, g) \to (N, h)$, of a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h), which is an extremal of the *energy functional* defined by

$$E(\varphi) = \int_M e(\varphi)\, v_g,$$

where $e(\varphi) := \frac{1}{2}|d\varphi|^2$ is called the energy density of φ. That is, for any variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$,

$$\left.\frac{d}{dt}\right|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V)v_g = 0, \tag{1}$$

where $V \in \Gamma(\varphi^{-1}TN)$ is a variation vector field along φ which is given by $V(x) = \frac{d}{dt}|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$, $(x \in M)$, and the *tension field* is given by $\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^m$ is a locally defined frame field on (M, g), and $B(\varphi)$ is the second fundamental form of φ defined by

$$\begin{aligned} B(\varphi)(X,Y) &= (\widetilde{\nabla} d\varphi)(X,Y) \\ &= (\widetilde{\nabla}_X d\varphi)(Y) \\ &= \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y) \\ &= \nabla^N_{d\varphi(X)} d\varphi(Y) - d\varphi(\nabla_X Y), \end{aligned} \tag{2}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Furthermore, ∇, and ∇^N, are connections on TM, TN of (M, g), (N, h), respectively, and $\overline{\nabla}$, and $\widetilde{\nabla}$ are the induced ones on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$, respectively. By (1), φ is harmonic if and only if $\tau(\varphi) = 0$.

The second variation formula is given as follows. Assume that φ is harmonic. Then,

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(\varphi_t) = \int_M h(J(V), V)v_g, \tag{3}$$

where J is an elliptic differential operator, called *Jacobi operator* acting on $\Gamma(\varphi^{-1}TN)$ given by

$$J(V) = \overline{\Delta} V - \mathcal{R}(V), \tag{4}$$

where $\overline{\Delta} V = \overline{\nabla}^* \overline{\nabla} V$ is the *rough Laplacian* and $\mathcal{R}$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by $\mathcal{R}V = \sum_{i=1}^m R^N(V, d\varphi(e_i))d\varphi(e_i)$, and R^N is the curvature tensor of (N, h) given by $R^N(U, V) = {\nabla^N}_U {\nabla^N}_V - {\nabla^N}_V {\nabla^N}_U - {\nabla^N}_{[U,V]}$ for U, $V \in \mathfrak{X}(N)$.

J. Eells and L. Lemaire proposed ([8]) polyharmonic (k-harmonic) maps and Jiang studied ([15]) the first and second variation formulas of bi-harmonic maps. Let us consider the *bi-energy functional* defined by

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g, \tag{5}$$

where $|V|^2 = h(V, V)$, $V \in \Gamma(\varphi^{-1}TN)$. Then, the first and second variation formulas are given as follows.

1 Theorem. (*the first variation formula*)

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi_t) = -\int_M h(\tau_2(\varphi), V) v_g, \tag{6}$$

where

$$\tau_2(\varphi) = J(\tau(\varphi)) = \overline{\Delta}\tau(\varphi) - \mathcal{R}(\tau(\varphi)), \tag{7}$$

J is given in (4)*.*

2 Definition. A smooth map φ of M into N is said to be *bi-harmonic* if $\tau_2(\varphi) = 0$.

3 Theorem. (*the second variation formula*) *Assume that φ is bi-harmonic. Then, we have*

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E_2(\varphi_t) = \int_M h(J_2(V), V) v_g, \tag{8}$$

where J_2 is a fourth order elliptic differential operator acting on $\Gamma(\varphi^{-1}TN)$ given by

$$J_2(V) = J(J(V)) - \mathcal{R}_2(V), \tag{9}$$

where J is given in (4)*, and $\mathcal{R}_2$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by*

$$\begin{aligned} \mathcal{R}_2(V) = {} & R^N(\tau(\varphi), V)\tau(\varphi) \\ & + 2\,\mathrm{trace} R^N(d\varphi(\cdot), \tau(\varphi))\overline{\nabla}_{\cdot} V + 2\,\mathrm{trace} R^N(d\varphi(\cdot), V)\overline{\nabla}_{\cdot}\tau(\varphi) \\ & + \mathrm{trace}\,(\nabla^N_{d\varphi(\cdot)} R^N)(d\varphi(\cdot), \tau(\varphi))V \\ & + \mathrm{trace}\,(\nabla^N_{\tau(\varphi)} R^N)(d\varphi(\cdot), V)d\varphi(\cdot). \end{aligned} \tag{10}$$

Here $\mathrm{trace} R^N(d\varphi(\cdot), \tau(\varphi))\overline{\nabla}_{\cdot} V$ *stands for* $\sum_{k=1}^m R^N(d\varphi(e_k), V)\overline{\nabla}_{e_k}\tau(\varphi)$*, where* $\{e_k\}_{k=1}^m$ *is a locally defined orthonormal frame field on* (M, g)*, etc.*

4 Definition. Assume that $\varphi : (M, g) \to (N, h)$ is a harmonic map. The operator J on $\Gamma(\varphi^{-1}TN)$ is a second order self-adjoint elliptic differential operator, so that it has a spectrum consisting of discrete eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ with their finite multiplicities. Denote by $E_{\lambda_1}, E_{\lambda_2}, \cdots$, the corresponding eigenspaces in $\Gamma(\varphi^{-1}TN)$. Then, let us recall the definitions of *index* and *nullity*,

$$\mathrm{Index}(\varphi) = \dim(\oplus_{\lambda<0} E_\lambda), \quad \mathrm{Nullity}(\varphi) = \dim E_0. \tag{11}$$

5 Definition. We give also the similar definitions of index and nullity for a bi-harmonic map. Assume that $\varphi : (M, g) \to (N, h)$ is a bi-harmonic map. The operator J_2 on $\Gamma(\varphi^{-1}TN)$ is a fourth order self-adjoint elliptic differential operator, so that it has a spectrum consisting of discrete eigenvalues $\mu_1 < \mu_2 < \cdots < \mu_k < \cdots$ with their finite multiplicities. Denote by $E^2_{\mu_1}, E^2_{\mu_2}, \cdots$, the corresponding eigenspaces in $\Gamma(\varphi^{-1}TN)$. Then, the definitions of *2-index* and *2-nullity* are given ([23], [18]) by

$$\mathrm{Index}_2(\varphi) = \dim(\oplus_{\mu<0} E^2_\mu), \quad \mathrm{Nullity}_2(\varphi) = \dim E^2_0. \tag{12}$$

6 Corollary. *Assume that* $\varphi : (M,g) \to (N,h)$ *is a harmonic map. Then,* $\mathrm{Index}_2(\varphi) = 0$ *and* $\mathrm{Nullity}_2(\varphi) = \mathrm{Nullity}(\varphi)$*.*

Proof. Indeed, if φ is harmonic, then, $\tau(\varphi) = 0$, so that

$$J_2(V) = J(J(V)), \quad \text{i.e., } \mathcal{R}_2(V) = 0 \tag{13}$$

for all $V \in \Gamma(\varphi^{-1}TN)$. It is clear by (8) and (13) that $\mathrm{Index}_2(\varphi) = 0$ which follows also by definition, and we have

$$\begin{aligned}\{V \in \Gamma(\varphi^{-1}TN);\, J_2(V) = 0\} &= \{V :\, J(J(V)) = 0\} \\ &= \{V :\, J(V) = 0\}\end{aligned} \tag{14}$$

which implies that $\mathrm{Nullity}_2(\varphi) = \mathrm{Nullity}(\varphi)$. *QED*

7 Remark. The second variational formula for a bi-harmonic map in [23] from (M,g) into the unit sphere $(N,h) = S^n(1)$ follows directly from Theorem 2 and the curvature formula of (N,h):

$$R^N(U,V)W = h(V,W)U - h(U,W)V, \quad U,V,W \in \mathfrak{X}(N).$$

2 Biharmonic maps into the unit sphere

In this section, we give the classification of all the biharmonic isometrically immersed hypersurfaces of the unit sphere with constant principal curvatures. In order to show it, we want to recall the following theorem.

8 Theorem. (*cf. Jiang [15]*) *Let* $\varphi : (M^m, g) \to S^{m+1}\left(\frac{1}{\sqrt{c}}\right)$ *be an isometric immersion of an* m*-dimensional compact Riemannian manifold* (M^m, g) *into the* $(m+1)$*-dimensional sphere with constant sectional curvature* $c > 0$*. Assume that the mean curvature of* φ *is a nonzero constant. Then,* φ *is biharmonic if and only if the square of the pointwise norm of* $B(\varphi)$ *is constant and* $\|B(\varphi)\|^2 = cm$*.*

This theorem due to Jiang ([15]) can be shown using the following two lemmas.

9 Lemma. (*Jiang*) *Let* $\varphi : (M^m, g) \to (N^n, h)$ *be an isometric immersion whose mean curvature vector field* $\mathbb{H} = \frac{1}{m}\tau(\varphi)$ *is parallel, i.e.,* $\nabla^\perp \mathbb{H} = 0$*, where* $\nabla^\perp$ *is the induced connection of the normal bundle* $T^\perp M$ *by* φ*. Then,*

$$\begin{aligned}\overline{\Delta}\tau(\varphi) &= \sum_{i=1}^m h(\overline{\Delta}\tau(\varphi), d\varphi(e_i))d\varphi(e_i) \\ &\quad - \sum_{i,j=1}^m h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_j))(\widetilde{\nabla}_{e_i} d\varphi)(e_j), \end{aligned} \tag{15}$$

where $\{e_i\}$ *is a locally defined orthonormal frame field of* (M, g).

Proof. Let us recall the definition of $\nabla^\perp$: For any section $\xi \in \Gamma(T^\perp M)$, we decompose $\overline{\nabla}_X \xi$ according to $TN|_M = TM \oplus T^\perp M$ as follows.

$$\overline{\nabla}_X \xi = \nabla^N_{\varphi_* X}\xi = \nabla^T_{\varphi_* X}\xi + \nabla^\perp_{\varphi_* X}\xi.$$

By the assumption $\nabla^\perp \mathbb{H} = 0$, i.e., $\nabla^\perp_{\varphi_* X}\tau(\varphi) = 0$ for all $X \in \mathfrak{X}(M)$, we have

$$\overline{\nabla}_X \tau(\varphi) = \nabla^T_{\varphi_* X}\tau(\varphi) \in \Gamma(\varphi_* TM). \tag{16}$$

Thus, for all $i = 1, \cdots, m$,

$$\overline{\nabla}_{e_i}\tau(\varphi) = \sum_{j=1}^m h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_j))d\varphi(e_j) \tag{17}$$

because $\{d\varphi(e_j)_x\}_{j=1}^m$ is an orthonormal basis with respect to h, of $\varphi_* T_x M$ $(x \in M)$.

Now let us calculate

$$\overline{\nabla}^* \overline{\nabla}\tau(\varphi) = -\sum_{i=1}^m \{\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\tau(\varphi) - \overline{\nabla}_{\nabla_{e_i}e_i}\tau(\varphi)\}. \tag{18}$$

Indeed, we have

$$\begin{aligned}\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\tau(\varphi) &= \sum_{j=1}^m \{h(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\tau(\varphi)) + h(\overline{\nabla}_{e_i}\tau(\varphi), \overline{\nabla}_{e_i} d\varphi(e_j))\}d\varphi(e_j) \\ &\quad + \sum_{j=1}^m h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_j))\overline{\nabla}_{e_i} d\varphi(e_i), \end{aligned} \tag{19}$$

and

$$\overline{\nabla}_{\nabla_{e_i}e_i}\tau(\varphi) = \sum_{j=1}^m h(\overline{\nabla}_{\nabla_{e_i}e_i}, d\varphi(e_j))d\varphi(e_j), \tag{20}$$

so that we have

$$\begin{aligned}\overline{\nabla}^*\overline{\nabla}\tau(\varphi) = &\sum_{j=1}^m h(\overline{\nabla}^*\overline{\nabla}\tau(\varphi), d\varphi(e_j))d\varphi(e_j)\\ &-\sum_{i,j=1}^m \{h(\overline{\nabla}_{e_i}\tau(\varphi), \overline{\nabla}_{e_i}d\varphi(e_j))\}d\varphi(e_j)\\ &\qquad + h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_j))\overline{\nabla}_{e_i}d\varphi(e_i)\}.\end{aligned} \tag{21}$$

Denoting $\nabla_{e_i}e_j = \sum_{k=1}^m \Gamma_{ij}^k e_k$, we have $\Gamma_{ij}^k + \Gamma_{ik}^j = 0$. Since $(\widetilde{\nabla}_{e_i}d\varphi)(e_j)$ $= \overline{\nabla}_{e_i}(d\varphi(e_j)) - d\varphi(\nabla_{e_i}e_j)$ is a local section of $T^{\perp}M$, we have for the the second term of the RHS of (21), for each fixed $i = 1, \cdots, m$,

$$\begin{aligned}\sum_{j=1}^m &h(\overline{\nabla}_{e_i}\tau(\varphi),\overline{\nabla}_{e_i}d\varphi(e_j))d\varphi(e_j)\\ &= \sum_{j=1}^m h(\overline{\nabla}_{e_i}\tau(\varphi), (\widetilde{\nabla}_{e_i}d\varphi)(e_j) + d\varphi(\nabla_{e_i}e_j))d\varphi(e_j)\\ &= \sum_{j=1}^m h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(\nabla_{e_i}e_j))d\varphi(e_j)\\ &= \sum_{j,k=1}^m h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_k))\, d\varphi(\Gamma_{ij}^k e_j)\\ &= -\sum_{j,k=1}^m h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_k))\, d\varphi(\Gamma_{ik}^j e_j)\\ &= -\sum_{k=1}^m h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_k))\, d\varphi(\nabla_{e_i}e_k).\end{aligned} \tag{22}$$

Substituting (22) into (21), we have the desired (15). *QED*

10 Lemma. (*Jiang*) *Under the same assumptions as in Lemma 1, we have*

$$\begin{aligned}\overline{\Delta}\tau(\varphi) = &-\sum_{j,k=1}^m h(\tau(\varphi), R^N(d\varphi(e_j), d\varphi(e_k))d\varphi(e_k))d\varphi(e_j)\\ &+\sum_{i,j=1}^m h(\tau(\varphi), (\widetilde{\nabla}_{e_i}d\varphi)(e_j))(\widetilde{\nabla}_{e_i}d\varphi)(e_j).\end{aligned} \tag{23}$$

Proof. Since $h(\tau(\varphi), d\varphi(e_j)) = 0$, differentiating it by e_i, we have

$$\begin{aligned} h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(e_j)) &= -h(\tau(\varphi), \overline{\nabla}_{e_i} d\varphi(e_j)) \\ &= -h(\tau(\varphi), \overline{\nabla}_{e_i} d\varphi(e_j) - d\varphi(\nabla_{e_i} e_j)) \\ &= -h(\tau(\varphi), (\widetilde{\nabla}_{e_i} d\varphi)(e_j)). \end{aligned} \tag{24}$$

For the first term of (15), we have for each $j = 1, \cdots, m$,

$$\begin{aligned} h(\overline{\Delta}\tau(\varphi), d\varphi(e_j)) - 2\sum_{i=1}^{m} h(\overline{\nabla}_{e_i}\tau(\varphi), \overline{\nabla}_{e_i} d\varphi(e_j)) \\ + h(\tau(\varphi), \overline{\Delta} d\varphi(e_j)) = 0, \end{aligned} \tag{25}$$

which follows by the expression (18) of $\overline{\Delta}\tau(\varphi)$, differentiating the first equation of (24) by e_i, and doing $h(\tau(\varphi), d\varphi(e_j)) = 0$ by $\nabla_{e_i} e_i$.

