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On a class of rational matrices and interpolating polynomials related to the discrete Laplace operator

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Abstract. Let $\tilde{\nabla}^2$ be the discrete Laplace operator acting on functions (or rational matrices) $f : \mathbf{Q}_L \rightarrow \mathbb{Q}$, where \mathbf{Q}_L is the two dimensional lattice of size L embedded in \mathbb{Z}_2 . Consider a rational $L \times L$ matrix \mathcal{H} , whose inner entries \mathcal{H}_{ij} satisfy $\tilde{\nabla}^2 \mathcal{H}_{ij} = 0$. The matrix \mathcal{H} is thus the classical finite difference five-points approximation of the Laplace operator in two variables. We give a constructive proof that \mathcal{H} is the restriction to \mathbf{Q}_L of a discrete harmonic polynomial in two variables for any $L > 2$. This result proves a conjecture formulated in the context of deterministic fixed-energy sandpile models in statistical mechanics.

Keywords: rational matrices, discrete Laplacian, discrete harmonic polynomials, sandpile

MSC 2000 classification: 11C99 (Polynomials and matrices)

Introduction and Motivation

An interesting class \mathcal{M}_L of $L \times L$ matrices \mathcal{H} with rational entries and a related vector space of polynomials in two variables arise in some theoretical

physics models, the so-called *deterministic fixed-energy sandpiles* (DFES) with Bak-Tang-Wiesenfeld (BTW) toppling rule [3].

Introduced for the first time in [4] by imposing a global energy conservation constraint on its dissipative counterpart [2], DFES is a deterministic cellular automaton, in which two-dimensional configurations (represented by square matrices with integer elements $z_{ij}(t)$) evolve in discrete time steps t according to a precise parallel updating rule.

The main feature of DFES is that, in contrast with the dissipative model, only a small part of an a priori huge configuration space is dynamically explored, and the system enters a periodic orbit after a surprisingly short transient. This is a clear indication of the existence of many *hidden conservation laws* (HCL) which split the wide configuration space into dynamically intransitive, and thus much smaller subspaces [3].

Few of those HCL were identified in a non-systematic way in [1] and can be represented in the form:

$$\Phi_L[f](t) = \left[\sum_{i,j} f(i,j)z_{ij}(t) \right] \bmod L \quad (1)$$

where the sum runs over the integer coordinates of the two-dimensional $L \times L$ lattice sites, $z_{ij}(t)$ is the integer value taken by the entry (i,j) at time t and $f(i,j)$ is a $L \times L$ matrix with rational entries. The interest is then in characterizing the *generating functions* (GF) of HCL, i.e. the class of inequivalent matrices f such that $\Phi_L[f](t)$ is a HCL ($\Phi_L[f](t+1) = \Phi_L[f](t)$ for all t).

Bagnoli et al. [1] gave the following three GF: $f_1 = i$, $f_2 = j$ and $f_3 = i^2 - j^2$. An intriguing observation is that, when thought as functions on the whole \mathbb{R}^2 ($f(x,y) : \mathbb{R}^2 \mapsto \mathbb{R}$), those three GF belong to a special vector space of polynomials in two variables, which we call *discrete harmonic polynomials* (see Def. 5). It is then appealing to conjecture that this should be a general feature of any GF of a HCL.

In fact, an exhaustive characterization of GF has been given in [3] from a completely different perspective, i.e. *without any reference to polynomials*, but working simply on the matrix representation of those GF.

It was proven in [3] that a functional of the form (1) is a HCL if and only if its GF is a *inner-harmonic matrix of size L* (see Def. 4)¹.

¹In appendix B of [3] the necessary and sufficient condition is expressed in terms of K-harmonicity, and strictly speaking this is not equivalent to inner-harmonicity. However, it can be proved that for every K-harmonic function there exists an inner-harmonic function which belongs to the same equivalence class, i.e. generates an equivalent HCL. Thus, it is not restrictive to work with inner-harmonic matrices, as we will do from now on.

The purpose of this paper is to provide a rigorous link between the exact characterization of HCL in terms of matrices [3] and the conjectured polynomial form for any GF. More precisely, we will prove that every inner-harmonic matrix of size L (i.e. any GF of a HCL in the sandpile context) can be represented (non uniquely) as the restriction to the two-dimensional discrete lattice of a discrete harmonic polynomial in two variables.

The paper is organized as follows. In Section 1 we set up notations and basic definitions, providing in particular the notions of i) *inner-harmonic matrix of size L* (Def. 4), in terms of the well-known five-points formula for the discretization of the Laplace operator on a 2d lattice, and ii) *discrete harmonic polynomial* (Def. 5). In Section 2, we enunciate the main theorem and provide the algorithmic procedure for finding the discrete harmonic polynomial which interpolates any given inner-harmonic matrix of size $L \geq 3$. In the same section, we provide a stepwise example of application, together with pointers to subsequent lemmas needed for the proof. Section 3 is devoted to conclusive remarks and hints for future works, while a basis of discrete harmonic polynomials up to degree 9 is given in the Appendix.

