

Note di Matematica
 Note Mat. 29 (2009), n. 2, 213-218
 ISSN 1123-2536, e-ISSN 1590-0932
 DOI 10.1285/i15900932v29n2p213
<http://siba-ese.unisalento.it>, © 2009 Università del Salento

Warfield Invariants of $V(RG)/G$

Peter V. Danchev ⁱ

13, General Kutuzov Str., block 7, floor 2, flat 4
 4003 Plovdiv, Bulgaria.
pvdanchev@yahoo.com

Received: 07/07/2008; accepted: 27/11/2008.

Abstract. Let R be a commutative unitary ring of prime characteristic p and let G be an Abelian group. We calculate only in terms of R and G (and their sections) Warfield p -invariants of the quotient group $V(RG)/G$, that is, the group of all normalized units $V(RG)$ in the group ring RG modulo G . This supplies recent results of ours in (Extr. Math., 2005), (Collect. Math., 2008) and (J. Algebra Appl., 2008).

Keywords: Warfield p -invariants, nice subgroups, cardinalities, inequalities, idempotents.

MSC 2000 classification: 16S34, 16U60, 20K21.

1 Introduction

Throughout the present article, suppose that R is a commutative unitary ring of prime characteristic p , fixed for the duration, and G is an Abelian group, written multiplicatively as is customary when discussing group rings, with p -primary component G_p and torsion part G_t . As usual, RG denotes the group ring of G over R with group of normalized invertible elements $V(RG)$ and its p -component of torsion $V_p(RG)$. Moreover, let us define inductively, $G^{p^0} = G$, $G^{p^\alpha} = (G^{p^{\alpha-1}})^p$ when α is isolated and $G^{p^\alpha} = \bigcap_{\beta < \alpha} G^{p^\beta}$ when α is limit. By analogy $R^{p^0} = R$, $R^{p^\alpha} = (R^{p^{\alpha-1}})^p$ when α is isolated and $R^{p^\alpha} = \bigcap_{\beta < \alpha} R^{p^\beta}$ when α is limit. We shall say that the ring R is perfect if $R = R^p$. For any set M , we let $|M|$ designate its cardinality, and ζ_d designate the primitive d -th root of unity whenever d is a positive integer.

All other unexplained explicitly notations and notions are standard and follow essentially the classical ones stated in ([5], [6] and [8]).

The goal of this paper, that we pursue, is to calculate only in terms of R and G Warfield p -invariants of $V(RG)/G$, defined for an arbitrary multiplicative Abelian group A in the following way (compare with [9]):

$$W_{\alpha,p}(A) = \text{rank}(A^{p^\alpha} / (A^{p^{\alpha+1}} A_p^{p^\alpha})),$$

ⁱThe author would like to thank the referee for the helpful suggestions made as well as to thank the Editor, Professor Fr. de Giovanni, for his efforts and patience in processing this work.

where α is an ordinal.

It easily follows that $W_{\alpha,p}(A) = |A^{p^\alpha}/(A^{p^{\alpha+1}}A_p^{p^\alpha})|$ when $|A^{p^\alpha}/(A^{p^{\alpha+1}}A_p^{p^\alpha})| \geq \aleph_0$ or $W_{\alpha,p}(A) = \log_p |A^{p^\alpha}/(A^{p^{\alpha+1}}A_p^{p^\alpha})|$ otherwise.

Our calculations illustrated in the sequel naturally arise for applicable purposes and are helpful for the isomorphism description of the factor-group $V(RG)/G$. In fact, Warfield p -invariants, together with Ulm-Kaplansky invariants, determine, up to isomorphism, p -mixed Warfield groups (e.g., [7]).

It is worthwhile noticing that in [1]-[4] we have computed Warfield p -invariants of $V(RG)$ under various restrictions on R and G . These computations will be used here because as it will be proved below, we can restrict in some instances Warfield p -invariants of $V(RG)/G$ to the Warfield p -invariants of $V(RG)$.

2 Preliminaries

Before stating and proving our main result, we need some preparatory machineries.

1 Lemma. *For every ordinal number α , the following two identities hold:*

- (a) $G \cap V^{p^\alpha}(RG) = G^{p^\alpha}$;
- (b) $(V(RG)/G)^{p^\alpha} = V^{p^\alpha}(RG)G/G$.

PROOF. (a) Since it is straightforward that $V^{p^\alpha}(RG) = V(R^{p^\alpha}G^{p^\alpha})$, the equality now follows without any difficulty.