For the second term of (15), we have by (16) and (24),

$$\begin{aligned} h(\overline{\nabla}_{e_i}\tau(\varphi), \overline{\nabla}_{e_i} d\varphi(e_j)) &= h(\overline{\nabla}_{e_i}\tau(\varphi), (\widetilde{\nabla}_{e_i} d\varphi)(e_j) + d\varphi(\nabla_{e_i} e_j)) \\ &= h(\overline{\nabla}_{e_i}\tau(\varphi), d\varphi(\nabla_{e_i} e_j)) \\ &= -h(\tau(\varphi), (\widetilde{\nabla}_{e_i} d\varphi)(\nabla_{e_i} e_j)). \end{aligned} \tag{26}$$

For the third term $h(\tau(\varphi), \overline{\Delta} d\varphi(e_j))$ of (25), we have

$$\begin{aligned} h(\tau(\varphi), \overline{\Delta} d\varphi(e_j)) &= \sum_{k=1}^{m} h(\tau(\varphi), R^N(d\varphi(e_j), d\varphi(e_k)) d\varphi(e_k)) \\ &\quad - 2\sum_{k=1}^{m} h(\tau(\varphi), (\widetilde{\nabla}_{e_k} d\varphi)(\nabla_{e_k} e_j)). \end{aligned} \tag{27}$$

Because, by making use of

$$(\widetilde{\nabla}_X d\varphi)(Y) = \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y)$$

and

$$h(\tau(\varphi), d\varphi(X)) = 0 \ (X, Y \in \mathfrak{X}(M)),$$

the LHS of (27) coincides with

$$\begin{aligned}
& h(\tau(\varphi), -\sum_{k=1}^{m}\{\overline{\nabla}_{e_k}\overline{\nabla}_{e_k} - \overline{\nabla}_{\nabla_{e_k}e_k}\}d\varphi(e_j)) \\
&\quad = h(\tau(\varphi), -\sum_{k=1}^{m}\{(\widetilde{\nabla}_{e_k}\widetilde{\nabla}_{e_k}d\varphi)(e_j) + 2(\widetilde{\nabla}_{e_k}d\varphi)(\nabla_{e_k}e_j) \\
&\qquad\qquad - (\widetilde{\nabla}_{\nabla_{e_k}e_k}d\varphi)(e_j)\}) \\
&\quad = h(\tau(\varphi), (\widetilde{\nabla}^*\widetilde{\nabla}d\varphi)(e_j)) - 2h(\tau(\varphi), (\widetilde{\nabla}_{e_k}d\varphi)(\nabla_{e_k}e_j)) \\
&\quad = h(\tau(\varphi), \Delta d\varphi(e_j) - Sd\varphi(e_j)) - 2h(\tau(\varphi), (\widetilde{\nabla}_{e_k}d\varphi)(\nabla_{e_k}e_j)),
\end{aligned} \tag{28}$$

where the last equality follows from the Weitzenböck formula for the Laplacian $\Delta = d\delta + \delta d$ acting on 1-forms on (M, g):

$$\Delta d\varphi = \widetilde{\nabla}^*\widetilde{\nabla}d\varphi + Sd\varphi. \tag{29}$$

Here, we have

$$\begin{aligned}
Sd\varphi(e_j) &:= \sum_{k=1}^{m}(\widetilde{R}(e_k, e_j)d\varphi)(e_k) \\
&= \sum_{k=1}^{m}\{R^N(d\varphi(e_k), d\varphi(e_j))d\varphi(e_k) - d\varphi(R^M(e_k, e_j)e_k)\},
\end{aligned} \tag{30}$$

and

$$\Delta d\varphi(e_j) = d\delta d\varphi(e_j) = -d\tau(\varphi)(e_j) = -\overline{\nabla}_{e_j}\tau(\varphi). \tag{31}$$

Substituting these into (28), and using $h(\tau(\varphi), d\varphi(X)) = 0$ for all $X \in \mathfrak{X}(M)$, (28) coincides with

$$\sum_{k=1}^{m}\{h(\tau(\varphi), R^N(d\varphi(e_j), d\varphi(e_k))d\varphi(e_k)) - 2h(\tau(\varphi), (\widetilde{\nabla}_{e_k}d\varphi)(\nabla_{e_k}e_j))\},$$

which implies (27).

Substituting (26) and (27) into (25), we have

$$\begin{aligned}
h(\overline{\Delta}\tau(\varphi), d\varphi(e_j)) = &-2\sum_{i=1}^{m} h(\tau(\varphi), (\widetilde{\nabla}_{e_i}d\varphi)(\nabla_{e_i}e_j)) \\
&- \sum_{k=1}^{m} h(\tau(\varphi), R^N(d\varphi(e_j), d\varphi(e_k))d\varphi(e_k))
\end{aligned}$$

$$
\begin{aligned}
&+ 2\sum_{k=1}^{m} h(\tau(\varphi), (\widetilde{\nabla}_{e_k} d\varphi)(\nabla_{e_k} e_j)) \\
&= \sum_{k=1}^{m} h(\tau(\varphi), R^N(d\varphi(e_j), d\varphi(e_k))d\varphi(e_k)). \qquad (32)
\end{aligned}
$$

Substituting (26) and (32) into (15), we have (23). QED

11 Lemma. *Let $\varphi : (M^m, g) \to (N^{m+1}, h)$ be an isometric immersion which is not harmonic. Then, the condition that $\|\tau(\varphi)\|$ is constant is equivalent to the one that*

$$
\overline{\nabla}_X \tau(\varphi) \in \Gamma(\varphi_* TM), \quad \forall X \in \mathfrak{X}(M), \qquad (33)
$$

that is, the mean curvature tensor is parallel with respect to $\nabla^{\perp}$.

Proof. Assume that φ is not harmonic. Then, if $\|\tau(\varphi)\|$ is constant,

$$
Xh(\tau(\varphi), \tau(\varphi)) = 2h(\overline{\nabla}_X \tau(\varphi), \tau(\varphi)) = 0 \qquad (34)
$$

for all $X \in \mathfrak{X}(M)$, so we have $\overline{\nabla}_X \tau(\varphi) \in \Gamma(\varphi_* TM)$ because $\dim M = \dim N - 1$ and $\tau(\varphi) \neq 0$ everywhere on M. The converse is true from the above equality (34). QED

Proof. By Lemma 3, the condition (16) holds under the condition that the mean curvature of φ is constant. So, we may apply Lemmas 1 and 2.

Since the curvature tensor R^N of $S^{m+1}\left(\frac{1}{\sqrt{c}}\right)$ is given by

$$
R^N(U, V)W = c\{h(V, W)U - h(W, U)V\}, \quad U, V, W \in \mathfrak{X}(N),
$$

$R^N(d\varphi(e_j), d\varphi(e_k))d\varphi(e_k)$ is tangent to $\varphi_* TM$. By (23) of Lemma 2,

$$
\overline{\Delta}\tau(\varphi) = \sum_{i,j=1}^{m} h(\tau(\varphi), (\widetilde{\nabla}_{e_i} d\varphi)(e_j))(\widetilde{\nabla}_{e_i} d\varphi)(e_j). \qquad (35)
$$

Furthermore, we have

$$
\begin{aligned}
\mathcal{R}(\tau(\varphi)) &= \sum_{i=1}^{m} R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) \\
&= c\sum_{i=1}^{m} \{h(d\varphi(e_i), d\varphi(e_i))\tau(\varphi) - h(d\varphi(e_i), \tau(\varphi))d\varphi(e_i)\} \\
&= cm\tau(\varphi). \qquad (36)
\end{aligned}
$$

Then, $\varphi : (M, g) \to S^{m+1}(\frac{1}{\sqrt{c}})$ is biharmonic if and only if

$$\begin{aligned}\tau_2(\varphi) &= \overline{\Delta}\tau(\varphi) - \mathcal{R}(\tau(\varphi)) \\ &= \sum_{i,j=1}^{m} h(\tau(\varphi), (\widetilde{\nabla}_{e_i} d\varphi)(e_j))(\widetilde{\nabla}_{e_i} d\varphi)(e_j) - cm\tau(\varphi) \\ &= 0. \qquad (37)\end{aligned}$$

If we denote by ξ, the unit normal vector field to $\varphi(M)$, the second fundamental form $B(\varphi)$ is of the form $B(\varphi)(e_i, e_j) = (\widetilde{\nabla}_{e_i} d\varphi)(e_j) = h_{ij}\xi$. Then, we have $\tau(\varphi) = \sum_{i=1}^{m} B(\varphi)(e_i, e_i) = \sum_{i=1}^{m} h_{ii}\,\xi$ and $\|B(\varphi)\|^2 = \sum_{i,j=1}^{m} h_{ij}h_{ij}$. Substituting these into (37), we have

$$\tau_2(\varphi) = \sum_{k=1}^{m} h_{kk} \left(\sum_{i,j=1}^{m} h_{ij}h_{ij} - cm \right) \xi = 0, \qquad (38)$$

That is, $\|B(\varphi)\|^2 = cm$ since $\sum_{k=1}^{m} h_{kk} \neq 0$. QED

3 Biharmonic isoparametric hypersurfaces

First, we prepare the necessary materials on isoparametric hypersurfaces M in the unit sphere $S^n(1)$ following Münzner ([22]) or Ozeki and Takeuchi ([24]).

Let $\varphi : (M, g) \to S^n(1)$ be an isometric immersion of (M, g) into the unit sphere $S^n(1)$ and denote by (N, h), the unit sphere $S^n(1)$ with the canonical metric. Assume that $\dim M = n-1$. The shape operator A_ξ is a linear operator of T_xM into itself defined by

$$g(A_\xi X, Y) = h(\varphi_*(\nabla_X Y), \xi),\ X, Y \in \mathfrak{X}(M),$$

where ξ is the unit normal vector field along M. The eigenvalues of A_ξ are called the *principal curvatures*. M is called *isoparametric* if all the principal curvatures are constant in $x \in M$. It is known that there exists a homogeneous polynomial F on $\mathbb{R}^{n+1}$ of degree g whose restriction to $S^n(1)$, denoted by f, called *isoparametric function*, M is given by $M = f^{-1}(t)$ for some $t \in I = (-1, 1)$. For each $t \in I$, $\xi_t = \frac{\nabla f}{\sqrt{g(\nabla f, \nabla f)}}$ is a smooth unit normal vector field along $M_t = f^{-1}(t)$, and all the distinct principal curvatures of M_t with respect to ξ_t are given as

$$k_1(t) > k_2(t) > \cdots > k_{g(t)}(t)$$

with their multiplicities $m_j(t)$ $(j = 1, \cdots, g(t))$. And $g = g(t)$ is constant in t, and is should be $g = 1, 2, 3, 4,$ or 6. Furthermore, it holds that

$$\begin{aligned} m_1(t) &= m_3(t) = \cdots = m_1, \\ m_2(t) &= m_4(t) = \cdots = m_2, \\ k_j(t) &= \cot\left(\frac{(j-1)\pi + \cos^{-1} t}{g}\right) \quad (j = 1, \cdots, g). \end{aligned} \tag{39}$$

where m_1 and m_2 are constant in $t \in I$. We also have

$$\|B(\varphi)\|^2 = \|A_\xi\|^2 = \sum_{j=1}^{g(t)} m_j(t) k_j(t)^2. \tag{40}$$

Indeed, if we denote by λ_i $(i = 1, \cdots, m$ $(m = \dim M)$, all the principal curvatures counted with their multiplicities, we may choose orthonormal eigenvectors $\{X_i\}_{i=1}^m$ of T_xM in such a way that $A_\xi X_i = \lambda_i X_i$ $(i = 1, \cdots, m)$. Then, we have $h(B(X_i, X_j), \xi) = g(A_\xi(X_i), X_j) = \lambda_i \delta_{ij}$, and $\|B(X_i, X_j)\|^2 = {\lambda_i}^2 \delta_{ij}$, so that

$$\|B(\varphi)\|^2 = \|A_\xi\|^2 = \sum_{i,j=1}^{m} \|B(X_i, X_j)\|^2 = \sum_{j=1}^{m} {\lambda_j}^2. \tag{41}$$

Then, by using Theorem 3 and (40), we have

12 Theorem. *Let* $\varphi : (M, g) \to S^n(1)$ *be an isometric immersion* $(\dim M = n-1)$ *which is isoparametric. Then,* (M, g) *is biharmonic if and only if* (M, g) *is one of the following:*

(i) $M = S^{n-1}\left(\frac{1}{\sqrt{2}}\right) \subset S^n(1)$, (*a small sphere*)

(ii) $M = S^{n-p}\left(\frac{1}{\sqrt{2}}\right) \times S^{p-1}\left(\frac{1}{\sqrt{2}}\right) \subset S^n(1)$, *with* $n - p \neq p - 1$ (*the Clifford torus*), *or*

(iii) $\varphi : (M, g) \to S^n(1)$ *is harmonic, i.e., minimal.*

For a proof, see [12].

4 Biharmonic maps into the complex projective space

In the following two sections, we show the classification of all homogeneous real hypersurfaces in the complex n-dimensional projective space $\mathbb{C}P^n(c)$ with positive constant holomorphic sectional curvature $c > 0$ which are *biharmonic*. To do it, we first need the following theorem analogue to Theorem 3 which characterizes the biharmonic maps.