1 Definitions

We define \mathbf{Q}_L as the two dimensional lattice embedded in \mathbb{Z}_2 , i.e.:

$$\mathbf{Q}_L = \{(i, j) \in \mathbb{Z}_2 | 0 \leq i, j \leq L - 1\} \quad (2)$$

1 Definition. The *inner sublattice* \mathbf{Q}_L^\dagger of \mathbf{Q}_L is the set:

$$\mathbf{Q}_L^\dagger = \{(i, j) \in \mathbf{Q}_L | 1 \leq i, j \leq L - 2\} \quad (3)$$

The discrete Laplace operator is defined as the classical finite difference five-points second order formula for the approximation of the Laplace operator:

2 Definition. Let $f : \mathbf{Q}_L \rightarrow \mathbb{Q}$. The discrete Laplace operator $\tilde{\nabla}^2$ acts on f as:

$$(\tilde{\nabla}^2 f^\dagger)(i, j) = 4f(i, j) - f(i - 1, j) - f(i + 1, j) - f(i, j - 1) - f(i, j + 1) \quad (4)$$

where $f^\dagger := f|_{\mathbf{Q}_L^\dagger}$.

The generalization to functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is straightforward (consider $f \equiv f^\dagger$ in this case).

3 Definition. Let \mathcal{M}_L be the set of rational $L \times L$ matrices and $\mathfrak{F} = \{f | f : \mathbf{Q}_L \rightarrow \mathbb{Q}\}$. We define the invertible map $\Psi : \mathfrak{F} \rightarrow \mathcal{M}_L$ (*L-correspondence*) through the following:

$$\Psi(h) := \mathcal{H} \quad (5)$$

where $h(-1 + j, -i + L) := \mathcal{H}_{i,j}$.

Through Ψ , the lower left corner of \mathcal{H} is mapped to the point $(0, 0)$.

4 Definition. A $L \times L$ rational matrix \mathcal{H}_1 is called *inner-harmonic matrix of size L* ($L > 2$) if the following property holds ($h_1 = \Psi^{-1}(\mathcal{H}_1)$):

$$(\tilde{\nabla}^2 h_1^\dagger)(i, j) = 0 \quad (6)$$

as in the following example, where we restrict for simplicity to integer entries:

$$\tilde{\mathcal{H}} = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 2 & 0 & 2 & 1 \\ 1 & 7 & 0 & 6 & -4 & 6 & 2 \\ 1 & 25 & -14 & 26 & -28 & 24 & 1 \\ 2 & 106 & -107 & 140 & -158 & 117 & 2 \\ 2 & 504 & -660 & 799 & -861 & 600 & 1 \\ 1 & 2568 & -3836 & 4577 & -4685 & 3143 & 0 \end{pmatrix} \quad (7)$$

5 Definition. A polynomial $P(x, y)$ is called *discrete harmonic polynomial* if $(\tilde{\nabla}^2 P)(x, y) = 0 \quad \forall (x, y) \in \mathbb{R}^2$.

Examples of discrete harmonic polynomials are $P_1(x, y) = x^2 - y^2$, $P_2(x, y) = x^3 - 3xy^2$, $P_3(x, y) = xy$.

The set of discrete harmonic polynomials of degree g will be denoted as \mathbb{D}_g^* .

6 Definition. We say that a polynomial $P(x, y)$ *interpolates* a $L \times L$ matrix \mathcal{H} if $P(i, j) = h(i, j)$, where $\Psi(h) = \mathcal{H}$. In this case, we write $P \doteq \mathcal{H}$.

Note that:

7 Remark. Discrete harmonic polynomials are generally *not* harmonic in \mathbb{R}^2 , i.e. solutions of the continuum Laplace equation $\nabla^2 P = 0$. Generally speaking, every polynomial $\mathcal{P}(x, y)$ in two variables belongs to one of the following classes:

- $\mathcal{P}(x, y)$ is neither harmonic nor discrete harmonic. Example: $\mathcal{P}(x, y) = x^3 + y^3$
- $\mathcal{P}(x, y)$ is harmonic but not discrete harmonic. Example: $\mathcal{P}(x, y) = x^4 - 6x^2y^2 + y^4$
- $\mathcal{P}(x, y)$ is discrete harmonic but not harmonic. Example: $\mathcal{P}(x, y) = x^4 - 2x^2 - 6x^2y^2 + y^4$
- $\mathcal{P}(x, y)$ is both harmonic and discrete harmonic. Example: $\mathcal{P}(x, y) = xy$

8 Remark. Given a discrete harmonic polynomial $P(x, y)$, it obviously interpolates an inner-harmonic matrix \mathcal{H}_L on \mathbf{Q}_L . For example, the polynomial $P(x, y) = x^3 - 3xy^2$ interpolates the following matrix on \mathbf{Q}_7 :

$$\mathcal{H}_7 = \begin{pmatrix} 216 & 198 & 144 & 54 & -72 & -234 & -432 \\ 125 & 110 & 65 & -10 & -115 & -250 & -415 \\ 64 & 52 & 16 & -44 & -128 & -236 & -368 \\ 27 & 18 & -9 & -54 & -117 & -198 & -297 \\ 8 & 2 & -16 & -46 & -88 & -142 & -208 \\ 1 & -2 & -11 & -26 & -47 & -74 & -107 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (8)$$

The converse is not trivial for any $L > 2$: while it is straightforward to find an interpolating polynomial $\Phi(x, y)$ for any given inner-harmonic matrix through any of the known Polynomial Interpolation formulas in two variables [5], the resulting Φ is generally *not* discrete harmonic in \mathbb{R}^2 (and incidentally neither harmonic). This can be seen easily by referring to the widely used Bilinear Interpolation formula (see e.g. [8]), the extension to the two-dimensional lattice of the well-known Lagrange interpolation formula in 1d:

$$\Phi(x, y) = \sum_{h,k} z_{hk} \prod_{\substack{j=0 \\ j \neq h}}^{L-1} \frac{x-j}{h-j} \prod_{\substack{r=0 \\ r \neq k}}^{L-1} \frac{y-r}{k-r} \quad (9)$$

where $z_{hk} = \mathcal{H}_{h,k}$, the sum runs over the sites of the matrix and the products over rows and columns respectively. Note that $\deg(\Phi) = 2(L-1)$.