(b) It suffices to show that $\cap_{\beta < \alpha}(V^{p^\beta}(RG)G) = [\cap_{\beta < \alpha}(V^{p^\beta}(RG))]G = V^{p^\alpha}(RG)G$ for each limit α . In fact, take $x \in \cap_{\beta < \alpha}(V^{p^\beta}(RG)G) = \cap_{\beta < \alpha}(V(R^{p^\beta}G^{p^\beta})G)$, hence $x = (r_1a_1 + \dots + r_sa_s)g = (f_1b_1 + \dots + f_sb_s)h = \dots$, where $r_i \in R^{p^\beta}$, $a_i \in G^{p^\beta}$, $f_i \in R^{p^\gamma}$, $b_i \in G^{p^\gamma}$, $i \in [1, s]$, $\beta < \gamma < \alpha$; $g, h \in G$. Now, we obtain that $r_i = f_i$ and $ga_i = hb_i$, whence $a_i a_j^{-1} = b_i b_j^{-1} \in G^{p^\gamma}$. Writing $x = ga_1(r_1 + \dots + r_s a_s a_1^{-1})$, we observe that $x \in GV(R^{p^\gamma}G^{p^\gamma}) = GV^{p^\gamma}(RG)$.

Since the support is finite whereas the number of equalities is not because α is infinite being limit, we may assume that all relations are of the above type. That is why, $x \in (\cap_{\gamma < \alpha} V^{p^\gamma}(RG))G = V^{p^\alpha}(RG)G$ as required. QED

The next assertion appeared in ([1], Lemma 2). Nevertheless, for the reader's convenience and for the completeness of the exposition we shall provide a proof.

2 Lemma. *For each ordinal number α the following equality holds:*

$$G^{p^\alpha} \cap (V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)) = G^{p^{\alpha+1}}G_p^{p^\alpha}.$$

PROOF. Since it is routinely checked that $V^{p^\alpha}(RG) = V(R^{p^\alpha}G^{p^\alpha})$, we may write $g = uv$, where $g \in G^{p^\alpha}$, $u \in V^{p^{\alpha+1}}(RG) = V(R^{p^{\alpha+1}}G^{p^{\alpha+1}})$ and $v \in V_p^{p^\alpha}(RG) = V_p(R^{p^\alpha}G^{p^\alpha})$. Therefore, $g(r_1a_1 + \dots + r_sa_s) = f_1b_1 + \dots + f_sb_s$ and $r_i = f_i$ with $ga_i = b_i$, for each $i \in [1, s]$, where $r_i \in R^{p^{\alpha+1}}$, $a_i \in G^{p^{\alpha+1}}$ and $f_i \in R^{p^\alpha}$, $b_i \in G^{p^\alpha}$. Since $f_1b_1 + \dots + f_sb_s \in V_p(R^{p^\alpha}G^{p^\alpha})$, there is an index, say j , such that $b_j \in G_p^{p^\alpha}$. Thus $ga_j = b_j$ secures that $g = b_ja_j^{-1} \in G_p^{p^\alpha}G^{p^{\alpha+1}}$. \square

The next statement may be found in ([5], p. 157, Exercise 14) as well.

3 Lemma. [Dlab] *Let A be an Abelian multiplicative group with finite rank and $B \leq A$. Then B is neat in A (i.e., $B \cap pA = pB$) if and only if $r(A) = r(B) + r(A/B)$.*

The following corresponding claim is also useful.

4 Corollary. [[5], p. 105, Exercise 4] *If A is a multiplicative Abelian group and $B \leq A$ is a direct factor of A , then $r(A) = r(B) + r(A/B)$.*

3 Main Results

We are now in a position to prove the following

5 Theorem. *Suppose G is an Abelian group and R is a commutative unitary ring of prime characteristic p without zero divisors. Then, for each ordinal α , the following holds:*

$$(1) \quad W_{\alpha,p}(V(RG)/G) = W_{\alpha,p}(V(RG)) - W_{\alpha,p}(G)$$

when $W_{\alpha,p}(V(RG)/G) < \aleph_0$. Thus

$$(1') \quad W_{\alpha,p}(V(RG)/G) = \sum_{d \mid |G_t/G_p|} a(d) \cdot W_{\alpha,p}(G / \prod_{l \neq p} G_l) - W_{\alpha,p}(G)$$

where $a(d) = |\{g \in G_t/G_p : \text{order}(g) = d\}| / (R(\zeta_d) : R)$ provided that R is a perfect field.

$$(2) \quad W_{\alpha,p}(V(RG)/G) = W_{\alpha,p}(V(RG))$$

when $W_{\alpha,p}(V(RG)/G) \geq \aleph_0$. Thus

$$(2') \quad W_{\alpha,p}(V(RG)/G) = |G_t/G_p| W_{\alpha,p}(G)$$

provided that R is perfect.

