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# Congruences for (2, 3)-regular partition with designated summands

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**Abstract.** Let  $PD_{2,3}(n)$  count the number of partitions of  $n$  with designated summands in which parts are not multiples of 2 or 3. In this work, we establish congruences modulo powers of 2 and 3 for  $PD_{2,3}(n)$ . For example, for each  $n \geq 0$  and  $\alpha \geq 0$   $PD_{2,3}(6 \cdot 4^{\alpha+2}n + 5 \cdot 4^{\alpha+2}) \equiv 0 \pmod{2^4}$  and  $PD_{2,3}(4 \cdot 3^{\alpha+3}n + 10 \cdot 3^{\alpha+2}) \equiv 0 \pmod{3}$ .

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## 1 Introduction

A partition of a positive integer  $n$  is a non-increasing sequence of positive integers whose sum is  $n$ . A partition is (2, 3)-regular partition of  $n$  if none of the parts are divisible by 2 or 3.

Andrews, Lewis and Lovejoy [1] have investigated a new class of partition with designated summands are constructed by taking ordinary partitions and tagging exactly one of each part size. The total number of partitions of  $n$  with designated summands is denoted by  $PD(n)$ . Hence  $PD(4) = 10$ , namely

$4'$ ,  $3'+1'$ ,  $2'+2$ ,  $2+2'$ ,  $2'+1'+1$ ,  $2'+1+1'$ ,  $1'+1+1+1$ ,  $1+1'+1+1$ ,  
 $1+1+1'+1$ ,  $1+1+1+1'$ .

Andrews et al. [1] have derived the following generating function of  $PD(n)$ , namely

$$\sum_{n=0}^{\infty} PD(n)q^n = \frac{f_6}{f_1 f_2 f_3}, \quad (1)$$

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where

$$f_n := \prod_{j=1}^{\infty} (1 - q^{nj}), n \geq 1. \quad (2)$$

Andrews et al. [1] and N. D. Baruah and K. K. Ojah [3] have also studied  $PDO(n)$ , the number of partitions of  $n$  with designated summands in which all parts are odd. The generating function of  $PDO(n)$  is given by

$$\sum_{n=0}^{\infty} PDO(n)q^n = \frac{f_4 f_6^2}{f_1 f_3 f_{12}}. \quad (3)$$

Mahadeva Naika et al. [12] have studied  $PD_3(n)$ , the number of partitions of  $n$  with designated summands whose parts not divisible by 3 and the generating function is given by

$$\sum_{n=0}^{\infty} PD_3(n)q^n = \frac{f_6^2 f_9}{f_1 f_2 f_{18}}. \quad (4)$$

In [13] Mahadeva Naika et al. have established many congruences for  $PD_2(n)$ , the number of bipartitions of  $n$  with designated summands and the generating function is given by

$$\sum_{n=0}^{\infty} PD_2(n)q^n = \frac{f_6^2}{f_1^2 f_2^2 f_3^2}. \quad (5)$$

Motivated by the above works, in this paper, we defined  $PD_{2,3}(n)$ , the number of partitions of  $n$  with designated summands in which parts are not multiples of 2 or 3. For example  $PD_{2,3}(4) = 4$ , namely

$$1' + 1 + 1 + 1, \quad 1 + 1' + 1 + 1, \quad 1 + 1 + 1' + 1, \quad 1 + 1 + 1 + 1'.$$

The generating function of  $PD_{2,3}(n)$  is given by

$$\sum_{n=0}^{\infty} PD_{2,3}(n)q^n = \frac{f_4 f_6^2 f_9 f_{36}}{f_1 f_{12}^2 f_{18}^2}. \quad (6)$$

Following Ramanujan, for  $|ab| < 1$ , we define his general theta function  $f(a, b)$  as

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \quad (7)$$

The important special cases of  $f(a, b)$  are

$$\varphi(q) := f(q, q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2}, \quad (8)$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1} \tag{9}$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = f_1, \tag{10}$$

where the product representations arise from famous Jacobi’s triple product identity [5, p. 35, Entry 19]

$$f(a, b) = (-a; ab)_{\infty}(-b; ab)_{\infty}(ab; ab)_{\infty}. \tag{11}$$

In this paper, we list few formulas which helps to prove our main results in section 2. In section 3, we obtain several congruences modulo powers of 2 and congruences modulo 3 in section 4.

## 2 Preliminary results

We list few dissection formulas to prove our main results.

**Lemma 1.** [14, p. 212] *We have the following 5-dissection*

$$f_1 = f_{25} (a(q^5) - q - q^2/a(q^5)), \tag{12}$$

where

$$a := a(q) := \frac{(q^2, q^3; q^5)_{\infty}}{(q, q^4; q^5)_{\infty}}. \tag{13}$$

**Lemma 2.** *The following 2-dissection holds:*

$$\frac{f_9}{f_1} = \frac{f_{18}f_{12}^3}{f_{36}f_6f_2^2} + q \frac{f_{36}f_6f_4^2}{f_{12}f_2^3}. \tag{14}$$

Identity (2) is nothing but Lemma 3.5 in [16].