It is then possible to interpolate the following simple inner-harmonic matrix of size $L = 4$:

$$\mathcal{H}_4 = \begin{pmatrix} 27 & 18 & -9 & -54 \\ 8 & 2 & -16 & -46 \\ 1 & -2 & -11 & -26 \\ -3 & 0 & 0 & 0 \end{pmatrix} \quad (10)$$

The bilinear interpolating polynomial is the following:

$$\begin{aligned} \Phi_{\mathcal{H}_4}(x, y) = & -3 + \frac{11}{2}x - 3x^2 + \frac{1}{2}x^3 + \frac{11}{2}y - \frac{121}{12}xy + \\ & \frac{5}{2}x^2y - \frac{11}{12}x^3y + -3y^2 + \frac{11}{2}xy^2 - 3x^2y^2 + \frac{1}{2}x^3y^2 \\ & + \frac{3}{2}y^3 - \frac{11}{12}xy^3 + \frac{1}{2}x^2y^3 - \frac{1}{12}x^3y^3 \quad (11) \end{aligned}$$

and a straightforward calculation yields $\tilde{\nabla}^2(\Phi_{\mathcal{H}_4})(x, y) \neq 0$ in \mathbb{R}^2 .

In the following section, we shall provide the enunciation of the main result, a stepwise example of application of the algorithm, and a constructive proof of the main theorem.

2 Interpolation by discrete harmonic polynomials: main result and algorithm

We enunciate our main result:

9 Theorem. *Let \mathcal{H} be an inner-harmonic matrix of size $L > 2$. There exists a discrete harmonic polynomial $P(x, y)$ of degree less or equal to $2(L - 1)$ such that P interpolates \mathcal{H} on \mathbf{Q}_L .*

Before getting to the technical points, it is informative to provide an example of how our algorithmic procedure roughly works.

Let us consider the inner-harmonic matrix $\mathcal{H} := \mathcal{H}_4$ in (10).

First step:

Isolate the lower left (3×3) minor $\mathcal{H}^{(1)} \subset \mathcal{H}$:

$$\mathcal{H}^{(1)} = \begin{pmatrix} 8 & 2 & -16 \\ 1 & -2 & -11 \\ -3 & 0 & 0 \end{pmatrix} \quad (12)$$

Second step:

Apply Lemma 13 and find a discrete harmonic polynomial² $P^{(1)}(x, y) \doteq \mathcal{H}^{(1)}$:

$$\begin{aligned} P^{(1)}(x, y) = & -3 + \frac{15}{4}x - \frac{1}{8}x^2 - \frac{3}{4}x^3 + \frac{1}{8}x^4 + \frac{15}{4}y + \\ & - \frac{27}{4}xy - \frac{3}{4}x^2y - \frac{1}{8}y^2 + \frac{9}{4}xy^2 - \frac{3}{4}x^2y^2 + \frac{1}{4}y^3 + \frac{1}{8}y^4 \end{aligned} \quad (13)$$

Third step:

Evaluate $P^{(1)}(x, y)$ on the lattice $\mathbf{Q}_L \equiv \mathbf{Q}_4$, obtaining the matrix $\hat{\mathcal{H}}_4$:

$$\hat{\mathcal{H}}_4 = \begin{pmatrix} 24 & \boxed{18} & -9 & -57 \\ 8 & 2 & -16 & -46 \\ 1 & -2 & -11 & \boxed{-26} \\ -3 & 0 & 0 & -3 \end{pmatrix} \quad (14)$$

²Note that this polynomial does NOT coincide with the bilinear interpolating polynomial we would get for the same matrix.

Note that i) $\hat{\mathcal{H}}_4 \neq \mathcal{H}$ ii) the sites $(1, 3) = 18$ and $(3, 1) = -26$ are uniquely determined by the discrete harmonicity requirement and thus coincide in the two matrices.

Fourth step:

In order to amend the other mismatching entries along the border, compute the four (L)-Polynomials (Lemma 14):

$$\begin{aligned} \xi_1(x, y) = & 384x - 656x^2 + 375x^3 - 65x^4 - 3x^5 + x^6 - 516y + 332xy + \\ & + 465x^2y - 440x^3y + 105x^4y - 6x^5y + 776y^2 - 1095xy^2 + \\ & + 360x^2y^2 + 30x^3y^2 - 15x^4y^2 - 225y^3 + 460xy^3 - 210x^2y^3 + \\ & + 20x^3y^3 - 55y^4 - 15xy^4 + 15x^2y^4 + 21y^5 - 6xy^5 - y^6 \end{aligned} \quad (15)$$

$$\begin{aligned} \xi_2(x, y) = & 240x - 386x^2 + 135x^3 + 25x^4 - 15x^5 + x^6 - 168y - 152xy + \\ & + 555x^2y - 280x^3y + 15x^4y + 6x^5y + 326y^2 - 255xy^2 + \\ & - 180x^2y^2 + 150x^3y^2 - 15x^4y^2 - 195y^3 + 260xy^3 - 30x^2y^3 + \\ & - 20x^3y^3 + 35y^4 - 75xy^4 + 15x^2y^4 + 3y^5 + 6xy^5 - y^6 \end{aligned} \quad (16)$$