PROOF. By definition we write

$$W_{\alpha,p} = \text{rank}((V(RG)/G)^{p^\alpha} / ((V(RG)/G)^{p^{\alpha+1}} (V(RG)/G)_p^{p^\alpha}).$$

But according to Lemma 1 we may write

$$(V(RG)/G)^{p^\alpha} = (V^{p^\alpha}(RG)G)/G,$$

$$(V(RG)/G)^{p^{\alpha+1}} = (V^{p^{\alpha+1}}(RG)G)/G$$

and $(V(RG)/G)^{p^\alpha} = (V^{p^\alpha}(RG)G)/G$. Therefore, using the modular law from [5], we obtain

$$\begin{aligned} & (V(RG)/G)^{p^\alpha} / ((V(RG)/G)^{p^{\alpha+1}} (V(RG)/G)^{p^\alpha}) \\ &= (V^{p^\alpha}(RG)G)/G / (V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)G)/G \\ &\cong (V^{p^\alpha}(RG)G) / (V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)G) \\ &\cong V^{p^\alpha}(RG) / (V^{p^\alpha}(RG) \cap [GV^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)]) \\ &= V^{p^\alpha}(RG) / [V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)(G \cap V^{p^\alpha}(RG))] \\ &= V^{p^\alpha}(RG) / (V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)G^{p^\alpha}) \\ &\cong V^{p^\alpha}(RG) / (V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)) \\ &\quad / (V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)G^{p^\alpha}) / (V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)). \end{aligned}$$

But

$$\begin{aligned} & (V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)G^{p^\alpha}) / (V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)) \\ &\cong G^{p^\alpha} / [G^{p^\alpha} \cap (V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG))] = G^{p^\alpha} / (G^{p^{\alpha+1}}G_p^{p^\alpha}) \end{aligned}$$

by using Lemma 2.

Furthermore, since $V^{p^\alpha}(RG) / (V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)G^{p^\alpha})$ is an epimorphic image of the quotient group $V^{p^\alpha}(RG) / (V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG))$, we observe that $W_{\alpha,p}(V(RG)/G) \leq W_{\alpha,p}(V(RG))$.

Next, we shall show that $W_{\alpha,p}(V(RG)/G) \geq W_{\alpha,p}(G)$ whenever $G_t \neq G_p$. In fact, we consider the element $e = (1/|C|) \sum_{c \in C} r_c c \in RC \leq RG_q \subseteq RG^{p^{\alpha+t}}$, for any $t \in \mathbb{N}$, where $|C| < \aleph_0$; clearly $|C|$ inverts in R since $\text{char}(R) = p$. It is not hard to verify that e is an idempotent, i.e., $e^2 = e$. Let $g, h \in G^{p^\alpha}$ with $gG^{p^{\alpha+1}}G_p^{p^\alpha} \neq hG^{p^{\alpha+1}}G_p^{p^\alpha}$. Construct the elements $x_g = eg + (1-e)$ and $x_h = eh + (1-e)$. Apparently, $x_g, x_h \in V(RG)$. We claim that $x_g G^{p^\alpha} V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG) \neq x_h G^{p^\alpha} V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)$. If not, $x_g x_h^{-1} = x_g x_{h^{-1}} = (eg + (1-e))(eh^{-1} + (1-e)) = egh^{-1} + (1-e) = ea + (1-e) \in G^{p^\alpha} V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)$, where we denote $a = gh^{-1} \notin G^{p^{\alpha+1}}G_p^{p^\alpha}$. By our assumption there exists a natural k such that $(ea + (1-e))^{p^k} = ea^{p^k} + (1-e) \in G^{p^{\alpha+k}} V^{p^{\alpha+k+1}}(RG) = G^{p^{\alpha+k}} V(R^{p^{\alpha+k+1}} G^{p^{\alpha+k+1}})$. Writing $e = \sum_{c \in C} f_c c$, we obtain that $\sum_{c \in C} f_c c a^{p^k} +$