13 Theorem. *Let (M,g) be a real $(2n-1)$-dimensional compact Riemannian manifold, and $\varphi : (M,g) \to \mathbb{C}P^n(c)$ be an isometric immersion with non-zero constant mean curvature. Then, the necessary and sufficient condition for φ to be biharmonic is*

$$\|B(\varphi)\|^2 = \frac{n+1}{2}c. \tag{42}$$

PROOF. By Lemma 3, the mean curvature vector field of φ is parallel with respect to $\nabla^\perp$, so we may apply Lemmas 1 and 2 in this case. Let us recall the fact that the curvature tensor of $(N,h) = \mathbb{C}P^n(c)$ is given by

$$\begin{aligned} R^N(U,V)W = \frac{c}{4}\big\{&h(V,W)U - h(U,W)V \\ &+ h(JV,W)JU - h(JU,W)JV + 2h(U,JV)JW\big\}, \end{aligned}$$

where J is the adapted almost complex tensor, and U,V and W are vector fields on $\mathbb{C}P^n(c)$. Then, we have

$$\begin{aligned} R^N(d\varphi(e_j), d\varphi(e_k))d\varphi(e_k) = \frac{c}{4}\big\{&d\varphi(e_j) - \delta_{jk}\, d\varphi(e_k) \\ &+ 3h(d\varphi(e_j), Jd\varphi(e_k))\, Jd\varphi(e_k)\big\}. \end{aligned} \tag{43}$$

Then, we have

$$\sum_{j,k=1}^m h\big(\tau(\varphi), R^N(d\varphi(e_j), d\varphi(e_k))d\varphi(e_k)\big)\, d\varphi(e_j) = 0. \tag{44}$$

Because the LHS of (44) coincides with

$$\begin{aligned} &\frac{3c}{4}\sum_{j,k=1}^m h\big(d\varphi(e_j), Jd\varphi(e_k)\big)\, h\big(\tau(\varphi), Jd\varphi(e_k)\big)\, d\varphi(e_j) \\ &\quad = \frac{3c}{4}\sum_{j,k=1}^m h\big(Jd\varphi(e_j), d\varphi(e_k)\big)\, h\big(J\tau(\varphi), d\varphi(e_k)\big)\, d\varphi(e_j) \\ &\quad = \frac{3c}{4}\sum_{j=1}^m h\Big(Jd\varphi(e_j), \sum_{k=1}^m h\big(J\tau(\varphi), d\varphi(e_k)\big)d\varphi(e_k)\Big)\, d\varphi(e_j) \\ &\quad = \frac{3c}{4}\sum_{j=1}^m h\big(Jd\varphi(e_j), J\tau(\varphi)\big)\, d\varphi(e_j) \\ &\quad = \frac{3c}{4}\sum_{j=1}^m h(d\varphi(e_j), \tau(\varphi))\, d\varphi(e_j) = 0. \end{aligned} \tag{45}$$

Here the third equality follows from that $J\tau(\varphi) \in \Gamma(\varphi_* TM)$ which is due to $h(J\tau(\varphi), \tau(\varphi)) = 0$, $0 \neq \tau(\varphi) \in T^\perp M$ and $\dim M = 2n-1$. Since $\{d\varphi(e_k)\}_{k=1}^m$ is an orthonormal basis of $\varphi_*(T_x M)$ at each

$$x \in M\,, J\tau(\varphi) = \sum_{k=1}^m h(J\tau(\varphi), d\varphi(e_k))d\varphi(e_k)\,.$$

By (23) in Lemma 2, we have

$$\overline{\Delta}\tau(\varphi) = \sum_{i,j=1}^m h\big(\tau(\varphi), (\widetilde{\nabla}_{e_i} d\varphi)(e_j)\big)\,(\widetilde{\nabla}_{e_i} d\varphi)(e_j). \tag{46}$$

Furthermore, we have

$$\mathcal{R}(\tau(\varphi)) = \frac{c}{4}(m+3)\tau(\varphi). \tag{47}$$

Because the LHS of (47) is equal to

$$\begin{aligned}\sum_{k=1}^m R^N(\tau(\varphi), d\varphi(e_k))d\varphi(e_k) &= \frac{c}{4}\big\{m\tau(\varphi) \\ &\qquad - 3\sum_{k=1}^m h(J\tau(\varphi), d\varphi(e_k))\, Jd\varphi(e_k)\big\} \\ &= \frac{c}{4}\{m\tau(\varphi) - 3J(J\tau(\varphi))\} \\ &= \frac{c}{4}(m+3)\tau(\varphi). \end{aligned}\tag{48}$$

Now the sufficient and necessary condition for φ to be biharmonic is that

$$\tau_2(\varphi) = \overline{\Delta}\tau(\varphi) - \mathcal{R}(\tau(\varphi)) = 0 \tag{49}$$

which is equivalent to

$$\sum_{i,j=1}^m h(\tau(\varphi), (\widetilde{\nabla}_{e_i} d\varphi)(e_j))(\widetilde{\nabla}_{e_i} d\varphi)(e_j) - \frac{c}{4}(m+3)\tau(\varphi) = 0. \tag{50}$$

Here, we may denote as

$$\begin{aligned} B(\varphi)(e_i, e_j) &= (\widetilde{\nabla}_{e_i} d\varphi)(e_j) = h_{ij}\,\xi \\ \tau(\varphi) &= \sum_{k=1}^m (\widetilde{\nabla}_{e_k} d\varphi)(e_k) = \sum_{k=1}^m h_{kk}\xi, \end{aligned}\tag{51}$$

where ξ is the unit normal vector field along $\varphi(M)$. Thus, the LHS of (50) coincides with

$$\begin{aligned}
&\sum_{i,j,k=1}^{m} h_{kk}h_{ij}h_{ij} - \frac{c}{4}(m+3)\sum_{k=1}^{m} h_{kk} \\
&\quad = \left(\sum_{k=1}^{m} h_{kk}\right)\left\{\sum_{i,j=1}^{m} h_{ij}h_{ij} - \frac{c}{4}(m+3)\right\} \\
&\quad = \|\tau(\varphi)\|^2 \left\{\|B(\varphi)\|^2 - \frac{c}{2}(n+1)\right\}, \qquad (52)
\end{aligned}$$

which yields the desired (42) due to the assumption that $\|\tau(\varphi)\|$ is a non-zero constant. QED

5 Biharmonic Homogeneous real hypersurfaces in the complex projective space

In this section, we classify all the *biharmonic* homogeneous real hypersurfaces in the complex projective space $\mathbb{C}P^n(c)$.

First, let us recall the classification theorem of all the homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$ due to R. Takagi (cf. [27]) based on a work by W.Y. Hsiang and H.B. Lawson ([11]). Let U/K be a symmetric space of rank two of compact type, and $\mathfrak{u} = \mathfrak{k}\oplus\mathfrak{p}$, the Cartan decomposition of the Lie algebra $\mathfrak{u}$ of U, and the Lie subalgebra $\mathfrak{k}$ corresponding to K. Let $\langle X, Y\rangle = -B(X,Y)$ $(X,Y \in \mathfrak{p})$ be the inner product on $\mathfrak{p}$, $\|X\|^2 = \langle X, X\rangle$, and $S := \{X \in \mathfrak{p};\ \|X\| = 1\}$, the unit sphere in the Euclidean space $(\mathfrak{p}, \langle\ ,\ \rangle)$, where B is the Killing form of $\mathfrak{u}$. Consider the adjoint action of K on $\mathfrak{p}$. Then, the orbit $\hat{M} = \mathrm{Ad}(K)A$ through any regular element $A \in \mathfrak{p}$ with $\|A\| = 1$ gives a homogeneous hypersurface in the unit sphere S. Conversely, any homogeneous hypersurface in S can be obtained in this way ([11]).

Let us take as U/K, a *Hermitian* symmetric space of compact type of rank two of complex dimension $(n+1)$, and identify $\mathfrak{p}$ with $\mathbb{C}^{n+1}$. Then, the adjoint orbit $\hat{M} = \mathrm{Ad}(K)A$ of K through any regular element A in $\mathfrak{p}$ is again a homogeneous hypersurface in the unit sphere S. Let $\pi : \mathbb{C}^{n+1} - \{\mathbf{0}\} = \mathfrak{p} - \{\mathbf{0}\} \to \mathbb{C}P^n$ be the natural projection. Then, the projection induces the Hopf fibration of S onto $\mathbb{C}P^n$, denoted also by π, and $\varphi :\ M := \pi(\hat{M}) \hookrightarrow \mathbb{C}P^n$ gives a homogeneous real hypersurface in the complex projective space $\mathbb{C}P^n(4)$ with constant holomorphic sectional curvature 4. Conversely, any homogeneous real hypersurface M in $\mathbb{C}P^n(4)$ is given in this way ([27]). Furthermore, all such hypersurfaces are classified into the following five types:

(1) A-type:
$\mathfrak{u} = \mathfrak{su}(p+2) \oplus \mathfrak{su}(q+2)$, $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(p+1) + \mathfrak{u}(1)) \oplus \mathfrak{s}(\mathfrak{u}(q+1) + \mathfrak{u}(1))$, where $0 \leq p \leq q$, $0 < q$, $p + q = n - 1$, and $\dim M = 2n - 1$.

(2) B-type: $\mathfrak{u} = \mathfrak{o}(m+2)$, $\mathfrak{k} = \mathfrak{o}(m) \oplus \mathbb{R}$, where $3 \leq m$, $\dim M = 2m - 3$.

(3) C-type: $\mathfrak{u} = \mathfrak{su}(m+2)$, $\mathfrak{k} = \mathfrak{s}(\mathfrak{o}(m) + \mathfrak{o}(2))$, where $3 \leq m$, and $\dim M = 4m - 3$.

(4) D-type: $\mathfrak{o}(10)$, $\mathfrak{u}(5)$, and $\dim M = 17$.

(5) E-type: $\mathfrak{u} = \mathfrak{e}_6$, $\mathfrak{k} = \mathfrak{o}(10) \oplus \mathbb{R}$, and $\dim M = 29$.

He also gave ([28], [29]) lists of the principal curvatures and their multiplicities of these M as follows:

(1) A-type: Assume that

$$U/K = \frac{SU(p+2) \times SU(q+2)}{S(U(p+1) \times U(1)) \times S(U(q+1) \times U(1)},$$

then, the adjoint orbit of K, $\mathrm{Ad}(K)A$ is given by the Riemannian product of two odd dimensional spheres,

$$\hat{M} = \hat{M}_{p,q} = S^{2p+1}(\cos u) \times S^{2q+1}(\sin u) \subset S^{2n+1}, \tag{53}$$

where $0 < u < \frac{\pi}{2}$. The projection $M_{p,q}(u) := \pi(\hat{M}_{p,q}(u))$ is a homogeneous real hypersurface of $\mathbb{C}P^n(4)$. The principal curvatures of $M_{p,q}$ with $0 \leq p \leq q$, $0 < q$, are given as

$$\begin{cases} \lambda_1 = -\tan u \quad (\text{with multiplicity } m_1 = 2p), \\ \qquad\qquad\quad (m_1 = 0 \text{ if } p = 0), \\ \lambda_2 = \cot u \quad (\text{with multiplicity } m_2 = 2q), \\ \lambda_3 = 2\cot(2u) \quad (\text{with multiplicity } m_3 = 1). \end{cases} \tag{54}$$

Thus, the mean curvature H of $M_{p,q}(u)$ is given by

$$\begin{aligned} H &= \frac{1}{2n-1}\{2q \cot u - 2p \tan u + 2\cot(2u)\} \\ &= \frac{1}{2n-1}\{(2q+1)\cot u - (2p+1)\tan u\}. \end{aligned} \tag{55}$$

The constant $\|B(\varphi)\|^2$ which is the sum of all the squares of principal curvatures with their multiplicities, is given by

$$\begin{aligned} \|B(\varphi)\|^2 &= 2q \cot^2 u + 2p \tan^2 u + 4\cot^2(2u) \\ &= (2q+1)\cot^2 u + (2p+1)\tan^2 u - 2. \end{aligned} \tag{56}$$

(2) B-type: Assume that $U/K = SO(m+2)/(SO(m) \times SO(2))$, $(m := n+1)$, and then, the adjoint orbit of K, $\mathrm{Ad}(K)A$ is given by

$$\hat{M} = \{SO(n+1) \times SO(2)\}/\{SO(n-1) \times \mathbb{Z}_2\} \subset S^{2n+1}.$$

The real hypersurface $\varphi : M \hookrightarrow \mathbb{C}P^n$ is a tube over a complex quadric with radius $\frac{\pi}{4} - u$ $(0 < u < \frac{\pi}{4})$ or a tube over a totally geodesic real projective space $\mathbb{R}P^n$ with radius u $(0 < u < \frac{\pi}{4})$. The principal curvatures of M are given as

$$\begin{cases} \lambda_1 = -\cot u \quad (\text{with multiplicity } m_1 = n-1), \\ \lambda_2 = \tan u \quad (\text{with multiplicity } m_2 = n-1), \\ \lambda_3 = 2\tan(2u) \quad (\text{with multiplicity } m_3 = 1). \end{cases} \tag{57}$$

Thus, the mean curvature of M is given by

$$\begin{aligned} H &= \frac{1}{2n-1}\{-(n-1)\cot u + (n-1)\tan u + 2\tan(2u)\} \\ &= -\frac{1}{2n-1} \cdot \frac{(n-1)t^4 - 2(n+1)t^2 + n - 1}{t(t^2-1)}, \end{aligned} \tag{58}$$

where $t = \cot u$. The constant $\|B(\varphi)\|^2$ is given by

$$\begin{aligned} \|B(\varphi)\|^2 &= (n-1)\cot^2 u + (n-1)\tan^2 u + 4\tan^2(2u) \\ &= (n-1)t^2 + \frac{n-1}{t^2} + \frac{16t^2}{(t^2-1)^2} \\ &= \frac{(n-1)(X-1)^2(X^2+1) + 16X^2}{X(X-1)^2}, \end{aligned} \tag{59}$$

where $X := t^2$.

(3) C-type: Assume that $U/K = SU(m+2)/S(U(m) \times U(2))$, $(n = 2m+1)$, and then, the adjoint orbit of K, $\mathrm{Ad}(K)A$ is given by

$$\hat{M} = S(U(m) \times U(2))/(T^2 \times SU(m-2)) \subset S^{2n+1}.$$

The real hypersurface $\varphi : M \hookrightarrow \mathbb{C}P^n$ is a tube over the Segre imbedding of $\mathbb{C}^1 \times \mathbb{C}P^m$ with radius u $(0 < u < \frac{\pi}{4})$. The principal curvatures of M are given by

$$\begin{cases} \lambda_1 = -\cot u \quad (\text{with multiplicity } m_1 = n-3), \\ \lambda_2 = \cot\left(\frac{\pi}{4} - u\right) \quad (\text{with multiplicity } m_2 = 2), \\ \lambda_3 = \cot\left(\frac{\pi}{2} - u\right) \quad (\text{with multiplicity } m_3 = n-3), \\ \lambda_4 = \cot\left(\frac{3\pi}{4} - u\right) \quad (\text{with multiplicity } m_4 = 2), \\ \lambda_5 = -2\cot(2u) \quad (\text{with multiplicity } m_5 = 1). \end{cases} \tag{60}$$

Then,

$$\lambda_1 = -t,\ \lambda_2 = \frac{t+1}{t-1},\ \lambda_3 = \frac{1}{t},\ \lambda_4 = -\frac{t-1}{t+1},\ \lambda_5 = -t + \frac{1}{t},$$

where $t = \cot u$. The mean curvature of M is given by

$$\begin{aligned} H &= \frac{1}{2n-1}\left\{(n-3)(-t) + 2\frac{t+1}{t-1} + (n-3)\frac{1}{t} - 2\frac{t-1}{t+1} - t + \frac{1}{t}\right\} \\ &= -\frac{(n-2)t^4 - 2(n+2)t^2 + n - 2}{t(t^2-1)}. \end{aligned} \tag{61}$$

The constant $\|B(\varphi)\|^2$ is given by

$$\begin{aligned} \|B(\varphi)\|^2 &= (n-3)t^2 + 2\left(\frac{t+1}{t-1}\right)^2 + (n-3)\frac{1}{t^2} \\ &\quad + 2\left(\frac{t-1}{t+1}\right)^2 + \left(-t + \frac{1}{t}\right)^2 \\ &= \frac{C(X)}{X(X-1)^2}, \end{aligned} \tag{62}$$

where

$$\begin{aligned} C(X) := (n-2)X^2(X-1)^2 + (n-2)(X-1)^2 \\ + 4X(X^2 + 6X + 1) - 2X(X-1)^2, \end{aligned} \tag{63}$$

and $X := t^2$.