$$\begin{aligned} \xi_3(x, y) = & 516x - 776x^2 + 225x^3 + 55x^4 - 21x^5 + x^6 - 348y - 332xy + \\ & + 1095x^2y - 460x^3y + 15x^4y + 6x^5y + 656y^2 - 465xy^2 + \\ & - 360x^2y^2 + 210x^3y^2 - 15x^4y^2 - 375y^3 + 440xy^3 - 30x^2y^3 + \\ & - 20x^3y^3 + 65y^4 - 105xy^4 + 15x^2y^4 + 3y^5 + 6xy^5 - y^6 \end{aligned} \quad (17)$$

$$\begin{aligned} \xi_4(x, y) = & 1644x - 2852x^2 + 1305x^3 - 35x^4 - 69x^5 + 7x^6 + \\ & - 1644y + 3225x^2y - 2130x^3y + 345x^4y + 2852y^2 + \\ & - 3225xy^2 + 690x^3y^2 - 105x^4y^2 - 1305y^3 + 2130xy^3 + \\ & - 690x^2y^3 + 35y^4 - 345xy^4 + 105x^2y^4 + 69y^5 - 7y^6 \end{aligned} \quad (18)$$

Those ξ_k have the remarkable properties to be i) discrete harmonic in \mathbb{R}^2 ii) almost everywhere 0 on \mathbf{Q}_L , except one single entry (two for ξ_4). In particular, $\xi_k \doteq \hat{\xi}_k$, where:

$$\hat{\xi}_1 = \begin{pmatrix} \gamma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (19)$$

$$\hat{\xi}_2 = \begin{pmatrix} 0 & 0 & 0 & \gamma_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (20)$$

$$\hat{\xi}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_3 \end{pmatrix} \quad (21)$$

$$\hat{\xi}_4 = \begin{pmatrix} 0 & 0 & \gamma_4 & 0 \\ 0 & 0 & 0 & -\gamma_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (22)$$

where, for the particular choice of the basis polynomials used to build up the $\xi(x, y)$ (see Lemma 14 for details), we have

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (-720, -720, 720, -720).$$

Fifth step:

Define the sought interpolating polynomial $P(x, y)$ for \mathcal{H} as:

$$P(x, y) = P^{(1)}(x, y) + \sum_{k=1}^4 z_k \xi_k(x, y) \quad (23)$$

where z_k are parameters to be determined, and compute $P(x, y)$ on \mathbf{Q}_L :

$$\hat{\mathcal{P}} = \begin{pmatrix} 24 - 720z_1 & 18 & -9 - 720z_4 & -57 - 720z_2 \\ 8 & 2 & -16 & -46 + 720z_4 \\ 1 & -2 & -11 & -26 \\ -3 & 0 & 0 & -3 + 720z_3 \end{pmatrix} \quad (24)$$

Sixth step:

Compute (z_1, z_2, z_3, z_4) by requiring $\hat{\mathcal{P}} \equiv \mathcal{H}$:

$$\begin{cases} 24 - 720z_1 & = 27 \\ -57 - 720z_2 & = -54 \\ -3 + 720z_3 & = 0 \\ -9 - 720z_4 & = -9 \end{cases}$$

which gives:

$$\begin{cases} z_1 & = -1/240 \\ z_2 & = -1/240 \\ z_3 & = 1/240 \\ z_4 & = 0 \end{cases} \quad (25)$$

Substituting (25) back into (23), the final result is obtained:

$$\begin{aligned}
P(x, y) = & -3 + \frac{69}{20}x + \frac{59}{60}x^2 - \frac{31}{16}x^3 + \frac{25}{48}x^4 - \frac{1}{80}x^5 - \frac{1}{240}x^6 + \frac{103}{20}y + \\
& - \frac{533}{60}xy - \frac{7}{16}x^2y + \frac{13}{12}x^3y - \frac{7}{16}x^4y + \frac{1}{40}x^5y - \frac{119}{60}y^2 + \\
& + \frac{95}{16}xy^2 - 3x^2y^2 + \frac{1}{8}x^3y^2 + \frac{1}{16}x^4y^2 + \frac{7}{16}y^3 - \frac{7}{6}xy^3 + \frac{7}{8}x^2y^3 + \\
& + \frac{1}{12}x^3y^3 + \frac{23}{48}y^4 - \frac{1}{16}xy^4 - \frac{1}{16}x^2y^4 - \frac{7}{80}y^5 + \frac{1}{40}xy^5 + \frac{1}{240}y^6 \quad (26)
\end{aligned}$$

Note the difference between (26) and (11) although they interpolate the very same matrix (10). The degree of P is $6 \equiv 2(L-1)$ as stated in Theorem 9.

This procedure can be iterated without difficulties up to interpolating inner-harmonic matrices of any size through a repeated application of Lemma 15.

Hereafter we shall provide several preliminary lemmas which are essential for the proof of the main result and have been hinted previously.

10 Lemma. *Let $k > 0$. Then \mathbb{D}_k^* is a vector space of dimension 2.*

PROOF. First, we easily prove the following statement: let \mathbb{P}_N be the set of two variables polynomials up to degree N and let $P_1(x, y) \in \mathbb{P}_N$. Then $P_2(x, y) = \tilde{\nabla}^2 P_1(x, y) \in \mathbb{P}_{N-2}$.