**Lemma 3.** *The following 3-dissection holds:*

$$\frac{f_4}{f_1} = \frac{f_{12}f_{18}^4}{f_3^3f_{36}^2} + q \frac{f_6^2f_9^3f_{36}}{f_3^4f_{18}^2} + 2q^2 \frac{f_6f_{18}f_{36}}{f_3^3}. \tag{15}$$

Identity (3) is nothing but Lemma 2.6 in [3].

**Lemma 4.** *The following 3-dissection holds:*

$$\frac{f_2}{f_1^2} = \frac{f_6^4f_9^6}{f_3^8f_{18}^3} + 2q \frac{f_6^3f_9^3}{f_3^7} + 4q^2 \frac{f_6^2f_{18}^3}{f_3^6}. \tag{16}$$

Equation (16) was proved by Hirschhorn and Sellers [10].

**Lemma 5.** [5, p. 49] *We have*

$$\varphi(q) = \varphi(q^9) + 2qf(q^3, q^{15}), \quad (17)$$

$$\psi(q) = f(q^3, q^6) + q\psi(q^9). \quad (18)$$

**Lemma 6.** *The following 2-dissections holds:*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_2 f_{16}^2}{f_8}, \quad (19)$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \quad (20)$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q \frac{f_2^2 f_8^4}{f_4^2}, \quad (21)$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}. \quad (22)$$

Lemma (6) is a consequence of dissection formulas of Ramanujan, collected in Berndt's book [5, p. 40, Entry 25].

**Lemma 7.** *The following 2-dissection holds:*

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q \frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}. \quad (23)$$

Xia and Yao [18] gave a proof of Lemma (7).

**Lemma 8.** *The following 2-dissections holds:*

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q \frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \quad (24)$$

$$\frac{f_1^2}{f_3^2} = \frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} - 2q \frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}. \quad (25)$$

Xia and Yao[17] proved (24) by employing an addition formula for theta functions. Replacing  $q$  by  $-q$  in (20) and then using the fact that  $(-q; -q)_\infty = \frac{f_2^3}{f_1 f_4}$ , we obtain (25).

**Lemma 9.** *The following 2-dissections holds:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4}, \quad (26)$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q \frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \quad (27)$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^7}. \quad (28)$$

Hirschhorn, Garvan and Borwein [8] proved (26) and (27). For proof of (28), see [4].

**Lemma 10.** *The following 2-dissections holds:*

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q \frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}} \quad (29)$$

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q \frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}. \quad (30)$$

Equation (29) was proved by Baruah and Ojah [3]. Replacing  $q$  by  $-q$  in (29) and using the fact that  $(-q; -q)_\infty = \frac{f_2^3}{f_1 f_4}$ , we get (30).

**Lemma 11.** *The following 3-dissection holds:*

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2}. \quad (31)$$

One can see this identity in [9].

**Lemma 12.** *(Cui and Gu [7, Theorem 2.2]). For any prime  $p \geq 5$ ,*

$$f_1 = \sum_{\substack{k=\frac{1-p}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}, \quad (32)$$

where

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod{6}, \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod{6}. \end{cases}$$

Furthermore, for  $\frac{-(p-1)}{2} \leq k \leq \frac{p-1}{2}$  and  $k \neq \frac{(\pm p-1)}{6}$ ,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

### 3 Congruences Modulo Powers of 2.

**Theorem 1.** For  $n \geq 1$  and  $\alpha \geq 0$ , then

$$PD_{2,3}(18n) \equiv 0 \pmod{4}, \quad (33)$$

$$PD_{2,3}(2 \cdot 3^{\alpha+3}n) \equiv 0 \pmod{4}. \quad (34)$$

*Proof.* We have

$$\sum_{n=0}^{\infty} PD_{2,3}(n)q^n = \frac{f_4 f_6^2 f_9 f_{36}}{f_1 f_{12}^2 f_{18}^2}. \quad (35)$$

Substituting (14) into (35), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(n)q^n = \frac{f_4 f_6 f_{12}}{f_2^2 f_{18}} + q \frac{f_4^3 f_6^3 f_{36}^2}{f_2^3 f_{12}^3 f_{18}^2}. \quad (36)$$

Extracting the even terms in the above equation

$$\sum_{n=0}^{\infty} PD_{2,3}(2n)q^n = \frac{f_2 f_3 f_6}{f_1^2 f_9}. \quad (37)$$

Substituting (16) into (37), we find

$$\sum_{n=0}^{\infty} PD_{2,3}(2n)q^n = \frac{f_6^5 f_9^5}{f_3^7 f_{18}^3} + 2q \frac{f_6^4 f_9^2}{f_3^6} + 4q^2 \frac{f_6^3 f_{18}^3}{f_3^5 f_9}. \quad (38)$$

Extracting the terms involving  $q^{3n}$  from both sides of (38) and replacing  $q^3$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n = \frac{f_2^5 f_3^5}{f_1^7 f_6^3}. \quad (39)$$