$1 - \sum_{c \in C} f_c c \in G^{p^{\alpha+k}} V(R^{p^{\alpha+k+1}} G^{p^{\alpha+k+1}})$; $f_c \in R$. Furthermore, $\sum_{c \in C} f_c c a^{p^k} + 1 - \sum_{c \in C} f_c c = d^{p^k} \sum_{v \in G^{p^\alpha}} f_v v^{p^{k+1}} = \sum_{v \in G^{p^\alpha}} f_v d^{p^k} v^{p^{k+1}}$, where $f_v \in R$ and $d \in G^{p^\alpha}$. Thus, $d^{p^k} v^{p^{k+1}} \in C \subseteq G^{p^{\alpha+k+1}}$ for some $v \in G^{p^\alpha}$, and hence $d^{p^k} \in G^{p^{\alpha+k+1}}$. Therefore, $ca^{p^k} \in G^{p^{\alpha+k+1}}$ and so $a^{p^k} \in G^{p^{\alpha+k+1}}$ because $c \in G^{p^{\alpha+k+1}}$. Now, $a^{p^k} = b^{p^{k+1}}$ with $b \in G^{p^\alpha}$, i.e., $(ab^{-p})^{p^k} = 1$ and $ab^{-p} \in G_p^{p^\alpha}$. Consequently, $a \in G^{p^{\alpha+1}} G_p^{p^\alpha}$ which is the desired contradiction.

Since $V^{p^\alpha}(RG)/(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG))$ is a group bounded by p , all its subgroups are pure and so they are direct factors (see, for example, [5], Theorem 27.5). That is why, by what we have just shown above, we may write $V^{p^\alpha}(RG)/(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)) \cong (G^{p^\alpha}/(G^{p^{\alpha+1}}G_p^{p^\alpha})) \times ((V(RG)/G)^{p^\alpha}/((V(RG)/G)^{p^{\alpha+1}}(V(RG)/G)^{p^\alpha}))$. Consequently, employing Lemma 3 and Corollary 4 (see also [5], p. 157, Exercise 14 and p. 105, Exercise 4), we deduce that

$$\text{rank}(V^{p^\alpha}(RG)/(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG))) = \text{rank}(G^{p^\alpha}/G^{p^{\alpha+1}}G_p^{p^\alpha}) + \text{rank}((V(RG)/G)^{p^\alpha}/((V(RG)/G)^{p^{\alpha+1}}(V(RG)/G)^{p^\alpha})),$$

i.e., $W_{\alpha,p}(V(RG)) = W_{\alpha,p}(G) + W_{\alpha,p}(V(RG)/G)$. By what we have already shown above when $G_t \neq G_p$, if $W_{\alpha,p}(V(RG)/G)$ is finite, then $W_{\alpha,p}(G)$ is finite, whence $W_{\alpha,p}(V(RG))$ is finite and thus $W_{\alpha,p}(V(RG)/G) = W_{\alpha,p}(V(RG)) - W_{\alpha,p}(G)$ whenever $G_t \neq G_p$. Note that when $G_t = G_p$ we know via [1] that $W_{\alpha,p}(V(RG)) = W_{\alpha,p}(G)$ and that $W_{\alpha,p}(V(RG)/G) = 0$. So, the same formula is true even in this case. Further, we apply [3] and [4] to complete point (1’).

Let us now $W_{\alpha,p}(V(RG)/G)$ be infinite; thus $G_t \neq G_p$. By virtue of the inequality $W_{\alpha,p}(V(RG)/G) \geq W_{\alpha,p}(G)$ established above we obtain that $W_{\alpha,p}(V(RG)/G) = W_{\alpha,p}(V(RG))$. Finally, we can apply [2] and [3] to conclude that point (2’) is valid. QED

References

- [1] P. V. DANCHEV: *Warfield invariants in abelian group rings*, Extr. Math., **20** (2005), 233–241.
- [2] P. V. DANCHEV: *Warfield invariants in abelian group algebras*, Collect. Math., **59** (2008), 255–262.
- [3] P. V. DANCHEV: *Warfield invariants in commutative group algebras*, J. Algebra Appl., **7** (2008), 337–346.
- [4] P. V. DANCHEV: *Warfield invariants in commutative group rings*, J. Algebra Appl., **8** (2009), 829–836.
- [5] L. FUCHS: *Infinite Abelian Groups*, v. I, Mir, Moskva (1974) (translated from English in Russian).

- [6] L. FUCHS: *Infinite Abelian Groups*, v. II, Mir, Moskva (1977) (translated from English in Russian).
- [7] R. HUNTER, F. RICHMAN: *Global Warfield groups*, Trans. Amer. Math. Soc., **266** (1981), 555–572.
- [8] S. LANG: *Algebra*, Mir, Moskva, (1968) (translated from English in Russian).
- [9] R. B. WARFIELD JR.: *Classification theorems for p -groups and modules over a discrete valuation ring*, Bull. Amer. Math. Soc., **78** (1972), 88–92.