In fact, we notice that the following properties hold:

$$\tilde{\nabla}^2(ax^n + by^m) = a\tilde{\nabla}^2(x^n) + b\tilde{\nabla}^2(y^m) \quad \text{Linearity} \quad (27)$$

$$\tilde{\nabla}^2(x^n y^m) = x^n \tilde{\nabla}^2(y^m) + y^m \tilde{\nabla}^2(x^n) \quad \text{Leibniz rule} \quad (28)$$

Furthermore, for every one-variable monomial in x (or y), it is straightforward to prove the following:

$$\tilde{\nabla}^2 x^n = \begin{cases} -2 \sum_{k=0}^{(n-2)/2} \binom{n}{2k} x^{2k} & \text{if } n \text{ is even} \\ -2 \sum_{k=0}^{(n-3)/2} \binom{n}{2k+1} x^{2k+1} & \text{if } n \text{ is odd} \end{cases} \quad (29)$$

Therefore, applying the Laplace operator to a one-variable monomial of degree n , we obtain a linear combination of one-variables monomials up to degree $n-2$. Thanks to (27) and (28), we can conclude that the same holds also for two-variables polynomials. \square

It is well-known that \mathbb{P}_N is a linear vector space, with $\dim(\mathbb{P}_N) = \sum_{n=0}^N (n+1) = \frac{(N+1)(N+2)}{2}$. According to the previous results, we call $\Pi_N : \mathbb{P}_N \rightarrow \mathbb{P}_{N-2}$ the following linear map:

$$\Pi_N(P(x, y)) = (\tilde{\nabla}^2 P)(x, y) \quad (30)$$

Then, we call $\mathbb{D}_N = \ker(\Pi_N)$, i.e. the following vector subspace of \mathbb{P}_N :

$$\mathbb{D}_N = \{P(x, y) \in \mathbb{P}_N : (\tilde{\nabla}^2 P)(x, y) = 0 \quad \forall (x, y) \in \mathbb{R}^2\} \quad (31)$$

The elements of \mathbb{D}_N are discrete harmonic polynomials. The dimension of \mathbb{D}_N can be found simply applying the Rank-nullity theorem to the map Π_N :

$$\begin{aligned} \dim(\mathbb{D}_N) &= \dim(\mathbb{P}_N) - \dim(\mathbb{P}_{N-2}) \\ &= \frac{(N+1)(N+2)}{2} - \frac{N(N-1)}{2} = 2N+1 \end{aligned} \quad (32)$$

Let \mathbb{D}_k^* be the following vector subspace of \mathbb{D}_N :

$$\mathbb{D}_k^* = \{P(x, y) \in \mathbb{D}_N \mid P\text{'s degree is exactly } k \leq N\} \quad (33)$$

Obviously, $\dim(\mathbb{D}_k^*) = \dim(\mathbb{D}_k) - \dim(\mathbb{D}_{k-1}) = 2$.

Therefore, for $k > 0$ we can always find two (and not more) linearly independent discrete harmonic polynomials, i.e. elements of \mathbb{D}_N , with the same degree k .

Following the standard algebraic procedure, it is quite easy to build up a complete basis $\mathcal{B}^* = \{e_1^*, \dots, e_{2N+1}^*\}$ for \mathbb{D}_N , starting from the canonical basis in \mathbb{P}_N :

$$\mathcal{B}_N = \{1, x, y, x^2, xy, y^2, \dots, y^N\}$$

Throughout this paper, we will refer to the basis $\{U_k(x, y)\}$ listed in the Appendix.

11 Lemma. *For every square matrix with an arbitrary fixed rational contour, there exists one and only one inner-harmonic completion.*

PROOF. Let $F_\Omega(i, j) : \mathbf{Q}_L \rightarrow \mathbb{Q}$ and let its $(4L - 4)$ border sites be forced to assume rational values z_k belonging to the set Ω .

In matrix form, we have:

$$F_\Omega = \begin{pmatrix} z_1 & z_2 & z_3 & \cdots & \cdots & z_L \\ z_{4L-4} & x_1 & x_2 & \cdots & x_{L-2} & z_{L+1} \\ z_{4L-5} & x_{L-1} & x_L & \cdots & x_{2(L-2)} & z_{L+2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{3L-2} & z_{3L-3} & \cdots & \cdots & z_{2L} & z_{2L-1} \end{pmatrix} \quad (34)$$

The nested $(L - 2) \times (L - 2)$ submatrix F_Ω^\dagger has unknown entries $x_j \in \mathbb{Q}$.

We prove that, for each set Ω , there exists one and only one submatrix F_Ω^\dagger with rational entries such that $F_\Omega(i, j)$ is inner-harmonic.

If we impose the inner-harmonicity condition on F_Ω , we get the linear system $\hat{\mathbf{A}}\vec{x} = \vec{\eta}(\{z_k\})$, where $\hat{\mathbf{A}}$ is the following $(L-2)^2 \times (L-2)^2$ matrix:

$$\hat{\mathbf{A}} = 4\hat{\mathbf{I}} - \hat{\mathbf{H}} = 4\hat{\mathbf{I}} - \begin{pmatrix} \mathbf{H} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{H} & \mathbf{I} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{H} & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \mathbf{I} \\ \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{I} & \mathbf{H} \end{pmatrix} \quad (35)$$