By the binomial theorem, it is easy to see that for positive integers  $k$  and  $m$ ,

$$f_{2k}^{2m} \equiv f_k^{4m} \pmod{4} \quad (40)$$

and

$$f_{2k}^{4m} \equiv f_k^{8m} \pmod{8}. \quad (41)$$

Invoking (41) into (39), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n \equiv \frac{f_1 f_2 f_3^5}{f_6^3} \pmod{8}. \quad (42)$$

Employing (31) into (42), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n \equiv \frac{f_3^4 f_9^4}{f_6^2 f_{18}^2} - q \frac{f_3^5 f_9 f_{18}}{f_6^3} - 2q^2 \frac{f_3^6 f_9^4}{f_6^4 f_9^2} \pmod{8}. \quad (43)$$

Extracting the terms involving  $q^{3n}$  from both sides of (43) and replacing  $q^3$  by  $q$ , we have

$$\sum_{n=0}^{\infty} PD_{2,3}(18n)q^n \equiv \frac{f_1^4 f_3^4}{f_2^2 f_6^2} \pmod{8}. \quad (44)$$

Congruence (33) follow from (40) and (44).

Equation (44) can be rewritten as

$$\sum_{n=0}^{\infty} PD_{2,3}(18n)q^n \equiv \frac{f_3^4}{f_6^2} \left( \frac{f_1^2}{f_2} \right)^2 \pmod{8}. \quad (45)$$

Replacing  $q$  by  $-q$  in (17) and using the fact that

$$\phi(-q) = \frac{f_1^2}{f_2}, \quad (46)$$

we find that

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q \frac{f_3 f_{18}^2}{f_6 f_9}. \quad (47)$$

Employing (47) into (45), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(18n)q^n \equiv \frac{f_3^4 f_9^4}{f_6^2 f_{18}^2} + 4q^2 \frac{f_3^6 f_{18}^4}{f_6^4 f_9^2} - 4q \frac{f_3^5 f_9 f_{18}}{f_6^3} \pmod{8}. \quad (48)$$

Extracting the terms involving  $q^{3n}$  from both sides of (48) and replacing  $q^3$  by  $q$ , we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(54n)q^n \equiv \frac{f_1^4 f_3^4}{f_2^2 f_6^2} \pmod{8}. \quad (49)$$

In view of the congruences (44) and (49), we get

$$PD_{2,3}(54n) \equiv PD_{2,3}(18n) \pmod{8}. \quad (50)$$

Utilizing (50) and by mathematical induction on  $\alpha$ , we arrive

$$PD_{2,3}(2 \cdot 3^{\alpha+3}n) \equiv PD_{2,3}(18n) \pmod{8}. \quad (51)$$

Using (33) into (51), we get (34).  $\square$

**Theorem 2.** For  $n \geq 0$  and  $\alpha \geq 0$ , we have

$$PD_{2,3}(72n + 42) \equiv 0 \pmod{4}, \quad (52)$$

$$PD_{2,3}(36n + 30) \equiv 0 \pmod{4}, \quad (53)$$

$$PD_{2,3}(144n + 120) \equiv 0 \pmod{4}, \quad (54)$$

$$PD_{2,3}(9 \cdot 4^{\alpha+3}n + 30 \cdot 4^{\alpha+2}) \equiv 0 \pmod{4}, \quad (55)$$

$$PD_{2,3}(54n + 18) \equiv 4 \cdot PD_{2,3}(18n + 6) \pmod{8}, \quad (56)$$

$$PD_{2,3}(54n + 36) \equiv 2 \cdot PD_{2,3}(18n + 12) \pmod{8}, \quad (57)$$

$$PD_{2,3}(36n + 30) \equiv 2 \cdot PD_{2,3}(72n + 60) \pmod{8}. \quad (58)$$

*Proof.* Extracting the terms involving  $q^{3n+1}$  from (48), dividing by  $q$  and then replacing  $q^3$  by  $q$ , we have

$$\sum_{n=0}^{\infty} PD_{2,3}(54n + 18)q^n \equiv -4 \frac{f_1^5 f_3 f_6}{f_2^3} \pmod{8}. \quad (59)$$

Extracting the terms involving  $q^{3n+1}$  from (43), dividing by  $q$  and then replacing  $q^3$  by  $q$ , we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(18n + 6)q^n \equiv -\frac{f_1^5 f_3 f_6}{f_2^3} \pmod{8}. \quad (60)$$

From (59) and (60), we arrive at (56).

Extracting the terms involving  $q^{3n+2}$  from (48), dividing by  $q^2$  and then replacing  $q^3$  by  $q$ , we find

$$\sum_{n=0}^{\infty} PD_{2,3}(54n + 36)q^n \equiv 4 \frac{f_1^6 f_6^4}{f_2^4 f_3^2} \pmod{8}. \quad (61)$$

Extracting the terms involving  $q^{3n+2}$  from (43), dividing by  $q^2$  and then replacing  $q^3$  by  $q$ , we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(18n + 12)q^n \equiv -2 \frac{f_1^6 f_6^4}{f_2^4 f_3^2} \pmod{8}. \quad (62)$$

In view of the congruences (61) and (62), we get (57).