(4) D-type: Assume that $U/K = O(10)/U(5)$, and then, the adjoint orbit of K, $\mathrm{Ad}(K)A$ is given by

$$\hat{M} = U(5)/(SU(2) \times SU(2) \times U(1)) \subset S^{19}.$$

The real hypersurface $\varphi : M \hookrightarrow \mathbb{C}P^9$ is a tube over the Plücker imbedding of $\mathrm{Gr}_2(\mathbb{C}^5)$ with radius u ($0 < u < \frac{\pi}{4}$). The principal curvatures of M are given by

$$\begin{cases} \lambda_1 = -\cot u \quad \text{(with multiplicity } m_1 = 4), \\ \lambda_2 = \cot\left(\frac{\pi}{4} - u\right) \quad \text{(with multiplicity } m_2 = 4), \\ \lambda_3 = \cot\left(\frac{\pi}{2} - u\right) \quad \text{(with multiplicity } m_3 = 4), \\ \lambda_4 = \cot\left(\frac{3\pi}{4} - u\right) \quad \text{(with multiplicity } m_4 = 4), \\ \lambda_5 = -2\cot(2u) \quad \text{(with multiplicity } m_5 = 1). \end{cases} \tag{64}$$

Then,

$$\lambda_1 = -t,\ \lambda_2 = \frac{t+1}{t-1},\ \lambda_3 = \frac{1}{t},\ \lambda_4 = -\frac{t-1}{t+1},\ \lambda_5 = -t + \frac{1}{t},$$

where $t = \cot u$. The mean curvature of M is given by

$$\begin{aligned} H &= \frac{1}{17}\left\{4(-t) + 4\frac{t+1}{t-1} + 4\frac{1}{t} - 4\frac{t-1}{t+1} - t + \frac{1}{t}\right\} \\ &= -\frac{5t^4 - 26t^2 + 5}{17t(t^2-1)} = -\frac{(5t^2-1)(t^2-5)}{17t(t^2-1)}. \end{aligned} \tag{65}$$

The constant $\|B(\varphi)\|^2$ is given by

$$\begin{aligned} \|B(\varphi)\|^2 &= 4t^2 + 4\left(\frac{t+1}{t-1}\right)^2 + 4\frac{1}{t^2} \\ &\quad + 4\left(\frac{t-1}{t+1}\right)^2 + \left(-t + \frac{1}{t}\right)^2 \\ &= \frac{D(X)}{X(X-1)^2}, \end{aligned} \tag{66}$$

where

$$D(X) := 11X^3 + 63X^2 + X + 5, \tag{67}$$

and $X := t^2$.

(5) E-type: Assume that $U/K = E_6/(\mathrm{Spin}(10) \times U(1)$, and then, the adjoint orbit of K, $\mathrm{Ad}(K)A$ is given by

$$\hat{M} = (\mathrm{Spin}(10) \times U(1))/(SU(4) \times U(1)) \subset S^{31}.$$

The real hypersurface $\varphi : M \hookrightarrow \mathbb{C}P^{15}$ is a tube over the canonical imbedding of $SO(10)/U(5) \subset \mathbb{C}P^{15}$ with radius u ($0 < u < \frac{\pi}{4}$). The principal curvatures of M are given by

$$\begin{cases} \lambda_1 = -\cot u \quad \text{(with multiplicity } m_1 = 8), \\ \lambda_2 = \cot\left(\frac{\pi}{4} - u\right) \quad \text{(with multiplicity } m_2 = 6), \\ \lambda_3 = \cot\left(\frac{\pi}{2} - u\right) \quad \text{(with multiplicity } m_3 = 8), \\ \lambda_4 = \cot\left(\frac{3\pi}{4} - u\right) \quad \text{(with multiplicity } m_4 = 6), \\ \lambda_5 = -2\cot(2u) \quad \text{(with multiplicity } m_5 = 1). \end{cases} \tag{68}$$

Then,

$$\lambda_1 = -t,\ \lambda_2 = \frac{t+1}{t-1},\ \lambda_3 = \frac{1}{t},\ \lambda_4 = -\frac{t-1}{t+1},\ \lambda_5 = -t + \frac{1}{t},$$

where $t = \cot u$. The mean curvature of M is given by

$$\begin{aligned} H &= \frac{1}{29}\left\{8(-t) + 6\frac{t+1}{t-1} + 8\,\frac{1}{t} - 6\frac{t-1}{t+1} - t + \frac{1}{t}\right\} \\ &= -\frac{9t^4 - 42t^2 + 9}{29t(t^2-1)}. \end{aligned} \tag{69}$$

The constant $\|B(\varphi)\|^2$ is given by

$$\begin{aligned} \|B(\varphi)\|^2 &= 8t^2 + 6\left(\frac{t+1}{t-1}\right)^2 + 8\frac{1}{t^2} \\ &\quad + 6\left(\frac{t-1}{t+1}\right)^2 + \left(-t + \frac{1}{t}\right)^2 \\ &= \frac{E(X)}{X(X-1)^2} - 2, \end{aligned} \tag{70}$$

where

$$E(X) := 21X^3 + 99X^2 - 9X + 9, \tag{71}$$

and $X := t^2$.

Now our main theorem is the following:

14 Theorem. *Let M be any homogeneous real hypersurface in $\mathbb{C}P^n(4)$, so that M is a tube of $A \sim E$ type.*

(I) *Then, for each type, there is a unique u with $0 < u < \frac{\pi}{4}$ in such a way that M is a tube of radius u and is minimal.*

(II) *Assume that M is biharmonic but not minimal. Then, M is one of type A, D or E. More precisely,*

(1) *in the case of A-type, M is a tube $M_{p,q}(u)$ of $\mathbb{C}P^p \subset \mathbb{C}P^n$ ($p \geq 0$ and $q = (n-1) - p$) of radius u with $0 < u < \frac{\pi}{2}$ of which $t = \cot u$ is a solution of the equation*

$$\cot u = \left\{\frac{p+q+3 \pm \sqrt{(p-q)^2 + 4(p+q+2)}}{1+2q}\right\}^{1/2}. \tag{72}$$

(2) *In the case of D-type, M is a tube of the Plücker imbedding $\mathrm{Gr}_2(\mathbb{C}^5) \subset \mathbb{C}P^9$ of radius u with $0 < u < \frac{\pi}{4}$ of which $t = \cot u$ is a unique solution of the equation*

$$41t^6 + 43t^4 + 41t^2 - 15 = 0. \tag{73}$$

I.e., $u = 1.0917\cdots$.

(3) *In the case of E-type, M is a tube of the imbedding* $SO(10)/U(5) \subset \mathbb{C}P^{15}$ *of radius u with* $0 < u < \frac{\pi}{4}$ *of which* $t = \cot u$ *is a unique solution of the equation*

$$13t^6 - 107t^4 + 43t^2 - 9 = 0. \tag{74}$$

I.e., $u = 0.343448\cdots$.

For a proof, see [12].

6 Biharmonic homogeneous real hypersurfaces in the quaternionic projective space

In this section, we show the classification of all the real hypersurfaces curvature adapted in the quaternionic projective space $\mathbb{H}P^n(4)$ which are *biharmonic*.

Let $(N, h) = \mathbb{H}P^n(c)$ be the quaternionic projective space with quaternionic sectional curvature $c > 0$. Then, the Riemannian curvature tensor is given by

$$R(U,V)W = \frac{c}{4}\Big\{h(V,W)U - h(U,W)V + \sum_{\alpha=1}^{3}\big(h(J_\alpha V,W)J_\alpha U - h(J_\alpha U,W)J_\alpha V + 2h(U,J_\alpha V)J_\alpha W\big)\Big\},$$

for vector fields U, V and W on $\mathbb{H}P^n(c)$. Here, J_α ($\alpha = 1, 2, 3$) are the locally defined adapted three almost complex tensors on $\mathbb{H}P^n(c)$ which satisfy $J_1J_2 = -J_2J_1 = J_3$. Then, we have the following theorem whose proof we omit since it is similar to that of Theorem 5.

15 Theorem. *Let* (M, g) *be a real* $(4n-1)$*-dimensional compact Riemannian manifold, and* $\varphi : (M, g) \to \mathbb{H}P^n(c)$ *an isometric immersion with constant non-zero mean curvature* $(n \geq 2)$*. Then, the necessary and sufficient condition for* φ *to be biharmonic is*

$$\|B(\varphi)\|^2 = (n+2)c. \tag{75}$$

Now, let us recall Berndt's classification ([2]) of all the real hypersurfaces (M, g) in the quaternionic projective space $\mathbb{H}P^n(4)$ which are *curvature adapted*, i.e., $J_\alpha\xi$ is a direction of the principal curvature for all $\alpha = 1, 2, 3$, where ξ is the unit normal vector field along M.

16 Theorem. (*Berndt* [*2*]) (I) *All the curvature adapted real hypersurfaces in* $\mathbb{H}P^n(4)$ *are one of the following:*

(1) *a geodesic sphere* $M(u)$ *of radius* u $(0 < u < \frac{\pi}{2})$,

(2) *a tube* $M(u)$ *of radius* u $(0 < u < \frac{\pi}{4})$ *of the complex projective space* $\mathbb{C}P^n \subset \mathbb{H}P^n(4)$, *and*

(3) *tubes* $M_k(u)$ *of radii* u $(0 < u < \frac{\pi}{4})$ *of the quaternionic projective subspaces* $\mathbb{H}P^k \subset \mathbb{H}P^n(4)$ *with* $1 \leq k \leq n-1$.

(II) *Furthermore, their principal curvatures are given as follows.*

(1) *The geodesic sphere* $M(u)$*:*

$$\begin{cases} \lambda_1 = \cot u \text{ (with multiplicity } m_1 = 4(n-1)), \\ \lambda_2 = 2\cot(2u) \text{ (with multiplicity } m_2 = 3). \end{cases} \tag{76}$$

(2) *The tube* $M(u)$ *of the complex projective space:*

$$\begin{cases} \lambda_1 = \cot u \text{ (with multiplicity } m_1 = 2(n-1)), \\ \lambda_2 = -\tan u \text{ (with multiplicity } m_2 = 2(n-1)), \\ \lambda_3 = 2\cot(2u) \text{ (with multiplicity } m_3 = 1), \\ \lambda_4 = -2\tan(2u) \text{ (with multiplicity } m_4 = 2). \end{cases} \tag{77}$$

(3) *The tubes* $M_k(u)$ *of the quaternionic projective spaces:*

$$\begin{cases} \lambda_1 = \cot u \text{ (with multiplicity } m_1 = 4(n-k-1)), \\ \lambda_2 = -\tan u \text{ (with multiplicity } m_2 = 4k), \\ \lambda_3 = 2\cot(2u) \text{ (with multiplicity } m_3 = 3). \end{cases} \tag{78}$$

Then, we obtain the following theorem.

17 Theorem. *For all the three classes* (1), (2) *and* (3) *of Theorem 8, harmonic (i.e, minimal), and biharmonic but not harmonic real hypersurfaces* $M(u)$ *or* $M_k(u)$ *in* $\mathbb{H}P^n(4)$ *with radii* u *are given as follows:*

(1) *The geodesic sphere* $M(u)$: *The necessary and sufficient condition for* $M(u)$ *is to be harmonic* (*i.e., minimal*) *is that* $t = \cot u$ $(0 < u < \frac{\pi}{2})$ *satisfies*

$$t = \sqrt{\frac{3}{4n-1}}, \tag{79}$$

and to be biharmonic but not harmonic is that $t = \cot u$ $(0 < u < \frac{\pi}{2})$ *satisfies*

$$(4n-1)t^4 - 2(2n+7)t^2 + 3 = 0. \tag{80}$$

Both the equations (79) *and* (80) *have always solutions.*

(2) *The tube $M(u)$ of radius u $(0 < u < \frac{\pi}{4})$ of the complex projective space*: *The necessary and sufficient condition for $M(u)$ is to be harmonic* (*i.e., minimal*) *is that*

$$(2n-1)t^4 - (4n+5)t^2 + 2(n-1) = 0, \tag{81}$$

and to be biharmonic but not harmonic is that

$$(2n-1)t^8 - 8(n+1)t^6 - (6n+11)t^4 - 2(2n-1)t^2 - 12 = 0. \tag{82}$$

Both the (81) *and* (82) *have always solutions.*

(3) *The tubes $M_k(u)$ of radii u $(0 < u < \frac{\pi}{4})$ of the quaternionic projective subspaces*: *The necessary and sufficient conditions for $M_k(u)$ to be harmonic* (*i.e., minimal*) *is that*

$$t = \sqrt{\frac{4k+3}{4n-4k-1}}, \tag{83}$$

and to be biharmonic but not harmonic is that

$$(4n-4k-1)t^4 - 2(2n+4)t^2 + 4k + 3 = 0. \tag{84}$$

Both the (83) *and* (84) *have always solutions.*

For a proof, see [12].