$\hat{\mathbf{I}}$ is the identity matrix $(L-2)^2 \times (L-2)^2$, \mathbf{I} is the identity matrix $(L-2) \times (L-2)$, $\mathbf{0}$ is the null matrix $(L-2) \times (L-2)$ and \mathbf{H} is a well-known matrix describing the Hamiltonian of nearest-neighbor hopping on a one-dimensional lattice (see [7] and references therein):

$$\mathbf{H} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad (36)$$

The vector $\vec{\eta}$ depends on the fixed contour values. Its entries are of the following forms:

$$\eta_j = \begin{cases} z_\alpha + z_\beta & \text{if } x_j \text{ is a corner site of } F_\Omega^\dagger \\ z_\gamma & \text{if } x_j \text{ is a border site of } F_\Omega^\dagger, \text{ but not a corner site} \\ 0 & \text{otherwise} \end{cases}$$

Since the matrix (35) is diagonal predominant [6], the system admits one and only one solution in \mathbb{Q} . \square

12 Corollary. *If $\Omega = \{0, \dots, 0\}$, then F_Ω is the null matrix $L \times L$.*

13 Lemma. *Given a 3×3 inner-harmonic matrix \mathcal{A} , it is always possible to find a discrete harmonic polynomial $P(x, y)$ with rational coefficients and degree 4 such that $P \doteq \mathcal{A}$ on \mathbf{Q}_3 .*

PROOF. For $L = 3$, there are 8 sites along the contour. Choose the following set of discrete harmonic polynomials³ (see Appendix):

$$\{U_0(x, y), \dots, U_6(x, y)\} \cup \{U_8(x, y)\} \quad (37)$$

³Obviously, infinitely many other choices are equally possible.

The sought polynomial $P(x, y)$ satisfying the Lemma may be written as a linear combination of the polynomials in (37), with unknown coefficients α_j ($j = 1, \dots, 8$).

The condition that $P \doteq \mathcal{A}$ translates into a linear system with 8 equations in the unknowns α_j , whose matrix of coefficient for the choice (37) is:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & -1 \\ 1 & 0 & 2 & 0 & 4 & 0 & 8 & 8 \\ 1 & 1 & 0 & 0 & -1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 & 3 & -11 & 2 & -15 \\ 1 & 2 & 0 & 0 & -4 & 8 & 0 & 16 \\ 1 & 2 & 1 & 2 & -3 & 2 & -11 & -9 \\ 1 & 2 & 2 & 4 & 0 & -16 & -16 & -72 \end{pmatrix} \quad (38)$$

The determinant of \mathbf{M} is nonzero. Thus the polynomial interpolating the contour (and for Lemma 11 also the central site) always exists and has degree 4. $\overline{\text{QED}}$

14 Lemma. *For every $L \geq 3$, there exist four discrete harmonic (L)-polynomials ξ_1, ξ_2, ξ_3 and ξ_4 , whose degree is less or equal to $2L$, such that $\xi_k \doteq Z^{(k)}$, $k = 1, 2, 3, 4$. The entries of the matrices $Z^{(k)}$, $k = 1, 2, 3, 4$ are all 0 except:*

- (1) the entry $(0, L)$ for $Z^{(1)}$;
- (2) the entry (L, L) for $Z^{(2)}$;
- (3) the entry $(L, 0)$ for $Z^{(3)}$;
- (4) the entries $(L - 1, L)$ and $(L, L - 1)$ for $Z^{(4)}$;

As it was evident from the example of application, the (L)-polynomials have the following effect. Given a $(L + 1) \times (L + 1)$ inner-harmonic matrix \mathcal{G} and a discrete harmonic polynomial $P(x, y)$ interpolating the lower-left *minor* ($L \times L$) of \mathcal{G} , those polynomials neutralize the mismatch between 4 sites along the border of \mathcal{G} and the values assumed by $P(x, y)$ on \mathbf{Q}_{L+1} .

We prove now the existence of $\xi_1(x, y)$. For the others, the procedure is completely analogue.

Consider a set of $4L$ linearly independent discrete harmonic polynomials $P_{k,s}(x, y)$, where $k = 1, \dots, 2L$ is the degree, and $s = 1, 2$ which do not contain the constant term.

We write the sought $\xi_1(x, y)$ in the form

$$\xi_1(x, y) = \alpha_1 P_{1,1}(x, y) + \alpha_2 P_{1,2}(x, y) + \dots + \alpha_{4L} P_{2L,2}(x, y) \quad (39)$$

To determine the $4L$ unknowns α_j , we require that $\xi_1(x, y)$ should be zero on i) the border sites of the lower-left minor \mathcal{M} of $Z^{(1)}$ ii) four other points in \mathbb{Z}_2 , precisely: $(L-1, L), (L, L), (L, 0), (L+1, L)$.

This translates into a linear homogeneous system \mathcal{S} in $4L$ equations for the $4L$ unknowns $\alpha_1, \dots, \alpha_{4L}$. Note that the site $(L, L-1)$ is automatically zero due to the harmonicity condition.

The first row of the matrix of coefficients for \mathcal{S} is given by:

$$P_{1,1}(0,0), P_{1,2}(0,0), \dots, P_{2L,2}(0,0) \quad (40)$$

and these values are all zero because the polynomials do not contain the constant term.

Thus, the determinant is zero and the homogeneous system has an infinite non-zero solutions set $\{\alpha_1, \dots, \alpha_{4L}\}$. Since the polynomials $P_{k,s}(x, y)$ are linearly independent, the obtained polynomial cannot be identically zero by definition.

Now, let ξ_1 be defined by a non-zero solution $\{\alpha_1, \dots, \alpha_{4L}\}$. Being zero along the contour of \mathcal{M} , it is zero inside M because of the Corollary 12.