From (60), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(18n + 6)q^n \equiv 7 \frac{f_1^5 f_3 f_6}{f_2^3} \pmod{8}. \quad (63)$$



Invoking (41) into (63), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+6)q^n \equiv 7 \frac{f_2 f_3 f_6}{f_1^3} \pmod{8}. \quad (64)$$

Employing (28) into (64), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+6)q^n \equiv 7 \frac{f_4^6 f_6^4}{f_2^8 f_{12}^2} + 21q \frac{f_4^2 f_6^2 f_{12}^2}{f_2^6} \pmod{8}. \quad (65)$$

Extracting the terms involving  $q^{2n}$  from (65) and then replacing  $q^2$  by  $q$ , we have

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+6)q^n \equiv 7 \frac{f_2^6 f_3^4}{f_1 f_6^2} \pmod{8}. \quad (66)$$

Invoking (40) into (66), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+6)q^n \equiv 3f_2^2 \pmod{4}. \quad (67)$$

Extracting the terms involving  $q^{2n+1}$  from (67), we get (52).

Extracting the terms involving  $q^{2n+1}$  from (65), dividing by  $q$  and then replacing  $q^2$  by  $q$ , we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+24)q^n \equiv 5 \frac{f_2^2 f_3^2 f_6^2}{f_1^6} \pmod{8}. \quad (68)$$

Invoking (41) into (68), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+24)q^n \equiv 5 \frac{f_6^2}{f_2^2} (f_1 f_3)^2 \pmod{8}. \quad (69)$$

Employing (30) into (69), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+24)q^n \equiv 5 \frac{f_8^4 f_{12}^8}{f_4^4 f_{24}^4} + 5q^2 \frac{f_4^8 f_6^4 f_{24}^4}{f_2^4 f_8^4 f_{12}^4} - 10q \frac{f_4^2 f_6^2 f_{12}^2}{f_2^2} \pmod{8}. \quad (70)$$

Extracting the terms involving  $q^{2n+1}$  from (70), dividing by  $q$  and then replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(72n+60)q^n \equiv 6 \frac{f_2^2 f_3^2 f_6^2}{f_1^2} \pmod{8}. \quad (71)$$

Invoking (41) into equation (62), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+12)q^n \equiv 6 \frac{f_3^8}{f_1^2 f_3^2} \pmod{8}. \quad (72)$$

Invoking (40) into (72), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+12)q^n \equiv 2 \frac{f_6^3}{f_2} \pmod{4}. \quad (73)$$

Congruence (53) follows extracting the terms involving  $q^{2n+1}$  from (73).

Which implies that

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+12)q^n \equiv 2 \frac{f_3^3}{f_1} \pmod{4}. \quad (74)$$

Substituting (26) into (74), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+12)q^n \equiv 2 \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + 2q \frac{f_{12}^3}{f_4} \pmod{4}. \quad (75)$$

Which implies,

$$\sum_{n=0}^{\infty} PD_{2,3}(72n+48)q^n \equiv 2 \frac{f_6^3}{f_2} \pmod{4}. \quad (76)$$

Congruence (54) follows extracting the terms involving  $q^{2n+1}$  from (76).

From equation (76) and (73), we have

$$PD_{2,3}(72n+48) \equiv PD_{2,3}(18n+12) \pmod{4}. \quad (77)$$

By mathematical induction on  $\alpha$ , we arrive at

$$PD_{2,3}(18 \cdot 4^{\alpha+1} + 3 \cdot 4^{\alpha+2}) \equiv PD_{2,3}(18n+12) \pmod{4}. \quad (78)$$

Using (54) into (78), we get (55).

Equation (72) can be rewritten as

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+12)q^n \equiv 6 \left( \frac{f_3^3}{f_1} \right)^2 \pmod{8}. \quad (79)$$

Employing (26) into (79), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+12)q^n \equiv 6 \frac{f_4^6 f_6^4}{f_2^4 f_{12}^2} + 6q^2 \frac{f_{12}^6}{f_4^2} + 12q \frac{f_4^2 f_6^2 f_{12}^2}{f_2} \pmod{8}. \quad (80)$$

Extracting the terms involving  $q^{2n+1}$  from (80), dividing by  $q$  and then replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n + 30)q^n \equiv 12 \frac{f_2^2 f_3^2 f_6^2}{f_1^2} \pmod{8}. \quad (81)$$

From (71) and (81), we get (58).  $\square$

**Theorem 3.** *For each  $n \geq 0$  and  $\alpha \geq 0$ , we have*

$$PD_{2,3}(72 \cdot 25^{\alpha+1}n + 6 \cdot 25^{\alpha+1}) \equiv PD_{2,3}(72n + 6) \pmod{4}, \quad (82)$$

$$PD_{2,3}(360(5n + i) + 150) \equiv 0 \pmod{4}, \quad (83)$$

where  $i = 1, 2, 3, 4$ .