7 Biharmonic maps into a manifold of nonpositive curvature

In this section, we show answers in case of bounded geometry, to the following conjectures proposed by B.Y. Chen ([5]), and R. Caddeo, S. Montaldo and P. Piu ([4]):

B.Y. Chen's Conjecture. *Any biharmonic submanifold of the Euclidean space is harmonic.*

or more generally,

R. Caddeo, S. Montaldo and P. Piu's conjecture. *The only biharmonic submanifolds of a complete Riemanian manifold whose curvature is nonpositive are the minimal ones.*

18 Example. Let $\varphi : (\mathbb{R}^m, g_0) \ni x = (x_1, \ldots, x_m) \mapsto (\varphi_1, \ldots, \varphi_n) \in (\mathbb{R}^n, h_0)$ be a smooth mapping given by

$$\varphi_i(x) = \sum_{j=1}^{m} {x_j}^4 - m\, {x_i}^4 \quad (i = 1, \cdots, m), \tag{85}$$

and $\varphi_j(x)$ $(j = m+1,\dots,n)$ are at most linear, where $(\mathbb{R}^m, g_0)$ and $(\mathbb{R}^n, h_0)$ are the standard Euclidean spaces, respectively. Then, we have

$$\begin{cases} \tau(\varphi) = \Delta\varphi = (\Delta\varphi_1,\dots,\Delta\varphi_n), \\ \tau_2(\varphi) = \Delta(\Delta\varphi) = 0, \end{cases} \tag{86}$$

where

$$\Delta\varphi_i = 12\left(\sum_{j=1}^m {x_j}^2 - m\,{x_i}^2\right) \quad (i = 1,\dots,m). \tag{87}$$

Furthermore, we have

$$\|\tau(\varphi)\|^2 = 12^2\, m\left(m\sum_{j=1}^m {x_j}^4 - \left(\sum_{j=1}^m {x_j}^2\right)^2\right) \geq 0, \tag{88}$$

$$\|\overline{\nabla}\tau(\varphi)\|^2 = 24^2\, m(m-1)\left(\sum_{j=1}^m {x_j}^2\right)^2. \tag{89}$$

However, we show

19 Theorem. *Let $\varphi : (M,g) \to (N,h)$ be a biharmonic map from a complete Riemannian manifold (M,g) of bounded sectional curvature, $|\mathrm{Riem}^M| \leq C$ into a Riemannian manifold (N,h) of nonpositive curvature, i.e.,* $\mathrm{Riem}^N \leq 0$. *Assume that the tension field $\tau(\varphi)$ satisfies*

$$\|\tau(\varphi)\| \in L^2(M), \;\; \textit{and} \quad \|\overline{\nabla}\tau(\varphi)\| \in L^2(M). \tag{90}$$

Then, $\varphi : (M,g) \to (N,h)$ is harmonic.

20 Corollary. *Let $\varphi : (M,g) \to (N,h)$ be a biharmonic isometric immersion from a complete Riemannian manifold (M,g) of bounded sectional curvature $|\mathrm{Riem}^M| \leq C$ into a Riemannian manifold (N,h) of nonpositive curvature, i.e.,* $\mathrm{Riem}^N \leq 0$. *Assume that the second fundamental form $\tau(\varphi)$ satisfies that*

$$\|\tau(\varphi)\| \in L^2(M), \;\; \textit{and} \quad \|\overline{\nabla}\tau(\varphi)\| \in L^2(M). \tag{91}$$

Then $\varphi : (M,g) \to (N,h)$ is harmonic.

For a proof, we use a cut off function λ_R $(0 < R < \infty)$ on a complete Riemannian manifold (M,g) as follows ([7]). Let μ be a real valued C^∞ function on $\mathbb{R}$ satisfying the following conditions:

$$\begin{cases} 0 \leq \mu(t) \leq 1 \quad (t \in \mathbb{R}), \\ \mu(t) = 1 \qquad (t \leq 1), \\ \mu(t) = 0 \qquad (t \geq 2), \\ |\mu'| \leq C, \text{ and } |\mu''| \leq C, \end{cases} \tag{92}$$

where $\mu'(t)$ and $\mu''(t)$ stand for the derivations of the first and second order of $\mu(t)$ with respect to t, respectively. Then, for all $R > 0$, the function defined by

$$\lambda_R(x) = \mu\left(\frac{r(x)}{R}\right), \quad (x \in M)$$

is said to be a *cut off function* on (M, g), where

$$r(x) = d(x_0, x), \quad (x \in M)$$

for some fixed point x_0 in M and $d(x, y)$, $(x, y \in M)$ is the Riemannian distance function of (M, g).

Then, we have

$$\int_M \Delta(\lambda_R\, e_2(\varphi))v_g = \int_M \operatorname{div}(X)\, v_g = 0, \tag{93}$$

where $X = \nabla(\lambda_R\, e_2(\varphi))$ is a C^∞ vector field on M with compact support. We calculate the left hand side, and use the Weitzenböck formula, the biharmonicity of φ and $\mathrm{Riem}^N \leq 0$, we have $\tau(\varphi) = 0$. For details, see [12].

8 The first and second variational formulas for bi-Yang-Mills fields

From this section, we begin to state interesting phenomena on bi-Yang-Mills fields which are closely related to biharmonic maps. We will recall the Yang-Mills setting ([3]) and the definition of bi-Yang-Mills fields following Bejan and Urakawa ([1]). We give the second variation formula and isolation phenomena of bi-Yang-Mills fields.

Let us start with the Yang-Mills setting following [3]. Let (E, h) be a real vector bundle of rank r with an inner product h over an m-dimensional compact Riemannian manifold (M, g). Let $\mathcal{C}(E, h)$ be the space of all C^∞-connections of E satisfying the compatibility condition:

$$Xh(s, t) = h(\nabla_X s, t) + h(s, \nabla_X t), \quad s,\, t \in \Gamma(E),$$

for all $X \in \mathfrak{X}(M)$, where $\Gamma(E)$ stands for the space of all C^∞-sections of E. For $\nabla \in \mathcal{C}(E,h)$, let R^∇ be its curvature tensor defined by

$$R^\nabla(X,Y)s = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]}s,$$

for all $X, Y \in \mathfrak{X}(M)$, $s \in \Gamma(E)$. Let $F = \mathrm{End}(E,h)$ be the bundle of endomorphisms of E which are skew symmetric with respect to the inner product h on E. We define the inner product $\langle\, ,\, \rangle$ on F by

$$\langle \varphi, \psi \rangle = \sum_{i=1}^r h(\varphi u_i, \psi u_i), \quad \varphi, \psi \in F_x,$$

where $\{u_i\}_{i=1}^r$ is an orthonormal basis of E_x with respect to h $(x \in M)$. Let us also consider the space of F-valued k-forms on M, denoted by $\Omega^k(F) = \Gamma(\wedge^k T^*M) \otimes F)$, which admits a global inner product $(\, ,\,)$ given by

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle v_g,$$

where the pointwise inner product $\langle \alpha, \beta \rangle$ is given by

$$\langle \alpha, \beta \rangle = \sum_{i_1 < \cdots < i_k} \langle \alpha(e_{i_1}, \ldots, e_{i_k}), \beta(e_{i_1}, \ldots, e_{i_k}) \rangle$$

and $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame field on (M, g).

For every $\nabla \in \mathcal{C}(E,h)$, let $d^\nabla : \Omega^k(F) \to \Omega^{k+1}(F)$ be the exterior differentiation with respect to ∇ (cf. [3]), and the adjoint operator $\delta^\nabla : \Omega^{k+1}(F) \to \Omega^k(F)$ given by

$$\delta^\nabla \alpha = (-1)^{k+1} * d^\nabla * \alpha, \quad \alpha \in \Omega^{k+1} F),$$

where $* : \Omega^p(F) \to \Omega^{m-p}(F)$ is the extension of the usual Hodge star operator on (M, g). Then, it holds that

$$(d^\nabla \alpha, \beta) = (\alpha, \delta^\nabla \beta), \quad \alpha \in \Omega^k(F),\ \beta \in \Omega^{k+1}(F).$$

Now let us recall the bi-Yang-Mills functional (see [1]) and Yang-Mills one (see [3]):

21 Definition.

$$\mathcal{YM}_2(\nabla) = \frac{1}{2} \int_M \|\delta^\nabla R^\nabla\|^2 v_g, \quad \nabla \in \mathcal{C}(E,h), \tag{94}$$

$$\mathcal{YM}(\nabla) = \frac{1}{2} \int_M \|R^\nabla\|^2 v_g, \quad \nabla \in \mathcal{C}(E,h), \tag{95}$$

where $\|\delta^\nabla R^\nabla\|$, (resp. $\|R^\nabla\|$) is the norm of $\delta^\nabla R^\nabla \in \Omega^1(F)$ (resp. $R^\nabla \in \Omega^2(F)$) relative to each $\langle\, ,\, \rangle$.

Then, the bi-Yang-Mills fields and the Yang-Mills ones are critical points of the above functionals as follows.

22 Definition. A connection $\nabla \in \mathcal{C}(E,h)$ is said to be a *bi-Yang-Mills field* (resp. *Yang-Mills field*) if for any smooth one-parameter family ∇^t ($|t| < \epsilon$) with $\nabla^0 = \nabla$,

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{YM}_2(\nabla^t) = 0, \quad \left(resp.\ \frac{d}{dt}\bigg|_{t=0} \mathcal{YM}(\nabla^t) = 0 \right). \tag{96}$$

Then, the first variation formulas are given as

23 Theorem. *([1], [3]) Let $\alpha = \frac{d}{dt}|_{t=0}\nabla^t \in \Omega^1(F)$. Then, we have*

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{YM}_2(\nabla^t) = \int_M \langle (\delta^\nabla d^\nabla + \mathcal{R}^\nabla)(\delta^\nabla R^\nabla), \alpha\rangle v_g, \tag{97}$$

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{YM}(\nabla^t) = \int_M \langle \delta^\nabla R^\nabla, \alpha\rangle v_g, \tag{98}$$

respectively. Here, $\mathcal{R}^\nabla(\beta) \in \Omega^1(F)$ ($\beta \in \Omega^1(F)$) is defined by

$$\mathcal{R}^\nabla(\beta)(X) = \sum_{j=1}^m [R^\nabla(e_j, X), \beta(e_j)], \quad X \in \mathfrak{X}(M). \tag{99}$$

Thus, ∇ is a bi-Yang-Mills field (resp. Yang-Mills one) if and only if

$$(\delta^\nabla d^\nabla + \mathcal{R}^\nabla)(\delta^\nabla R^\nabla) = 0 \quad (resp.\ \delta^\nabla R^\nabla = 0). \tag{100}$$

Thus, by this theorem, we have immediately

24 Corollary. *If ∇ is a Yang-Mills field, then it is also a bi-Yang-Mills one.*

The second variation formula for the Yang-Mills field was given by Bourguignon and Lawson ([3]) as follows.

25 Theorem. *Let $\nabla \in \mathcal{C}(E,h)$ be a Yang-Mills field, and ($|t| < \epsilon$), any smooth one-parameter family ∇^t with $\nabla^0 = \nabla$. Then,*

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{YM}(\nabla^t) = \int_M \langle (\delta^\nabla d^\nabla + \mathcal{R}^\nabla)(\alpha), \alpha\rangle, \tag{101}$$

where $\alpha = \frac{d}{dt}|_{t=0}\nabla^t \in \Omega^1(F)$. Therefore, if $\delta^\nabla \alpha = 0$, then,

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{YM}(\nabla^t) = \int_M \langle \mathcal{S}^\nabla(\alpha), \alpha\rangle v_g, \tag{102}$$

where $\mathcal{S}^\nabla$ is the second order selfadjoint elliptic differential operator acting on $\Omega^1(F)$ defined by

$$\mathcal{S}^\nabla(\alpha) = (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\alpha) + \mathcal{R}^\nabla(\alpha). \tag{103}$$

Now, we want to give the second variational formula for the bi-Yang-Mills field.

26 Theorem. *Let $\nabla \in \mathcal{C}(E,h)$ be a bi-Yang-Mills field and ∇^t ($|t| < \epsilon$), any smooth one-parameter family in $\mathcal{C}(E,h)$ with $\nabla^0 = \nabla$. Then, we have*

$$\begin{aligned} \frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{Y}M_2(\nabla^t) = \int_M &\langle (\delta^\nabla d^\nabla + \mathcal{R}^\nabla)^2(\alpha) \\ &+ 2\delta^\nabla([\alpha \wedge \delta^\nabla R^\nabla]) + \mathcal{R}(d^\nabla \delta^\nabla R^\nabla)(\alpha), \alpha\rangle v_g. \end{aligned} \tag{104}$$

If $\delta^\nabla \alpha = 0$, then (104) can be written as

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{Y}M_2(\nabla^t) = \int_M \langle \mathcal{S}_2^\nabla(\alpha), \alpha\rangle v_g. \tag{105}$$

Here, $\mathcal{S}_2^\nabla$ is the fourth order selfadjoint elliptic differential operator acting on $\Omega^1(F)$ defined by

$$\mathcal{S}_2^\nabla(\alpha) = \mathcal{S}^\nabla(\mathcal{S}^\nabla(\alpha)) + 2\delta^\nabla([\alpha \wedge \delta^\nabla R^\nabla]) + \mathcal{R}(d^\nabla \delta^\nabla R^\nabla)(\alpha). \tag{106}$$

Here, for α, $\beta \in \Omega^1(F)$, $[\alpha \wedge \beta] \in \Omega^2(F)$ is defined (cf. [3]) by

$$[\alpha \wedge \beta](X,Y) = [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)], \quad X, Y \in \mathfrak{X}(M),$$

and, for $\varphi \in \Omega^2(F)$, $\mathcal{R}(\varphi)(\alpha) \in \Omega^1(F)$ is defined by

$$\mathcal{R}(\varphi)(\alpha)(X) = \sum_{j=1}^m [\varphi(e_j, X), \alpha(e_j)], \quad X \in \mathfrak{X}(M). \tag{107}$$

Notice that $\mathcal{R}^\nabla = \mathcal{R}(R^\nabla)$ if we take $\varphi = R^\nabla$.

27 Remark. If ∇ is a Yang-Mills field, then it is an absolute minimum of $\mathcal{Y}M_2$ by definition, but one can also see its stability by means of the equality that

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{Y}M_2(\nabla^t) = \int_M \|\mathcal{S}_2^\nabla(\alpha)\|^2 v_g = \int_M \|\mathcal{S}^\nabla(\alpha)\|^2 v_g \tag{108}$$

due to the self-adjointness of $\mathcal{S}_2^\nabla$, (106) and $\delta^\nabla R^\nabla = 0$.

In order to see Theorem 13, we need some Lemmas.

28 Lemma. *Let $\nabla \in \mathcal{C}(E,h)$ be a connection in $\mathcal{C}(E,h)$, and ∇^t ($|t| < \epsilon$), any smooth one-parameter family in $\mathcal{C}(E,h)$ with $\nabla^0 = \nabla$. Then, for any $\beta \in \Omega^1(F)$, we have*

$$\left.\frac{d}{dt}\right|_{t=0} d^{\nabla^t}\beta = [\alpha \wedge \beta], \quad \left.\frac{d^2}{dt^2}\right|_{t=0} d^{\nabla^t}\beta = [\gamma \wedge \beta], \tag{109}$$

where $\alpha = \frac{d}{dt}|_{t=0}\nabla^t \in \Omega^1(F)$, and $\gamma = \frac{d^2}{dt^2}|_{t=0}\nabla^t \in \Omega^1(F)$.