PROOF. It is also zero by the discrete harmonicity relation on sites (k, L) , $k = 1, \dots, L+1$, and sites (L, k) , $k = 0, \dots, L+1$. Instead, it is required to be nonzero on the site $(0, L)$. Indeed, we can prove that this is the case by contradiction. Assume that $\xi_1(0, L) = 0$. We have:

$$\xi_1(j, k) = 0, \quad k = L, \quad j = 0, 1, \dots, L+1. \quad (41)$$

Let:

$$n = L/2 + 1 \quad m = L/2 \quad \text{for even } L \quad (42)$$

$$n = (L+1)/2 + 1 \quad m = (L+1)/2 - 1 \quad \text{for odd } L \quad (43)$$

Due to the harmonicity relation, there is an integer J , $0 < J < L+1$, such that:

$$\begin{aligned} \xi_1(J, L-1+i) &= 0 && \text{for every } i = 1, \dots, n \\ \xi_1(J, -k) &= 0 && \text{for every } k = 1, \dots, m \end{aligned}$$

This means that the one variable polynomial $\eta(y) = \xi_1(J, y)$ has $2L+1$ zeros: but this is absurd, since its degree in y is at most $2L$. Therefore $\xi_1(0, L) \neq 0$. \square

15 Lemma. *Let A be a inner-harmonic matrix of order L , and A' the $(L-1) \times (L-1)$ lower-left inner-harmonic minor of A . Let $\chi(x, y)$ be a discrete harmonic polynomial of degree h interpolating A' . Then, it is possible to define a discrete harmonic polynomial $\sigma(x, y)$, of degree $k = \max[2(L-1), h]$, interpolating A .*

PROOF. Define:

$$\begin{cases} s_1 & := \text{Site } (0, L-1) \\ s_2 & := \text{Site } (L-2, L-1) \\ s_3 & := \text{Site } (L-1, L-1) \\ s_4 & := \text{Site } (L-1, L-2) \\ s_5 & := \text{Site } (L-1, 0) \end{cases}$$

and denote $\chi(s_k) := \chi_k$ and $A(s_k) := a_k$ for simplicity.

We write the sought $\sigma(x, y)$ in the form:

$$\sigma(x, y) = \chi(x, y) + \sum_{k=1}^4 z_k \xi_k(x, y), \quad (44)$$

where the $\xi_k(x, y)$ are (L-1)-polynomials as defined in Lemma 14, and z_k are coefficients to be determined. The degree of each of the ξ_k is at most $2(L-1)$, confirming the statement of the Lemma about the degree of σ .

We note that $\sigma(x, y) \equiv \chi(x, y)$ on the sites of A' , since all the (L-1)-polynomials assume value 0 there.

The values assumed by the polynomial χ on the North and East borders of \mathbf{Q}_L are uniquely constrained by the harmonicity condition, except the five sites s_k . In general, $a_k \neq \chi_k$.

For example, for $L = 5$ we have the following schematic situation (compare with (14)):

$$A = \begin{pmatrix} \square & \blacksquare & \blacksquare & \diamond & \square \\ \cdot & \cdot & \cdot & \cdot & \diamond \\ \cdot & \cdot & \cdot & \cdot & \blacksquare \\ \cdot & \cdot & \cdot & \cdot & \blacksquare \\ \cdot & \cdot & \cdot & \cdot & \square \end{pmatrix} \quad (45)$$

where:

$$\begin{cases} \cdot & \Rightarrow \text{Sites in } A', \text{ where } \chi \doteq A \\ \blacksquare & \Rightarrow \text{Sites where } \chi \doteq A \text{ by harmonicity} \\ \square & \Rightarrow \text{Sites } (s_1, s_3, s_5) \text{ where } \chi \not\doteq A \\ \diamond & \Rightarrow \text{Sites } (s_2, s_4) \text{ where } \chi \not\doteq A, \text{ but mutually constrained by harmonicity} \end{cases}$$

Indeed, the discrete harmonicity condition, applied to A and χ , requires that:

$$\chi_2 + \chi_4 = a_2 + a_4 \quad (46)$$

Given that $\xi_k(s_1) = 0$ for $k \neq 1$ and $\xi_1(s_1) = \gamma \neq 0$ (Lemma 14), we get from equation (44):

$$\sigma(s_1) = \chi_1 + z_1\gamma = a_1 \quad (47)$$

This determines z_1 as $z_1 = (a_1 - \chi_1)/\gamma$.

The same procedure applies to the sites s_3 and s_5 , determining z_2 and z_3 : note the shift of indices, reflecting the fact that we have five sites and only four (L-1)-polynomials.

In fact, the polynomial ξ_4 has to be nonzero simultaneously on both sites s_2 and s_4 , and by harmonicity $\xi_4(s_2) = -\xi_4(s_4)$. This constraint, however, is compatible with the correct definition of z_4 and therefore of $\sigma(x, y)$.

Indeed, define $\omega = \xi_4(s_2) = -\xi_4(s_4)$. Equation (44) requires evidently that $\sigma(s_2) = \chi_2 + z_4\omega = a_2$ and $\sigma(s_4) = \chi_4 - z_4\omega = a_4$. Both equations are obviously satisfied by $z_4 = (a_2 - \chi_2)/\omega$ thanks to (46).

Thus, the coefficients z_1, \dots, z_4 in (44) are uniquely determined and the polynomial $\sigma(x, y)$ interpolating A exists. \square

We are now able to provide a proof of Theorem 9.

PROOF OF THEOREM 9. We only need an iterative (or “telescopic”) application of previous results: starting from \mathcal{H} , we drop the upper row and last column on the right, defining the minor $\mathcal{H}^{(1)}$.