*Proof.* From the equation (67), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(72n + 6)q^n \equiv 3f_1^2 \pmod{4}. \quad (84)$$

Employing (12) in the above equation, and then extracting the terms containing  $q^{5n+2}$ , dividing by  $q^2$  and replacing  $q^5$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(360n + 150)q^n \equiv 3f_5^2 \pmod{4}, \quad (85)$$

which yields

$$\sum_{n=0}^{\infty} PD_{2,3}(1800n + 150)q^n \equiv 3f_1^2 \equiv \sum_{n=0}^{\infty} PD_{2,3}(72n + 6)q^n \pmod{4}. \quad (86)$$

By induction on  $\alpha$ , we obtain (82). The congruence (83) follows by extracting the terms involving  $q^{5n+i}$  for  $i = 1, 2, 3, 4$  from both sides of (85).  $\square$

**Theorem 4.** *For each  $n \geq 0$  and  $\alpha \geq 0$ , we have*

$$PD_{2,3}(24n + 20) \equiv 0 \pmod{16}, \quad (87)$$

$$PD_{2,3}(6 \cdot 4^{\alpha+2}n + 5 \cdot 4^{\alpha+2}) \equiv 0 \pmod{16}. \quad (88)$$

*Proof.* Extracting the terms involving  $q^{3n+1}$  from (38), dividing by  $q$  and then replacing  $q^3$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+2)q^n = 2 \frac{f_2^4 f_3^2}{f_1^6}. \quad (89)$$

Invoking (41) into equation (89), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+2)q^n \equiv 2(f_1 f_3)^2 \pmod{16}. \quad (90)$$

Substituting (30) into (90), we arrive

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+2)q^n \equiv 2 \frac{f_2^2 f_8^4 f_{12}^8}{f_4^4 f_6^2 f_{24}^4} + 2q^2 \frac{f_4^8 f_6^2 f_{24}^4}{f_2^2 f_8^4 f_{12}^4} - 4q f_4^2 f_{12}^2 \pmod{16}. \quad (91)$$

Extracting the terms involving  $q^{2n+1}$  from (91), dividing by  $q$  and then replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+8)q^n \equiv 12f_2^2 f_6^2 \pmod{16}. \quad (92)$$

Extracting the terms involving  $q^{2n+1}$  from (92), we get (87).

Extracting the terms involving  $q^{2n}$  from (92) and replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+8)q^n \equiv 12(f_1 f_3)^2 \pmod{16}. \quad (93)$$

In view of the congruences (90) and (93), we get

$$PD_{2,3}(24n+8) \equiv 6 \cdot PD_{2,3}(6n+2) \pmod{16}. \quad (94)$$

Utilizing (94) and by mathematical induction on  $\alpha$ , we arrive

$$PD_{2,3}(6 \cdot 4^{\alpha+1} + 2 \cdot 4^{\alpha+1}) \equiv 6^{\alpha+1} \cdot PD_{2,3}(6n+2) \pmod{16}. \quad (95)$$

Using (87) into (95), we arrive (88).  $\square$

**Theorem 5.** For each  $n \geq 0$  and  $\alpha \geq 0$ , we have

$$PD_{2,3}(6 \cdot 4^{\alpha+1} n + 4^{\alpha+2}) \equiv PD_{2,3}(6n+4) \pmod{32}. \quad (96)$$

*Proof.* Extracting the terms involving  $q^{3n+2}$  from (38), dividing by  $q^2$  and then replacing  $q^3$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+4)q^n = 4 \frac{f_2^3 f_6^3}{f_1^5 f_3}. \quad (97)$$

Invoking (41) into (97), we arrive

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+4)q^n \equiv 4 \frac{f_1^3 f_6^3}{f_2 f_3} \pmod{32}. \quad (98)$$

Employing (27) into (98), we find

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+4)q^n \equiv 4 \frac{f_4^3 f_6^3}{f_2 f_{12}} - 12q \frac{f_2 f_6 f_{12}^3}{f_4} \pmod{32}. \quad (99)$$

Extracting the terms involving  $q^{2n}$  from (99) and replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+4)q^n \equiv 4 \frac{f_2^3 f_3^3}{f_1 f_6} \pmod{32}. \quad (100)$$

Employing (26) into (100), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+4)q^n \equiv 4 \frac{f_2 f_4^3 f_6}{f_{12}} + 4q \frac{f_2^3 f_{12}^3}{f_4 f_6} \pmod{32}. \quad (101)$$

Extracting the terms involving  $q^{2n+1}$  from (101), dividing by  $q$  and then replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+16)q^n \equiv 4 \frac{f_1^3 f_6^3}{f_2 f_3} \pmod{32}. \quad (102)$$

In view of the congruences (98) and (102), we obtain

$$PD_{2,3}(24n+16) \equiv PD_{2,3}(6n+4) \pmod{32}. \quad (103)$$

Utilizing (103) and by mathematical induction on  $\alpha$ , we get (96).  $\square$

**Theorem 6.** For  $n \geq 0$ , we have

$$PD_{2,3}(48n+34) \equiv 0 \pmod{8}, \quad (104)$$

$$PD_{2,3}(48n+46) \equiv 0 \pmod{8}, \quad (105)$$

$$PD_{2,3}(96n+52) \equiv 0 \pmod{8}, \quad (106)$$

$$PD_{2,3}(96n+76) \equiv 0 \pmod{8}. \quad (107)$$

*Proof.* Extracting the terms involving  $q^{2n+1}$  from (99), dividing by  $q$  and then replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+10)q^n \equiv 20 \frac{f_1 f_3 f_6^3}{f_2} \pmod{32}. \quad (108)$$