Proof. To see (109), we first notice that, for $X, Y \in \mathfrak{X}(M)$,

$$(d^{\nabla^t}\beta)(X,Y) = \nabla^t_X(\beta(Y)) - \nabla^t_Y(\beta(X)) - \beta([X,Y]), \tag{110}$$

$$\frac{d}{dt}\nabla^t_X(\beta(Y)) = \left[\left(\frac{d}{dt}\alpha^t\right)(X), \beta(Y)\right], \tag{111}$$

where we put $\nabla^t = \nabla + \alpha^t$, with $\alpha^t \in \Omega^1(F)$. Indeed, (110) is by definition, and for (111), we note for $u \in \Gamma(E)$,

$$\begin{aligned}(\nabla^t_X(\beta(Y))u &= \nabla^t_X(\beta(Y)u) - \beta(Y)(\nabla^t_X u)\\ &= \nabla_X(\beta(Y))(u) + \alpha^t(X)(\beta(Y)u) - \beta(Y)(\alpha^t(X)u),\end{aligned}$$

so that, by differentiating in t, we have

$$\begin{aligned}\frac{d}{dt}(\nabla^t_X(\beta(Y))u &= \left(\frac{d}{dt}\alpha^t\right)(X)(\beta(Y)u) - \beta(Y)\left(\left(\frac{d}{dt}\alpha^t\right)(X)u\right)\\ &= \left[\left(\frac{d}{dt}\alpha^t\right)(X), \beta(Y)\right]u,\end{aligned}$$

which implies (111).

Thus, we have (109) immediately by (110) and (111). QED

29 Lemma. *For all β_1, $\beta_2 \in \Omega^1(F)$, and $\varphi \in \Omega^2(F)$, we have*

$$\langle \varphi, [\beta_1 \wedge \beta_2]\rangle = \langle \mathcal{R}(\varphi)(\beta_2), \beta_1\rangle = \langle \beta_2, \mathcal{R}(\varphi)(\beta_1)\rangle. \tag{112}$$

Proof. For the first equality, we have

$$\begin{aligned}
\langle \varphi, [\beta_1 \wedge \beta_2] \rangle &= \sum_{i<j} \langle \varphi(e_i, e_j), [\beta_1 \wedge \beta_2](e_i, e_j) \rangle \\
&= \sum_{i<j} \varphi(e_i, e_j), [\beta_1(e_i), \beta_2(e_j)] - [\beta_1((e_j), \beta_2(e_i)] \rangle \\
&= \sum_{i,j=1}^{m} \langle \varphi(e_i, e_j), [\beta_1(e_i), \beta_2(e_j)] \rangle \\
&= \sum_{i=1}^{m} \left\langle \sum_{j=1}^{m} [\varphi(e_j, e_i), \beta_2(e_j)], \beta_1(e_i) \right\rangle \\
&= \sum_{i=1}^{m} \langle \mathcal{R}(\varphi)(\beta_2)(e_i), \beta_1(e_i) \rangle \\
&= \langle \mathcal{R}(\varphi)(\beta_2), \beta_1 \rangle,
\end{aligned}$$

since $\langle [\eta, \psi], \xi \rangle + \langle \psi, [\eta, \xi] \rangle = 0$ for all endomorphisms η, ψ, and ξ of E_x $(x \in M)$. By the same reason, for the second equality, we have

$$\begin{aligned}
\langle \mathcal{R}(\varphi)(\beta_2), \beta_1 \rangle &= \sum_{i=1}^{m} \left\langle \sum_{j=1}^{m} [\varphi(e_j, e_i), \beta_2(e_j)], \beta_1(e_i) \right\rangle \\
&= -\sum_{i,j=1}^{m} \langle \beta_2(e_j), [\varphi(e_j, e_i), \beta_1(e_i)] \rangle \\
&= \sum_{j=1} \langle \beta_2(e_j), \sum_{i=1}^{m} [\varphi(e_i, e_j), \beta_1(e_i)] \rangle \\
&= \sum_{j=1}^{m} \langle \beta_2(e_j), \mathcal{R}(\varphi)(\beta_1)(e_j) \rangle \\
&= \langle \beta_2, \mathcal{R}(\varphi)(\beta_1) \rangle,
\end{aligned}$$

thus, we obtain (112). QED

30 Lemma. *We have*

$$\left.\frac{d}{dt}\right|_{t=0} R^{\nabla^t} = d^{\nabla} \alpha, \quad \left.\frac{d^2}{dt^2}\right|_{t=0} R^{\nabla^t} = d^{\nabla} \gamma + [\alpha \wedge \alpha], \tag{113}$$

where $\alpha = \frac{d}{dt}|_{t=0} \nabla^t$, *and* $\gamma = \frac{d^2}{dt^2}|_{t=0} \nabla^t$.

Proof. If we write $\nabla^t = \nabla + \alpha^t$ with $\alpha^t \in \Omega^1(F)$, it is known (cf. (2.20) in [3], p.196) that

$$R^{\nabla^t} = R^{\nabla} + d^{\nabla}\alpha^t + \frac{1}{2}[\alpha^t \wedge \alpha^t]. \tag{114}$$

Differentiating (114), and doing it twice at $t = 0$, we have (113). QED

31 Lemma. *We have*

$$\left.\frac{d}{dt}\right|_{t=0} \delta^{\nabla^t}\varphi = \mathcal{R}(\varphi)(\alpha), \quad \varphi \in \Omega^2(F). \tag{115}$$

In particular, we have

$$\left.\frac{d}{dt}\right|_{t=0} \delta^{\nabla^t} R^{\nabla} = \mathcal{R}^{\nabla}(\alpha). \tag{116}$$

Furthermore, we have

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \delta^{\nabla^t} R^{\nabla} = \mathcal{R}^{\nabla}(\gamma), \tag{117}$$

where $\mathcal{R}(\varphi)$ *is given by* (107), $\alpha = \frac{d}{dt}|_{t=0}\nabla^t$ *and* $\gamma = \frac{d^2}{dt^2}|_{t=0}\nabla^t$.

Proof. For all $\beta \in \Omega^1(F)$, we have

$$\begin{aligned}
\left(\left.\frac{d}{dt}\right|_{t=0} \delta^{\nabla^t}\varphi, \beta\right) &= \left.\frac{d}{dt}\right|_{t=0} \left(\delta^{\nabla^t}\varphi, \beta\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\varphi, d^{\nabla^t}\beta\right) \\
&= \left(\varphi, \left.\frac{d}{dt}\right|_{t=0} d^{\nabla^t}\beta\right) \\
&= (\varphi, [\alpha \wedge \beta]) \qquad \text{(by Lemma 4)} \\
&= (\mathcal{R}(\varphi)(\alpha), \beta) \qquad \text{(by Lemma 5)},
\end{aligned}$$

which implies (115). If we take $\varphi = R^{\nabla}$ in (115), we have (116). By the same way, we have

$$\begin{aligned}
\left(\left.\frac{d^2}{dt^2}\right|_{t=0} \delta^{\nabla^t}\varphi, \beta\right) &= \left.\frac{d^2}{dt^2}\right|_{t=0} \left(\delta^{\nabla^t}\varphi, \beta\right) \\
&= \left.\frac{d^2}{dt^2}\right|_{t=0} \left(\varphi, d^{\nabla^t}\beta\right) \\
&= \left(\varphi, \left.\frac{d^2}{dt^2}\right|_{t=0} d^{\nabla^t}\beta\right) \\
&= (\varphi, [\gamma \wedge \beta]) \qquad \text{(by Lemma 4)} \\
&= (\mathcal{R}(\varphi)(\gamma), \beta) \qquad \text{(by Lemma 5)},
\end{aligned}$$

which implies (117). QED

32 Lemma. *We have*

$$\left.\frac{d}{dt}\right|_{t=0} \delta^{\nabla^t} R^{\nabla^t} = \mathcal{R}^{\nabla}(\alpha) + \delta^{\nabla} d^{\nabla} \alpha, \tag{118}$$

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} \delta^{\nabla^t} R^{\nabla^t} &= \mathcal{R}^{\nabla}(\gamma) + 2\mathcal{R}(d^{\nabla}\alpha)(\alpha) \\ &\quad + \delta^{\nabla} d^{\nabla} \gamma + \delta^{\nabla}[\alpha \wedge \alpha],\end{aligned} \tag{119}$$

where $\mathcal{R}(\varphi)$ *is given by (107),* $\alpha = \frac{d}{dt}|_{t=0} \nabla^t$ *and* $\gamma = \frac{d^2}{dt^2}|_{t=0} \nabla^t$.

Proof. For the first equation, we have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \delta^{\nabla^t} R^{\nabla^t} &= \left.\frac{d}{dt}\right|_{t=0} \delta^{\nabla^t} R^{\nabla} + \delta^{\nabla} \left.\frac{d}{dt}\right|_{t=0} R^{\nabla^t} \\ &= \mathcal{R}^{\nabla}(\alpha) + \delta^{\nabla} d^{\nabla} \alpha,\end{aligned} \tag{120}$$

by (116) and (113). For the second equation,

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} \delta^{\nabla^t} R^{\nabla^t} &= \left(\left.\frac{d^2}{dt^2}\right|_{t=0} \delta^{\nabla^t}\right) R^{\nabla} + 2\left(\left.\frac{d}{dt}\right|_{t=0} \delta^{\nabla^t}\right)\left(\left.\frac{d}{dt}\right|_{t=0} R^{\nabla^t}\right) \\ &\quad + \delta^{\nabla}\left(\left.\frac{d^2}{dt^2}\right|_{t=0} R^{\nabla^t}\right) \\ &= \mathcal{R}^{\nabla}(\gamma) + 2\mathcal{R}(d^{\nabla}\alpha)(\alpha) + \delta^{\nabla} d^{\nabla}\gamma + \delta^{\nabla}[\alpha \wedge \alpha],\end{aligned}$$

by (117), (115), (113). We have Lemma 8. QED

Now we are in position to give a proof of Theorem 13.

Proof. The second derivative of the integrand of $\mathcal{YM}_2(\nabla^t)$ in t at $t = 0$ is given by

$$\begin{aligned}\frac{1}{2}\left.\frac{d^2}{dt^2}\right|_{t=0} \|\delta^{\nabla^t} R^{\nabla^t}\|^2 &= \left\langle \left.\frac{d^2}{dt^2}\right|_{t=0} \delta^{\nabla^t} R^{\nabla^t}, \delta^{\nabla} R^{\nabla} \right\rangle \\ &\quad + \left\langle \left.\frac{d}{dt}\right|_{t=0} \delta^{\nabla^t}, \left.\frac{d}{dt}\right|_{t=0} \delta^{\nabla^t}, \right\rangle \\ &= \left\langle \mathcal{R}^{\nabla}(\gamma) + 2\mathcal{R}(d^{\nabla}\alpha)(\alpha) + \delta^{\nabla} d^{\nabla}\gamma + \delta^{\nabla}[\alpha \wedge \alpha], \delta^{\nabla} R^{\nabla} \right\rangle \\ &\quad + \left\langle \mathcal{R}^{\nabla}(\alpha) + \delta^{\nabla} d^{\nabla}\alpha, \mathcal{R}^{\nabla}(\alpha) + \delta^{\nabla} d^{\nabla}\alpha \right\rangle,\end{aligned} \tag{121}$$

by (118) and (119) in Lemma 8. By integrating (121) over M, we have

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{YM}_2(\nabla^t) &= \big(\mathcal{R}^\nabla(\gamma)+2\mathcal{R}(d^\nabla\alpha)(\alpha)+\delta^\nabla d^\nabla\gamma \\ &\qquad +\delta^\nabla[\alpha\wedge\alpha],\delta^\nabla R^\nabla\big) \\ &\quad +\big(\mathcal{R}^\nabla(\alpha)+\delta^\nabla d^\nabla\alpha,\mathcal{R}^\nabla(\alpha)+\delta^\nabla d^\nabla\alpha\big) \\ &= \big(2\mathcal{R}(d^\nabla\alpha)(\alpha)+\delta^\nabla[\alpha\wedge\alpha],\delta^\nabla R^\nabla\big) \\ &\quad +\big(\mathcal{R}^\nabla(\alpha)+\delta^\nabla d^\nabla\alpha,\mathcal{R}^\nabla(\alpha)+\delta^\nabla d^\nabla\alpha\big), \end{aligned} \tag{122}$$

because

$$\big(\mathcal{R}^\nabla(\gamma)+\delta^\nabla d^\nabla\gamma,\delta^\nabla R^\nabla\big)=\big(\gamma,\big(\mathcal{R}^\nabla+\delta^\nabla d^\nabla\big)\left(\delta^\nabla R^\nabla\right)\big)=0,$$

since ∇ is a bi-Yang-Mills filed, i.e., $\big(\mathcal{R}^\nabla+\delta^\nabla d^\nabla\big)\left(\delta^\nabla R^\nabla\right)=0$.

Furthermore, for the first term of the RHS of (122), we have

$$\begin{aligned}(2\mathcal{R}(d^\nabla\alpha)(\alpha)+\delta^\nabla[\alpha\wedge\alpha],\delta^\nabla R^\nabla) &= 2(\mathcal{R}(d^\nabla\alpha)(\alpha),\delta^\nabla R^\nabla) \\ &\quad +([\alpha\wedge\alpha],d^\nabla\delta^\nabla R^\nabla) \\ &= 2(d^\nabla\alpha,[\alpha\wedge\delta^\nabla R^\nabla])+(\alpha,\mathcal{R}(d^\nabla\delta^\nabla R^\nabla)(\alpha)) \\ &= (\alpha,2\delta^\nabla[\alpha\wedge\delta^\nabla R^\nabla]+\mathcal{R}(d^\nabla\delta^\nabla R^\nabla)(\alpha)),\end{aligned} \tag{123}$$

by Lemma 5, and, we have that the second term of the RHS of (122) coincides with

$$(\alpha,(\mathcal{R}^\nabla+\delta^\nabla d^\nabla)^2(\alpha)), \tag{124}$$

since $\mathcal{R}^\nabla+\delta^\nabla d^\nabla$ is selfadjoint with respect to the global inner product $(\,,\,)$. Due to (122), (123) and (124), we obtain the desired (104). *QED*

Due to Theorems 12 and 13, one can define the indices, nullities and stability of bi-Yang-Mills fields (Yang-Mills ones) as follows.

