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Banach–Steinhaus type theorems in locally convex spaces for linear bounded operators

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Abstract. Banach–Steinhaus type results are established for linear bounded operators between locally convex spaces without barrelledness.

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Introduction

In the past, all of Banach-Steinhaus type results have been established only for some special classes of locally convex spaces, e.g.,barrelled spaces ([2],[3],[4]), s-barrelled spaces ([5]), strictly s-barrelled spaces ([6]), etc. Recently, Cui Chengri and Songho Han ([1]) have obtained a Banach-Steinhaus type result which is valid for every locally convex space as follows

1 Theorem. *Let* $(X, \lambda), (Y, \mu)$ *be locally convex spaces and* $T_n : X \to Y$ *bounded linear operators,* $n \in \mathbf{N}$*. If weak-*$\lim_n T_n y = Ty$ *exists at each* $y \in X$*, then the limit operator* T *send* $\eta(X, X^b)-$*bounded sets into bounded sets.*

In this paper we would like to obtain the same result by taking the topology λ in place of $\eta(X, X^b)$.

Let (X, λ) and (Y, μ) be locally convex spaces. Assume that the locally convex topology μ is generated by the family $(q_\beta)_{\beta \in I}$ of semi-norms on Y.

An operator $T : X \to Y$ is said to be sequentially continuous if $\{x_n\}$ is a sequence in X such that $x_n \to x$ then $Tx_n \to Tx$; T is said to be bounded if T sends bounded sets into bounded sets. Clearly, continuous operators are sequentially continuous, and sequentially continuous operators are bounded but in general, converse implications fail. Let X', X^s and X^b denote the families

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of continuous linear functionals, sequentially continuous linear functionals and bounded linear functionals on X, respectively. In general, the inclusions $X' \subset X^s \subset X^b$ are strict.

For a linear dual pair (E, F) let $\beta(E, F)$ denote the strongest (E, F) polar topology on E which is just the topology of uniform convergence on $\sigma(F, E)$-bounded subsets of F.

Let $\mathcal{C}(X_\lambda)$, $\mathcal{B}(X_\lambda)$ and $\mathcal{C}_0(X_\lambda)$ denote the families of conditionally $\lambda-$ sequentially compact sets, bounded sets in (X, λ) and convergent sequences in (X, λ) to 0, respectively.

Let $\sigma \subset \mathcal{B}(X_\lambda)$ such that $\bigcup\limits_{C \in \sigma} C = X$.

Let ζ be the topology on X^b generated by the family of semi-norms

$$P_C(f) = \sup_{y \in C} \mid f(y) \mid, \quad C \in \sigma.$$

Let $\eta(X, X^s_\zeta)$ denote the topology of uniform convergence on conditionally $(X^s, \zeta)-$ sequentially compacts sets of X^s.

Remark that $\eta(X, X^s_\zeta)$ is coarser than $\eta(X, X^b)$. It follows immediately

2 Proposition. *For every locally convex space X the following conditions are equivalent.*
(1) For every locally convex space Y and for every sequence $\{T_n\}_n$ of bounded linear operators from X into Y such that for every $C \in \sigma$ $\mu - \lim\limits_n T_n x = Tx$ uniformly in $x \in C$, the limit operator T is also (λ, μ)-bounded.
(2) (X^b, ζ) is sequentially complete.

Proof. (1)$\Rightarrow$ (2). Let $\{f_n\}$ be a $X^b_\zeta-$ Cauchy sequence in X^b. then, there exists a linear functional f such that for every $C \in \sigma$ $\lim\limits_n f_n(x) = f(x)$ uniformly in $x \in C$. Consequently, $f \in X^b$ by (1).

(2)$\Rightarrow$ (1). Let Y be a locally convex space and $\{T_n\}$ a sequence of bounded linear operators from X into Y such that for every $C \in \sigma$ $\mu - \lim\limits_n T_n x = Tx$ uniformly in $x \in C$. Suppose that B is a bounded subset of X_λ and $y' \in Y'$. Then there exists $\beta_0 \in I$ and $c_1 > 0$ such that for every $z \in Y$ $\mid y'(z) \mid \leq c_1 q_{\beta_0}(z)$. Therefore, for every $C \in \sigma$ $\lim\limits_n y'(T_n x) = y'(Tx)$ uniformly in $x \in C$. Since $y' \circ T_n \in X^b$ for all $n \in \mathbf{N}$, $y' \circ T \in X^b$ by (2). Therefore, $\{y'(Tx) : \ x \in B\}$ is bounded. Since $y' \in Y'$ is arbitrary, $T(B)$ is $\mu-$bounded by the classical Mackey theorem. QED

The proof of proposition 1 gives the following

3 Proposition. *For every locally convex space X the following conditions are equivalent.*
(1) For every locally convex space Y and for every sequence T_n of sequentially

continuous linear operators from X into Y such that for every $C \in \sigma$ $\mu-\lim_n T_n y = Ty$ uniformly in $y \in C$, the limit operator T is also (λ,μ)-bounded.
(2) X^s_ζ is sequentially complete.