Substituting (30) into (108), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+10)q^n \equiv 20 \frac{f_6^2 f_8^2 f_{12}^4}{f_4^2 f_{24}^2} - 20q \frac{f_4^4 f_6^4 f_{24}^2}{f_2^2 f_8^2 f_{12}^2} \pmod{32}. \quad (109)$$

Extracting the terms involving  $q^{2n}$  from (109) and replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+10)q^n \equiv 20 \frac{f_3^2 f_4^2 f_6^4}{f_2^2 f_{12}^2} \pmod{32}. \quad (110)$$

Invoking (40) into (110), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+10)q^n \equiv 4f_2^2 f_3^2 \pmod{16}. \quad (111)$$

By the binomial theorem, it is easy to see that for positive integers  $k$  and  $m$ ,

$$f_{2k}^m \equiv f_k^{2m} \pmod{2} \quad (112)$$

Invoking (112) into (111), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+10)q^n \equiv 4f_4 f_6 \pmod{8}. \quad (113)$$

Congruences (104) follows that extracting the terms involving  $q^{2n+1}$  from (113).

Extracting the terms involving  $q^{2n+1}$  from (109), dividing by  $q$  and then replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+22)q^n \equiv 12 \frac{f_2^4 f_3^4 f_{12}^2}{f_1^2 f_4^2 f_6^2} \pmod{32}. \quad (114)$$

Invoking (40) into (114), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+22)q^n \equiv 12 \frac{f_3^4 f_6^2}{f_1^2} \pmod{16}. \quad (115)$$

Invoking (112) into (115), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(24n + 22)q^n \equiv 4 \frac{f_6^2 f_{12}}{f_2} \pmod{8}. \quad (116)$$

Extracting the terms involving  $q^{2n+1}$  from (116), we get (105).

Extracting the terms involving  $q^{2n}$  from (101) and replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n + 4)q^n \equiv 4 \frac{f_1 f_2^3 f_3}{f_6} \pmod{32}. \quad (117)$$

Substituting (30) into (117), we find

$$\sum_{n=0}^{\infty} PD_{2,3}(24n + 4)q^n \equiv 4 \frac{f_2^4 f_8^2 f_{12}^4}{f_4^2 f_6^2 f_{24}^2} - 4q \frac{f_2^2 f_4^4 f_{24}^2}{f_8^2 f_{12}^2} \pmod{32}. \quad (118)$$

Extracting the terms involving  $q^{2n}$  from (118) and replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(48n + 4)q^n \equiv 4 \frac{f_1^4 f_4^2 f_6^4}{f_2^2 f_3^2 f_{12}^2} \pmod{32}. \quad (119)$$

Invoking (40) into (119), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(48n + 4)q^n \equiv 4 \frac{f_4^2}{f_3^2} \pmod{16}. \quad (120)$$

Invoking (112) into (120), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(48n + 4)q^n \equiv 4 \frac{f_8}{f_6} \pmod{8}. \quad (121)$$

Congruences (106) obtained by extracting the term involving  $q^{2n+1}$  from (121).

Extracting the terms involving  $q^{2n+1}$  from (118), dividing by  $q$  and then replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(48n + 28)q^n \equiv 28 \frac{f_1^2 f_2^4 f_{12}^2}{f_4^2 f_6^2} \pmod{32}. \quad (122)$$

Invoking (40) into (122), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(48n + 28)q^n \equiv 12 f_1^2 f_6^2 \pmod{16}. \quad (123)$$

Invoking (112) into (123), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+28)q^n \equiv 4f_2f_{12} \pmod{8}. \quad (124)$$

Extracting the terms involving  $q^{2n+1}$  from (124), we get (107).  $\square$

**Theorem 7.** For any prime  $p \equiv 5$ ,  $\alpha \geq 1$  and  $n \geq 0$ , we have

$$\sum_{n=0}^{\infty} PD_{2,3}(48p^{2\alpha}n+10p^{2\alpha})q^n \equiv 4f_2f_3 \pmod{8}. \quad (125)$$

*Proof.* Extracting the terms involving  $q^{2n}$  from (113) and replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+10)q^n \equiv 4f_2f_3 \pmod{8}. \quad (126)$$

Define

$$\sum_{n=0}^{\infty} f(n)q^n = f_2f_3 \pmod{8}. \quad (127)$$

Combining (126) and (127), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+10)q^n \equiv 4 \sum_{n=0}^{\infty} f(n)q^n \pmod{8}. \quad (128)$$