33 Definition. Assume that $\nabla\in\mathcal{C}(E,h)$ is a bi-Yang-Mills field (resp. a Yang-Mills field), and let us denote by E^2_λ (resp. E_λ) the eigenspace of $\mathcal{S}^\nabla_2$ (resp. $\mathcal{S}^\nabla$) on $\Omega^1(F)$ with the eigenvalue λ. Since $\mathcal{S}^\nabla_2$ (resp. $\mathcal{S}^\nabla$) is a selfadjoint elliptic differential operator, and preserves $\mathrm{Ker}(\delta^\nabla)$ invariant, the restriction of $\mathcal{S}^\nabla_2$ (resp. $\mathcal{S}^\nabla$) to $\mathrm{Ker}(\delta^\nabla)$ has a discrete spectrum consisting of distinct eigenvalues $\lambda_1^2<\lambda_2^2<\cdots<\lambda_i^2<\cdots\to\infty$ (resp. $\lambda_1<\lambda_2<\cdots<\lambda_i<\cdots\to\infty$) with their corresponding finite dimensional eigenspaces $E^2_{\lambda_i}$ (resp. E_{λ_i}). Then, the *index* and *nullity* of ∇ are defined by

$$\mathrm{Index}_2(\nabla) = \dim\left(\bigoplus_{\lambda<0} E^2_\lambda\right), \quad \mathrm{Nullity}_2(\nabla) = \dim(E^2_0), \tag{125}$$

$$\mathrm{Index}(\nabla) = \dim\left(\bigoplus_{\lambda<0} E_\lambda\right), \quad \mathrm{Nullity}(\nabla) = \dim(E_0), \tag{126}$$

respectively. Here, E^2_λ, and E_λ are the eigenspaces of $\mathcal{S}^\nabla_2$, and $\mathcal{S}^\nabla$ with the eigenvalue λ, respectively.

Then, due to Theorem 13, we obtain the similar result for bi-Yang-Mills fields as Corollary 1 for biharmonic maps.

34 Corollary. *Assume that* $\nabla \in \mathcal{C}(E, h)$ *is a Yang-Mills field. Then,* $\mathrm{Index}_2(\nabla) = 0$ *and* $\mathrm{Nullity}_2(\nabla) = \mathrm{Nullity}(\nabla)$*.*

Proof. Indeed, if ∇ is a Yang-Mills field, then, $\delta^\nabla R^\nabla = 0$, so that

$$\mathcal{S}^\nabla_2(\alpha) = \mathcal{S}(\alpha) \tag{127}$$

for all $\alpha \in \Omega^1(F)$. It is clear by (104), and (127) that $\mathrm{Index}_2(\nabla) = 0$ which follows also by definition, and we have

$$\begin{aligned}\{\alpha \in \Omega^1(F) \cap \mathrm{Ker}(\delta^\nabla);\, \mathcal{S}_2(\alpha) = 0\} &= \{\alpha;\, \mathcal{S}^\nabla(\mathcal{S}^\nabla(\alpha)) = 0\} \\ &= \{\alpha;\, \mathcal{S}^\nabla(\alpha) = 0\}\end{aligned} \tag{128}$$

which implies that $\mathrm{Nullity}_2(\nabla) = \mathrm{Nullity}(\nabla)$. QED

9 Isolation phenomena for bi-Yang-Mills fields

In this section, we finally show very interesting phenomena which assert that Yang-Mills fields are isolated among all bi-Yang-Mills fields over compact Riemannian manifolds with positive Ricci curvature.

35 Theorem. (*bounded isolation phenomena*) *Let* (M, g) *a compact Riemannian of which Ricci curvature is bounded below by a positive constant* $k > 0$*, i.e.,* $\mathrm{Ric} \geq k\,\mathrm{Id}$*. Assume that* $\nabla \in \mathcal{C}(E, h)$ *is a bi-Yang-Mills field with* $\|R^\nabla\| < \frac{k}{2}$ *pointwisely everywhere on* M*. Then,* ∇ *is a Yang-Mills field.*

36 Theorem. (L^2*-isolation phenomena*) *Let* (M, g) *be a four dimensional compact Riemannian manifold of which Ricci curvature is bounded below by a* *positive constant* $k > 0$*, i.e.,* $\mathrm{Ric} \geq k\,\mathrm{Id}$*. Assume that* $\nabla \in \mathcal{C}(E,h)$ *is a bi-Yang-Mills field satisfying that*

$$\|R^\nabla\|_{L^2} < \frac{1}{2}\min\left\{\frac{\sqrt{c_1}}{18},\ \frac{k}{2}\,\mathrm{Vol}(M,g)^{1/2}\right\}. \tag{129}$$

Then, ∇ *is a Yang-Mills field. Here,* c_1 *is the isoperimetric constant of* (M,g) *given by*

$$c_1 = \inf_{W\subset M} \frac{\mathrm{Vol}_3(W)^4}{(\min\{\mathrm{Vol}(M_1),\,\mathrm{Vol}(M_2)\})^3}, \tag{130}$$

where $W \subset M$ *runs over all the hypersurfaces in* M*, and* $\mathrm{Vol}_3(W)$ *is the three dimensional volume of* W *with respect to the Riemannian metric on* W *induced from* g*, and the complement of* W *in* M *has a disjoint union of* M_1 *and* M_2*.*

To prove Theorem 14, we need the following Weitzenböck formula.

37 Lemma. *Assume that* $\nabla \in \mathcal{C}(E,h)$ *is a bi-Yang-Mills field. Then,*

$$\begin{aligned}\frac{1}{2}\Delta\|\delta^\nabla R^\nabla\|^2 &= \langle 2\mathcal{R}^\nabla(\delta^\nabla R^\nabla) + \delta^\nabla R^\nabla \circ \mathrm{Ric}, \delta^\nabla R^\nabla\rangle \\ &\quad + \sum_{i=1}^m \|\nabla_{e_i}(\delta^\nabla R^\nabla)\|^2.\end{aligned} \tag{131}$$

Here, $\Delta f = \sum_{i=1}^m ({e_i}^2 - \nabla_{e_i} e_i) f$ *is the Laplacian acting on smooth functions* f *on* M*, and, for all* $\alpha \in \Omega^1(F)$*,*

$$(\alpha \circ \mathrm{Ric})(X) := \alpha(\mathrm{Ric}(X)),\ \ X \in \mathfrak{X}(M), \tag{132}$$

where Ric *is the Ricci transform of* (M,g)*.*

Proof. Indeed, for the LHS of (131), we have

$$\frac{1}{2}\Delta\|\delta^\nabla R^\nabla\|^2 = \langle -\nabla^*\nabla(\delta^\nabla R^\nabla), \delta^\nabla R^\nabla\rangle + \sum_{i=1}^m \langle \nabla_{e_i}(\delta^\nabla R^\nabla), \nabla_{e_i}(\delta^\nabla R^\nabla)\rangle. \tag{133}$$

Let us recall the Weitzenböck formula (cf. [3], p.199, Theorem (3.2)) that

$$\begin{aligned}\Delta^\nabla \alpha &= (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)\alpha \\ &= \nabla^*\nabla\alpha + \alpha \circ \mathrm{Ric} + \mathcal{R}^\nabla(\alpha), \quad \alpha \in \Omega^1(F).\end{aligned} \tag{134}$$

It holds that

$$\delta^\nabla(\delta^\nabla R^\nabla) = 0. \tag{135}$$

Because for all $\varphi \in \Gamma(F)$,

$$(\delta^\nabla(\delta^\nabla R^\nabla), \varphi) = \int_M \langle R^\nabla, d^\nabla(d^\nabla \varphi)\rangle v_g.$$

But, by using the formula (2.9) in [3], p. 194, the integrand of the RHS coincides with

$$\begin{aligned}
\langle R^\nabla, d^\nabla(d^\nabla \varphi)\rangle &= \sum_{i<j}\sum_{s=1}^{r} \langle R^\nabla(e_i,e_j)u_s, (R^\nabla(e_i,e_j)\varphi)(u_s)\rangle \\
&= \sum_{i<j}\sum_{s=1}^{r} \langle R^\nabla(e_i,e_j)u_s, (R^\nabla(e_i,e_j)(\varphi(u_s)) - \varphi(R^\nabla(e_i,e_j)u_s)\rangle \\
&= \sum_{i<j} \langle R^\nabla(e_i,e_j), [R(e_i,e_j),\varphi]\rangle \\
&= -\sum_{i<j} \langle [R^\nabla(e_i,e_j), R^\nabla(e_i,e_j)], \varphi\rangle = 0.
\end{aligned}$$

since $\langle \psi, [\eta,\xi]\rangle = -\langle [\eta,\psi],\xi\rangle$ for all η, ψ, $\xi \in F = \mathrm{End}(E,h)$.

Now ∇ is a bi-Yang-Mills field, $(\delta^\nabla d^\nabla + \mathcal{R}^\nabla)(\delta^\nabla R^\nabla) = 0$, so that we have

$$\begin{aligned}
-\mathcal{R}^\nabla(\delta^\nabla R^\nabla) &= \delta^\nabla d^\nabla(\delta^\nabla R^\nabla) \\
&= \Delta^\nabla(\delta^\nabla R^\nabla) \quad \text{(by (135))} \\
&= \nabla^*\nabla(\delta^\nabla R^\nabla) + \delta^\nabla R^\nabla \circ \mathrm{Ric} + \mathcal{R}^\nabla(\delta^\nabla R^\nabla),
\end{aligned}$$

by (134). Thus, we have

$$-\nabla^*\nabla(\delta^\nabla R^\nabla) = 2\mathcal{R}^\nabla(\delta^\nabla R^\nabla) + \delta^\nabla R^\nabla \circ \mathrm{Ric}. \tag{136}$$

Substituting (136) into the first term of the RHS of (133), we have (131). QED

Proof. By Integrating (131) over M, and by Green's theorem, we have

$$2\int_M \langle \mathcal{R}^\nabla(\delta^\nabla R^\nabla), \delta^\nabla R^\nabla\rangle v_g + \int_M \langle \delta^\nabla R^\nabla \circ \mathrm{Ric}, \delta^\nabla R^\nabla\rangle v_g + \int_M \sum_{i=1}^{m} \langle \nabla_{e_i}(\delta^\nabla R^\nabla), \nabla_{e_i}(\delta^\nabla R^\nabla)\rangle v_g = 0. \tag{137}$$

We use the inequalities

$$|\langle \mathcal{R}^\nabla(\alpha), \alpha\rangle| \le \|R^\nabla\|\,\|\alpha\|^2,\ \alpha \in \Omega^1(F), \tag{138}$$

and

$$\langle \delta^\nabla R^\nabla \circ \mathrm{Ric}, \delta^\nabla R^\nabla\rangle \ge k\,\|\delta^\nabla R^\nabla\|^2. \tag{139}$$

For details, see [12].

QED

10 Appendix: The Euler-Lagrange equations of k-harmonic maps

Eells and Lemaire ([8]) proposed the notion of k-harmonic maps. In this final section, we give their Euler-Lagrange equations for the k-harmonic maps.

38 Definition. Let $k \geq 1$. The *k-energy functional* on the space of smooth maps from a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h) is defined by

$$E_k(f) = \frac{1}{2}\int_M \|(d+\delta)^k f\|^2 \, v_g, \quad f \in C^\infty(M, N). \tag{140}$$

Then, f is k-harmonic if it is a critical point of E_k, i.e., for all variation $\{f_t\}$ of f with $f_0 = f$,

$$\frac{d}{dt}\bigg|_{t=0} E_k(f_t) = 0. \tag{141}$$

In this Appendix A, we show

39 Theorem. *Let $k = 2, 3, \cdots$. Then, we have*

$$\frac{d}{dt}\bigg|_{t=0} E_k(f_t) = -\int_M \langle \tau_k(f), V\rangle \, v_g, \tag{142}$$

where

$$\tau_k(f) := J(W_f) = \overline{\Delta}(W_f) - \mathcal{R}(W_f), \tag{143}$$

and

$$W_f = \underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{k-2}\tau(f) \in \Gamma(f^{-1}TN). \tag{144}$$

The proof goes by a similar way as Section Three of Jiang's paper [15]. Before going into the proofs, we prepare some notations. We retain the following notations. Let ∇, ∇', and $\overline{\nabla}$ be the Levi-Civita connection of $(M, g$, (N, h) and the induced connection of $f^{-1}TN$ from ∇', respectively. Variation $\{f_t\}$ yields a C^∞ map

$$\begin{aligned} F \,:\, M \times I_\epsilon \to N \\ F(p, t) = f_t(p), \quad p \in M, t \in I_\epsilon = (-\epsilon, \epsilon). \end{aligned} \tag{145}$$

Taking the usual Euclidean metric on I_ϵ, with respect to the product Riemannian metric on $M \times I_\epsilon$, we denote by $\boldsymbol{\nabla}$, $\overline{\boldsymbol{\nabla}}$, and $\widetilde{\boldsymbol{\nabla}}$, the induced Riemannian connections on $T(M \times I_\epsilon)$, $F^{-1}TN$, and $T^*(M \times I_\epsilon) \otimes F^{-1}TN$, respectively. If

$\{e_i\}$ is a locally defined orthonormal frame on (M,g), $\{e_i, \frac{\partial}{\partial t}\}$ is also a locally defined orthonormal frame on $M\times I_\epsilon$, and it holds that

$$\boldsymbol{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}=0,\ \boldsymbol{\nabla}_{e_i}e_j=\nabla_{e_i}e_j,\ \boldsymbol{\nabla}_{\frac{\partial}{\partial t}}e_i=\boldsymbol{\nabla}_{e_i}\frac{\partial}{\partial t}=0. \tag{146}$$

It also holds that

$$\begin{aligned}(\widetilde{\nabla}_{e_i}df_t)(e_j)=&\nabla'_{df_t(e_i)}df_t(e_j)-df_t(\nabla_{e_i}e_j)=(\widetilde{\boldsymbol{\nabla}}_{e_i}dF)(e_j),\\ (\widetilde{\nabla}_{e_k}\widetilde{\nabla}_{e_i}df_t)(e_j)&=\nabla'_{df_t(e_k)}((\widetilde{\nabla}_{e_i}df_t)(e_j))-(\widetilde{\nabla}_{e_i}df_t)(\nabla_{e_k}e_j)\\ &=(\widetilde{\boldsymbol{\nabla}}_{e_k}\widetilde{\boldsymbol{\nabla}}_{e_i}dF)(e_j),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned} \tag{147}$$

etc. Here, we used the abbreviated symbol $\widetilde{\nabla}$ on $T^*M\otimes f_t^{-1}TN$ in which we omitted t.