Assume now that σ satisfies also the condition $\mathcal{C}(X_\lambda) \subset \sigma$. It follows immediately

4 Proposition. *X^s_ζ is sequentially complete.*

Proof. Suppose that $\{A_n\}_n$ is Cauchy sequence in X^s_ζ. Then,

$$\forall \varepsilon > 0 \;\; \forall C \in \sigma \;\; \exists n_0 \in \mathbf{N}$$

such that

$$\forall n, m \geq n_0 \;\; \forall y \in C \quad | A_n y - A_m y | < \varepsilon. \tag{1}$$

On the other hand, $\forall y \in X$ $\{A_n y\}_n$ is Cauchy sequence in $\mathbf{R}$. Consequently, $A_n y \to Ay$ in $\mathbf{R}$, as $n \to \infty$. Letting $m \to \infty$ in (1), it follows that

$$\forall \varepsilon > 0 \;\; \forall C \in \sigma \;\; \exists n_0 \in \mathbf{N}$$

such that

$$\forall n \geq n_0 \;\; \forall y \in C \quad | A_n y - Ay | \leq \varepsilon. \tag{2}$$

We will show now that $A \in X^s$.

Let $\{x_n\}_n \in \mathcal{C}_0(X_\lambda)$. Pick any $\varepsilon > 0$. As $\{x_n\} \in \mathcal{C}(X_\mu) \subset \sigma$, then there exists $n_0 \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ and forall $y \in C$

$$| A_{n_0} x_n - A x_n | \leq \frac{\varepsilon}{2}.$$

On the other hand, there exists $n_1 \in \mathbf{N}$ such that $\forall n > n_1 \;\; | A_{n_0} x_n | \leq \frac{\varepsilon}{2}$.

In this case

$$\forall n > n_1 \;\; | A x_n | \leq \varepsilon.$$

Consequently, $A \in X^s$. Thus, X^s_ζ is sequentially complete. QED

5 Proposition. *Let $(X,\lambda), (Y,\mu)$ be locally convex spaces and $T_n : X \to Y$ sequentially continuous linear operators, $n \in \mathbf{N}$. If for every $C \in \sigma$ $\mu-\lim_n T_n z = Tz$ uniformly in $z \in C$, then the limit operator T send $\eta(X, X^s_\zeta)-$bounded sets into bounded sets.*

Proof. Let $y' \in Y'$, $C \in \sigma$. Then there exists $\beta_0 \in I$ and $c_1 > 0$ such that for every $z \in Y$ $| y'(z) | \leq c_1 q_{\beta_0}(z)$. Consequently, $\sup_{z \in C} | y'(T_n z) - y'(Tz) | \to 0$.

Since X^s_ζ is sequentially complete, then $\{y' \circ T_n : \ n \in \mathbf{N}\}$ is conditionally $(X^s, \zeta)-$sequentially compact.

Suppose that B is a $\eta(X, X^s_\zeta)-$bounded subset of X and $\{x_k\} \subset B$. Then $\exists c > 0 \ \forall k \in \mathbf{N} \ \forall n \in \mathbf{N} \mid y'(T_n x_k) \mid \leq c$. Fix a $k \geq k_0$ and $\varepsilon > 0$. Since $\lim\limits_n y'(T_n x_k) = y'(T x_k)$ there is an $n_0 \in \mathbf{N}$ such that $\mid y'(T_{n_0} x_k) - y'(T x_k) \mid < \frac{\varepsilon}{2}$. Therefore,

$$\mid y'(Tx_k) \mid \leq \mid y'(Tx_k) - y'(T_{n_0} x_k) \mid + \mid y'(T_{n_0} x_k) \mid < \frac{\varepsilon}{2} + c.$$

This shows that $\{y'(Tx) : x \in B\}$ is bounded. Since $y' \in Y'$ is arbitrary, $T(B)$ is $\mu-$bounded by the classical Mackey theorem. Thus, we achieve the proof. *QED*

Let us denote by $\theta(X, X^s_\zeta)$ the topology of uniform convergence on $(X^s, \zeta)-$Cauchy sequences. A subset B of X is said to be $\theta(X, X^s_\zeta)-$bounded if for every $X^s_\zeta-$Cauchy sequence $\{f_n\}$ there exists $c > 0$ such that for every sequence $\{x_k\}$ in $B \mid f_n(x_k) \mid \leq c \ \ \forall n \in \mathbf{N} \ \ \forall k \in \mathbf{N}$.

Then the proof of proposition 4 gives the following.

6 Proposition. *Let $(X, \lambda), (Y, \mu)$ be locally convex spaces and $T_n : X \to Y$ bounded linear operators, $n \in \mathbf{N}$. If for every $C \in \sigma$ $\mu - \lim\limits_n T_n y = Ty$ uniformly in $y \in C$, then the limit operator T send $\theta(X, X^s_\zeta)$bounded sets into bounded sets.*

Now we have a useful proposition as follows.

7 Proposition. *Let $(X, \lambda), (Y, \mu)$ be locally convex spaces and $T_n : X \to Y$ sequentially continuous linear operators, $n \in \mathbf{N}$. If for every $C \in \sigma$ $\mu - \lim\limits_n T_n y = Ty$ uniformly in $y \in C$, then the limit operator T send $\lambda-$bounded sets into bounded sets.*

Proof. By propositions 2 and 3, we deduce the result. *QED*

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