Now, we consider the congruence equation

$$2 \cdot \frac{3k^2+k}{2} + 3 \cdot \frac{3m^2+m}{2} \equiv \frac{5p^2-5}{24} \pmod{p}, \quad (129)$$

which is equivalent to

$$(2 \cdot (6k+1))^2 + 6 \cdot (6m+1)^2 \equiv 0 \pmod{p},$$

where  $\frac{-(p-1)}{2} \leq k, m \leq \frac{p-1}{2}$  and  $p$  is a prime such that  $\left(\frac{-6}{p}\right) = -1$ . Since  $\left(\frac{-6}{p}\right) = -1$  for  $p \equiv 5 \pmod{6}$ , the congruence relation (129) holds if and only if both  $k = m = \frac{\pm p-1}{6}$ . Therefore, if we substitute (32) into (127) and then extracting the terms in which the powers of  $q$  are congruent to  $5 \cdot \frac{p^2-1}{24}$  modulo  $p$  and then divide by  $q^{5 \cdot \frac{p^2-1}{24}}$ , we find that

$$\sum_{n=0}^{\infty} f\left(pn + 5 \cdot \frac{p^2-1}{24}\right) q^{pn} = f_{2p}f_{3p},$$



which implies

$$\sum_{n=0}^{\infty} f\left(p^2n + 5 \cdot \frac{p^2 - 1}{24}\right) q^n = f_2 f_3 \quad (130)$$

and for  $n \geq 0$ ,

$$f\left(p^2n + pi + 5 \cdot \frac{p^2 - 1}{24}\right) = 0, \quad (131)$$

where  $i$  is an integer and  $1 \leq i \leq p - 1$ . By induction, we see that for  $n \geq 0$  and  $\alpha \geq 0$ ,

$$f\left(p^{2\alpha}n + 5 \cdot \frac{p^{2\alpha} - 1}{24}\right) = f(n). \quad (132)$$

Replacing  $n$  by  $p^{2\alpha}n + 5 \cdot \frac{p^{2\alpha} - 1}{24}$  in (128), we arrive at (125).  $\square$

**Corollary 1.** *For each  $n \geq 0$  and  $\alpha \geq 0$ , we have*

$$PD_{2,3}(3 \cdot 4^{\alpha+3}n + 34 \cdot 4^{\alpha+1}) \equiv 0 \pmod{8}, \quad (133)$$

$$PD_{2,3}(3 \cdot 4^{\alpha+3}n + 46 \cdot 4^{\alpha+1}) \equiv 0 \pmod{8}, \quad (134)$$

$$PD_{2,3}(6 \cdot 4^{\alpha+3}n + 13 \cdot 4^{\alpha+2}) \equiv 0 \pmod{8}, \quad (135)$$

$$PD_{2,3}(6 \cdot 4^{\alpha+3}n + 19 \cdot 4^{\alpha+2}) \equiv 0 \pmod{8}. \quad (136)$$

*Proof.* Corollary (1) follows from the Theorem (5) and Theorem (6).  $\square$

**Theorem 8.** *For  $n \geq 0$ , we have*

$$PD_{2,3}(12n + 11) \equiv 0 \pmod{4}, \quad (137)$$

$$PD_{2,3}(24n + 19) \equiv 0 \pmod{4}, \quad (138)$$

$$PD_{2,3}(24n + 17) \equiv 0 \pmod{4}, \quad (139)$$

$$PD_{2,3}(108n + 63) \equiv 0 \pmod{4}, \quad (140)$$

$$PD_{2,3}(108n + 99) \equiv 0 \pmod{4}, \quad (141)$$

$$PD_{2,3}(216n + 27)q^n \equiv 2\psi(q) \pmod{4}, \quad (142)$$

$$PD_{2,3}(72n + 6) \equiv PD_{2,3}(36n + 3) \pmod{4}, \quad (143)$$

$$PD_{2,3}(96n + 28) \equiv 2 \cdot PD_{2,3}(24n + 7) \pmod{4}. \quad (144)$$

*Proof.* Extracting the odd terms in (36), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(2n + 1)q^n = \frac{f_2^3 f_3^3 f_{18}^2}{f_1^3 f_6^3 f_9^2}. \quad (145)$$

Invoking (40) into (145), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(2n+1)q^n \equiv \frac{f_1 f_2 f_3^3 f_9^2}{f_6^3} \pmod{4}. \quad (146)$$

Substituting (31) into (146), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(2n+1)q^n \equiv \frac{f_3^2 f_9^6}{f_6^2 f_{18}^2} - q \frac{f_3^3 f_9^3 f_{18}}{f_6^3} - 2q^2 \frac{f_3^4 f_{18}^4}{f_6^4} \pmod{4}. \quad (147)$$

Extracting the terms involving  $q^{3n}$  from (147) and replacing  $q^3$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+1)q^n \equiv \frac{f_1^2 f_3^6}{f_2^2 f_6^2} \pmod{4}. \quad (148)$$

Invoking (40) into (148), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+1)q^n \equiv \frac{f_3^2}{f_1^2} \pmod{4}. \quad (149)$$

Employing (24) into (149), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+1)q^n \equiv \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q \frac{f_4 f_6^2 f_8 f_{24}}{f_2^2 f_{12}} \pmod{4}. \quad (150)$$

Extracting the terms involving  $q^{2n+1}$  from (150), dividing by  $q$  and then replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+7)q^n \equiv 2 \frac{f_2 f_3^2 f_4 f_{12}}{f_1^4 f_6} \pmod{4}. \quad (151)$$

Invoking (112) into (151), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+7)q^n \equiv 2f_2 f_{12} \pmod{4}. \quad (152)$$

Extracting the terms involving  $q^{2n+1}$  from (152), we obtain (137).