To show Theorem 16, we need the following lemma:

40 Lemma. *Let $k=2,3,\cdots$. The k energy functional E_k is given as follows.*

Case 1: $k=2\ell$, $\ell=1,2\cdots$ (k is even).

$$E_{2\ell}(f)=\frac{1}{2}\int_M \|\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{\ell-1}\,\tau(f)\|^2\,v_g. \tag{148}$$

Case 2: $k=2\ell+1$, $\ell=1,2,\cdots$ (k is odd).

$$E_{2\ell+1}(f)=\frac{1}{2}\int_M \|\overline{\nabla}\left(\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{\ell-1}\,\tau(f)\right)\|^2\,v_g. \tag{149}$$

Proof. To get the lemma, we only notice that

$$(d+\delta)^k f=\begin{cases}\underbrace{\delta d\cdots\delta d}_{\ell}\,f & (k=2\ell,\ \ell=1,2,\cdots)\\ d\ \underbrace{\delta d\cdots\delta d}_{\ell}\,f & (k=2\ell+1,\ \ell=1,2,\cdots),\end{cases} \tag{150}$$

$$\begin{cases}\delta df=-\tau(f),\\ d\delta df=-d(\tau(f))=-\overline{\nabla}\tau(f),\end{cases} \tag{151}$$

and also

$$\delta d\delta df=-\delta(d(\tau(f)))=-\overline{\Delta}\tau(f). \tag{152}$$

Then, inductively, we have

$$\begin{cases} \underbrace{\delta d \cdots \delta d}_{\ell} f = (-1)^{\ell-1} \underbrace{\overline{\Delta} \cdots \overline{\Delta}}_{\ell-1} \tau(f), \\ d\, \underbrace{\delta d \cdots \delta d}_{\ell} f = (-1)^{\ell} \overline{\nabla} \left(\underbrace{\overline{\Delta} \cdots \overline{\Delta}}_{\ell-1} \tau(f) \right), \end{cases} \tag{153}$$

which implies Lemma 10. QED

Let us recall the following lemma (cf. Lemma 3.2 in [15])

41 Lemma. *For all $V \in \Gamma(F^{-1}TN)$, we have*

$$\int_M \langle (\widetilde{\boldsymbol{\nabla}}_{e_i} \widetilde{\boldsymbol{\nabla}}_{e_i} dF)(\frac{\partial}{\partial t}) - (\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_i} e_i} dF)(\frac{\partial}{\partial t}), V \rangle v_g$$
$$= \int_M \langle dF(\frac{\partial}{\partial t}), \overline{\boldsymbol{\nabla}}_{e_k} \overline{\boldsymbol{\nabla}}_{e_k} \overline{\boldsymbol{\nabla}}_{e_k} V - \overline{\boldsymbol{\nabla}}_{\nabla_{e_k} e_k} V \rangle v_g.$$

Proof. Assume that $k = 2\ell$, $\ell = 1, 2, \cdots$. Then, by Lemma 10,

$$\frac{d}{dt} E_{2\ell}(f_t) = \frac{1}{2} \int_M \frac{d}{dt} \langle \underbrace{\overline{\boldsymbol{\Delta}} \cdots \overline{\boldsymbol{\Delta}}}_{\ell-1} \tau(f_t), \underbrace{\overline{\boldsymbol{\Delta}} \cdots \overline{\boldsymbol{\Delta}}}_{\ell-1} \tau(f_t) \rangle v_g$$
$$= \int_M \langle \overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} (\overline{\boldsymbol{\Delta}} \cdots \overline{\boldsymbol{\Delta}}((\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(e_i))), \overline{\boldsymbol{\Delta}} \cdots \overline{\boldsymbol{\Delta}}((\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(e_i))) \rangle v_g. \tag{154}$$

Here, we used (146) , (147),

$$\overline{\boldsymbol{\Delta}} V = -\overline{\boldsymbol{\nabla}}_{e_i} \overline{\boldsymbol{\nabla}}_{e_i} V + \overline{\boldsymbol{\nabla}}_{\nabla_{e_i} e_i} V, \quad V \in \Gamma(T^{-1}TN),$$

and $\tau(f_t) = \widetilde{\nabla}_{e_i} df_t)(e_i)$. Here, let us recall the equation (3.13) in [15]

$$\overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}} ((\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(e_i)) = (\widetilde{\boldsymbol{\nabla}}_{e_i} \widetilde{\boldsymbol{\nabla}}_{e_i} dF)(\frac{\partial}{\partial t}) - (\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_i} e_i} dF)(\frac{\partial}{\partial t})$$
$$+ R^N(dF(\frac{\partial}{\partial t}), dF(e_i)) dF(e_i) \tag{155}$$

by noticing our curvature convention. Then, (154) coincides with

$$
\begin{aligned}
&\int_M \langle \underbrace{\overline{\boldsymbol{\Delta}}\cdots\overline{\boldsymbol{\Delta}}}_{\ell-1}(\overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}((\widetilde{\boldsymbol{\nabla}}_{e_i}dF)(e_i))), \underbrace{\overline{\boldsymbol{\Delta}}\cdots\overline{\boldsymbol{\Delta}}}_{\ell-1}((\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j)))\rangle v_g \\
&\quad = \int_M \langle (\widetilde{\boldsymbol{\nabla}}_{e_i}\widetilde{\boldsymbol{\nabla}}_{e_i}dF)(\frac{\partial}{\partial t}) - (\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_i}e_i}dF)(\frac{\partial}{\partial t}) \\
&\qquad + R^N(dF(\frac{\partial}{\partial t}), dF(e_i))dF(e_i), \underbrace{\overline{\boldsymbol{\Delta}}\cdots\overline{\boldsymbol{\Delta}}}_{2(\ell-1)}(\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j))\rangle v_g \\
&\quad = \int_M \langle dF(\frac{\partial}{\partial t}), (\overline{\boldsymbol{\nabla}}_{e_k}\overline{\boldsymbol{\nabla}}_{e_k} - \overline{\boldsymbol{\nabla}}_{\nabla_{e_k}e_k})\big(\underbrace{\overline{\boldsymbol{\Delta}}\cdots\overline{\boldsymbol{\Delta}}}_{2(\ell-1)}((\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j))\big)\rangle v_g \\
&\quad + \int_M \langle R^N(dF(\frac{\partial}{\partial t}), dF(e_i))dF(e_i), \underbrace{\overline{\boldsymbol{\Delta}}\cdots\overline{\boldsymbol{\Delta}}}_{2(\ell-1)}((\widetilde{\boldsymbol{\nabla}}_{e_j}dF)(e_j))\rangle v_g. \qquad (156)
\end{aligned}
$$

If we put $t = 0$, we have

$$
\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} E_{2\ell}(f_t) &= -\int_M \langle V, \overline{\Delta}\big(\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{2(\ell-1)}\tau(f)\big)\rangle v_g \\
&\quad + \int_M \langle R^N(V, df(e_i))df(e_i), \underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{2(\ell-1)}\tau(f)\rangle v_g \\
&= \int_M \big(\langle V, -\overline{\Delta}W_f\rangle + \langle V, R^N(W_f, df(e_i))df(e_i)\rangle\big) v_g \\
&= -\int_M \langle V, J(W_f)\rangle v_g. \qquad (157)
\end{aligned}
$$

For $E_{2\ell+1}$, $\ell = 1, 2, \cdots$, we have by Lemma 10,

$$
\begin{aligned}
\frac{d}{dt}E_{2\ell+1}(f_t) &= \int_M \langle \overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}(\overline{\boldsymbol{\nabla}}_{e_k}(\underbrace{\overline{\boldsymbol{\Delta}}\cdots\overline{\boldsymbol{\Delta}}}_{\ell-1}((\widetilde{\boldsymbol{\nabla}}_{e_i}dF)(e_i))), \\
&\qquad \overline{\boldsymbol{\nabla}}_{e_k}(\underbrace{\overline{\boldsymbol{\Delta}}\cdots\overline{\boldsymbol{\Delta}}}_{\ell-1}((\widetilde{\boldsymbol{\nabla}}_{e_i}dF)(e_i)))\rangle v_g \\
&= \int_M \langle \overline{\boldsymbol{\nabla}}_{e_k}(\underbrace{\overline{\boldsymbol{\Delta}}\cdots\overline{\boldsymbol{\Delta}}}_{\ell-1}(\overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}(\widetilde{\boldsymbol{\nabla}}_{e_i}dF)(e_i))), \\
&\qquad \overline{\boldsymbol{\nabla}}_{e_k}(\underbrace{\overline{\boldsymbol{\Delta}}\cdots\overline{\boldsymbol{\Delta}}}_{\ell-1}((\widetilde{\boldsymbol{\nabla}}_{e_i}dF)(e_i)))\rangle v_g \\
&= \int_M \langle \overline{\boldsymbol{\nabla}}_{\frac{\partial}{\partial t}}(\widetilde{\boldsymbol{\nabla}}_{e_i}dF)(e_i))), \underbrace{\overline{\boldsymbol{\Delta}}\cdots\overline{\boldsymbol{\Delta}}}_{2\ell-1}((\widetilde{\boldsymbol{\nabla}}_{e_i}dF)(e_i)))\rangle v_g. \qquad (158)
\end{aligned}
$$

Furthermore, the RHS of (158) coincides with the following:

$$\int_M \langle (\widetilde{\boldsymbol{\nabla}}_{e_i}\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(\frac{\partial}{\partial t}) - (\widetilde{\boldsymbol{\nabla}}_{\nabla_{e_i}e_i} dF)(\frac{\partial}{\partial t}) + R^N(dF(\frac{\partial}{\partial t}), dF(e_i))dF(e_i),$$
$$\underbrace{\overline{\boldsymbol{\Delta}}\cdots\overline{\boldsymbol{\Delta}}}_{2\ell-1}((\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(e_i)))\rangle v_g$$
$$= \int_M \langle dF(\frac{\partial}{\partial t}), (\overline{\boldsymbol{\nabla}}_{e_k}\overline{\boldsymbol{\nabla}}_{e_k} - \overline{\boldsymbol{\nabla}}_{\nabla_{e_k}e_k})\{\underbrace{\overline{\boldsymbol{\Delta}}\cdots\overline{\boldsymbol{\Delta}}}_{2\ell-1}((\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(e_i)))\}\rangle v_g$$
$$+ \int_M \langle R^N(dF(\frac{\partial}{\partial t}), dF(e_i))dF(e_i), \underbrace{\overline{\boldsymbol{\Delta}}\cdots\overline{\boldsymbol{\Delta}}}_{2\ell-1}((\widetilde{\boldsymbol{\nabla}}_{e_i} dF)(e_i)))\}\rangle v_g. \quad (159)$$

Thus, we have

$$\frac{d}{dt}\bigg|_{t=0} E_{2\ell+1}(f_t) = \int_M \langle V, -\overline{\Delta}\{\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{2\ell-1}\tau(f)\}\rangle v_g$$
$$+ \int_M \langle R^N(V, df(e_i))df(e_i), \underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{2\ell-1}\tau(f)\rangle v_g$$
$$= \int_M \langle V, -\overline{\Delta}(\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{2\ell-1}\tau(f))) + R^N(\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{2\ell-1}\tau(f), df(e_i))df(e_i)\rangle v_g$$
$$= \int_M \langle V, -\overline{\Delta}W_f + R^N(W_f, df(e_i))df(e_i)\rangle v_g$$
$$= -\int_M \langle V, J(W_f)\rangle v_g, \quad (160)$$

where $W_f := \underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{2\ell-1}\tau(f)$. Therefore, we obtain Theorem 16. *QED*

References

[1] C.L. Bejan, H. Urakawa: *Yang-Mills fields analogue of biharmonic maps*. In: Topics in Almost Hermitian Geometry and Related Fields, World Scientific, (2005), 41–49.

[2] J. Berndt: *Real hypersurfaces in quaternionic space space forms*, J. reine Angew. Math., **419** (1991), 9–26.

[3] J.P. Bourguignon, H.B. Lawson: *Stability and isolation phenomena for Yang-Mills fields*, Commun. Math. Phys. **79** (1981), 189–230.

[4] R. Caddeo, S. Montaldo, P. Piu: *On biharmonic maps*, Contemp. Math., **288** (2001), 286–290.

[5] B.Y. Chen: *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math., **17** (1991), 169–188.

[6] Q. Chen, Y. L. Xin: *Gauss maps of hypersurfaces in the unit sphere* In: Conference of Differential Geometry in Shanghai (1983).

[7] J. Dodziuk, Min-Oo: *An L_2-isolation theorem for Yang-Mills fields over complete manifolds*, Compositio Math., **47** (1982), 165–169.

[8] J. Eells, L. Lemaire: *Selected topics in harmonic maps*, CBMS, **50**, Amer. Math. Soc, 1983.

[9] J. Eells, L. Lemaire: *A report on harmonic maps*, Bull. London Math. Soc., **10** (1978), 1–68.

[10] M.P. Gaffney: *A special Stokes' theorem for complete Riemannian manifolds*, Ann. Math., **60** (1954), 140–145.

[11] W.Y. Hsiang, H.B. Lawson: *Minimal submanifolds of low cohomogeneity*, J. Differ. Geom., **5** (1971), 1–38.

[12] T. Ichiyama, J. Inoguchi, H. Urakawa: *Classifications and isolation phenomena of biharmonic maps and bi-Yang-Mills fields*, to appear in Note di Mat., 2009.

[13] J. Inoguchi: *Submanifolds with harmonic mean curvature in contact 3-manifolds*, Colloq. Math., **100** (2004), 163–179.

[14] S. Ishihara, S. Ishikawa: *Notes on relatively harmonic immersions*, Hokkaido Math. J. **4**(1975), 234–246.

[15] G.Y. Jiang: *2-harmonic maps and their first and second variational formula*, Chinese Ann. Math. , **7A** (1986), 388–402 (the English translation by H. Urakawa, 2009, this volume).

[16] A. Kasue: *Riemannian Geometry*, in Japanese, Baihu-kan, Tokyo, 2001.

[17] S. Kobayashi, K. Nomizu: *Foundation of Differential Geometry, Vol. I, II*, J. Wiley (1963, 1969).

[18] E. Loubeau, C. Oniciuc: *The index of biharmonic maps in spheres*, Compositio Math., **141** (2005), 729–745.

[19] E. Loubeau and C. Oniciuc: *On the biharmonic and harmonic indices of the Hopf map*, Trans. Amer. Math. Soc., **359** (2007), no. 11, 5239–5256.

[20] Min-Oo: *An L_2-isolation theorem for Yang-Mills fields*, Compositio Math., **47** (1982), 153–163.

[21] S. Montaldo, C. Oniciuc: *A short survey on biharmonic maps between Riemannian manifolds*, Rev. Un. Mat. Argentina **47** (2006), 1–22.

[22] H.F. Münzner: *Isoparametrische Hyperfläche in Sphären*, Math. Ann., **251** (1980), 57–71.

[23] C. Oniciuc: *On the second variation formula for biharmonic maps to a sphere*, Publ. Math. Debrecen., **67** (2005), 285–303.

[24] H. Ozeki, M. Takeuchi: *On some types of isoparametric hypersurfaces in sphere I, II*, Tohoku Math. J., **27, 28**, (1975), (1976), 515–559, 7–55.

[25] T. Sasahara: *Legendre surfaces in Sasakian space forms whose mean curvature vectors are eigenvectors*, Publ. Math. Debrecen, **67** (2005), 285–303.

[26] R.T. Smith: *The second variation for harmonic mappings*, Proc. Amer. Math. Soc., **47** (1975), 229–236.

[27] R. Takagi: *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math., **10** (1973), 495–506.

[28] R. Takagi: *Real hypersurfaces in a complex projective space with constant principal curvature*, J. Math. Soc. Japan, **27** (1975), 43–53.

[29] R. Takagi: *Real hypersurfaces in a complex projective space with constant principal curvatures II*, J. Math. Soc. Japan, **27** (1975), 507–516.