If we can find a discrete harmonic polynomial $\chi(x, y) \doteq \mathcal{H}^{(1)}$, such that $\deg(\chi) \leq 2(L - 1)$, the Theorem follows via Lemma 15; otherwise, we drop the upper row and last column on the right of $\mathcal{H}^{(1)}$ again, and restart the procedure.

This process is consistent, because the minors iteratively defined continue to be inner-harmonic.

Suppose that we have finally found the minor $\mathcal{H}^{(n)}$ (whose size is $L - n$) of $\mathcal{H}^{(n-1)}$, admitting an interpolating polynomial $\chi_{(n)}(x, y)$ such that $\deg(\chi_{(n)}) \leq 2(L - 1)$. By Lemma 15, the minor $\mathcal{H}^{(n-1)}$ can be interpolated, and so on, up to interpolating \mathcal{H} .

Since at least for $L = 3$ the interpolating polynomial always exists (Lemma 13), in the worst possible case the telescopic algorithm will start from the 3×3 lower left minor of \mathcal{H} , and will eventually produce the desired result by repeated applications of Lemma 15. \square

3 Final remarks

In this note, we have developed a “telescopic” technique to interpolate an inner-harmonic matrix of size L by a discrete harmonic polynomial of degree less or equal to $2(L - 1)$.

The solution we have presented proves a conjecture about hidden conservation laws in the context of some statistical mechanics models, namely the so called fixed-energy sandpiles with deterministic BTW toppling rule.

We remark that the algorithmic procedure we devised should be regarded as a mere tool for the proof, and by no means is meant to provide a computationally efficient and robust interpolator for inner-harmonic matrices.

As a final point, we wish to give here a short survey on other related questions and problems which have not been addressed in this paper and could be worthy of further investigations.

- (1) *Discrete harmonic polynomials of minimal degree:* the constructive procedure outlined in section 2 does not lead to a uniquely defined interpolating polynomial. A natural question to ask is what the minimal attainable degree of such a polynomial is, and how to build it up.
- (2) *A related combinatorial problem:* Another class of matrices (\mathcal{M}_L^*) with integer entries and closely related to \mathcal{M}_L emerges in [3] and proves to be connected to deep symmetries of the evolving rule of that model.

The main features of \mathcal{M}_L^* are:

- Condition (6) holds *modulus* the size L of the matrix.
- Cyclical border conditions are imposed and condition (6) holds for border sites as well.
- Entries are bounded by an integer M .

An interesting problem in analytical combinatorics, with many possible consequences on the underlying physical issue, is to count the number of those matrices for fixed L and M .

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Appendix: list of Discrete Harmonic Polynomials

We report here a basis of discrete harmonic polynomials up to degree 9 that we used repeatedly throughout the paper:

$$U_0(x, y) = 1 \tag{48}$$

$$U_1(x, y) = y \tag{49}$$

$$U_2(x, y) = x \quad (50)$$

$$U_3(x, y) = xy \quad (51)$$

$$U_4(x, y) = x^2 - y^2 \quad (52)$$

$$U_5(x, y) = -3x^2y + y^3 \quad (53)$$

$$U_6(x, y) = x^3 - 3xy^2 \quad (54)$$

$$U_7(x, y) = x^3y - xy^3 \quad (55)$$

$$U_8(x, y) = x^4 - 2x^2 - 6x^2y^2 + y^4 \quad (56)$$

$$U_9(x, y) = 5x^4y - 10x^2y^3 - 10x^2y + y^5 \quad (57)$$

$$U_{10}(x, y) = x^5 - 10x^3y^2 + 5xy^4 - 10xy^2 \quad (58)$$

$$U_{11}(x, y) = x^5y - \frac{10}{3}x^3y^3 - \frac{10}{3}xy^3 + xy^5 \quad (59)$$

$$U_{12}(x, y) = -15x^4y^2 - 10x^4 + 10x^2 + 15x^2y^4 + 30x^2y^2 - y^6 + x^6 \quad (60)$$

$$U_{13}(x, y) = 35x^4y^3 + 70x^4y - 21x^2y^5 - 70x^2y^3 - 70x^2y + y^7 - 7x^6y \quad (61)$$

$$U_{14}(x, y) = -21x^5y^2 - 70x^3y^2 + 35x^3y^4 - 7xy^6 + 70xy^4 - 70xy^2 + x^7 \quad (62)$$

$$U_{15}(x, y) = -7x^5y^3 + 7x^3y^5 - \frac{70}{3}x^3y^3 - \frac{70}{3}xy^3 + xy^7 + 14xy^5 + x^7y \quad (63)$$

$$U_{16}(x, y) = -140x^4y^2 + 70x^4y^4 - 140x^4 + 166x^2 - 28x^2y^6 + 280x^2y^4 + 560x^2y^2 + y^8 - 28y^6 + x^8 - 28x^6y^2 \quad (64)$$

$$U_{17}(x, y) = 126x^5y^4 - 252x^5y^2 - 84x^3y^6 - 840x^3y^2 + 840x^3y^4 + 9xy^8 + -252xy^6 + 1260xy^4 - 1026xy^2 + x^9 - 36x^7y^2 \quad (65)$$

$$U_{18}(x, y) = 840x^4y^3 + 126x^4y^5 + 1260x^4y - 252x^2y^5 - 36x^2y^7 - 840x^2y^3 + -1026x^2y + y^9 + 9x^8y - 84x^6y^3 - 252x^6y \quad (66)$$

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