Extracting the terms involving  $q^{2n}$  from (152), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+7)q^n \equiv 2f_1 f_6 \pmod{4}. \quad (153)$$

Extracting the terms involving  $q^{2n}$  from (124) and replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(96n + 28)q^n \equiv 4f_1f_6 \pmod{8}. \quad (154)$$

In view of congruences (154) and (153), we obtain (144).

Extracting the terms involving  $q^{3n+1}$  from (147), dividing by  $q$  and then replacing  $q^3$  by  $q$ , we have

$$\sum_{n=0}^{\infty} PD_{2,3}(6n + 3)q^n \equiv 3\frac{f_1^3f_3^3f_6}{f_2^3} \pmod{4}. \quad (155)$$

Invoking (40) into (155), we find

$$\sum_{n=0}^{\infty} PD_{2,3}(6n + 3)q^n \equiv 3\frac{f_3^3f_6}{f_1f_2} \pmod{4}. \quad (156)$$

Employing (26) into (156), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n + 3)q^n \equiv 3\frac{f_4^3f_6^3}{f_2^3f_{12}} + 3q\frac{f_6f_{12}^3}{f_2f_4} \pmod{4}. \quad (157)$$

Extracting the terms involving  $q^{2n}$  from (157) and replacing  $q^2$  by  $q$ , we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(12n + 3)q^n \equiv 3\frac{f_2^3f_3^3}{f_1^3f_6} \pmod{4}. \quad (158)$$

Invoking (40) into (158), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(12n + 3)q^n \equiv 3\frac{f_1f_2f_3^3}{f_6} \pmod{4}. \quad (159)$$

Substituting (31) into (159), we find

$$\sum_{n=0}^{\infty} PD_{2,3}(12n + 3)q^n \equiv 3\frac{f_3^2f_9^4}{f_{18}^2} - 3q\frac{f_3^3f_9f_{18}}{f_6} - 6q^2\frac{f_3^4f_{18}^4}{f_6^2f_9^2} \pmod{4}. \quad (160)$$

Extracting the terms involving  $q^{3n}$  from (160) and replacing  $q^3$  by  $q$ , we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(36n + 3)q^n \equiv 3\frac{f_1^2f_3^4}{f_6^2} \pmod{4}. \quad (161)$$

Invoking (40) into (161), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+3)q^n \equiv 3f_1^2 \pmod{4}. \quad (162)$$

Extracting the terms involving  $q^{2n}$  from (67) and replacing  $q^2$  by  $q$ , we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(72n+6)q^n \equiv 3f_1^2 \pmod{4}. \quad (163)$$

In view of congruences (163) and (162), we obtain (143).

Extracting the terms involving  $q^{3n+2}$  from (160), dividing by  $q^2$  and then replacing  $q^3$  by  $q$ , we have

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+27)q^n \equiv 2 \frac{f_1^4 f_6^4}{f_2^2 f_3^2} \pmod{4}. \quad (164)$$

Invoking (40) into (164), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+27)q^n \equiv 2 \frac{f_6^4}{f_3^2} \pmod{4}. \quad (165)$$

Congruences (140) and (141) follows extracting the terms involving  $q^{3n+1}$  and  $q^{3n+2}$  from (165).

Invoking (112) into (165), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+27)q^n \equiv 2 \frac{f_{12}^2}{f_6} \pmod{4}. \quad (166)$$

Extracting the terms involving  $q^{6n}$  from (166) and replacing  $q^6$  by  $q$ , we get (142).

Extracting the terms involving  $q^{3n+2}$  from (147), dividing by  $q^2$  and then replacing  $q^3$  by  $q$ , we have

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+5)q^n \equiv 2 \frac{f_1^4 f_6^4}{f_2^4} \pmod{4}. \quad (167)$$

Invoking (40) into (167), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+5)q^n \equiv 2 \frac{f_6^4}{f_2^2} \pmod{4}. \quad (168)$$

Congruences (137) follows that extracting the terms involving  $q^{2n+1}$  from (168).

Extracting the terms involving  $q^{2n}$  from (168) and replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+5)q^n \equiv 2 \frac{f_3^4}{f_1^2} \pmod{4}. \quad (169)$$

Substitute (26) and (23) in (169)

$$\begin{aligned} & \sum_{n=0}^{\infty} PD_{2,3}(12n+5)q^n \\ & \equiv 2 \frac{f_4^4 f_6^3 f_{16} f_{24}^2}{f_2^4 f_8 f_{12}^2 f_{48}} + 2q \frac{f_4^3 f_6^3 f_8^2 f_{48}}{f_2^4 f_{12} f_{16} f_{24}} + 2q \frac{f_6 f_{12}^2 f_{16} f_{24}^2}{f_2^2 f_8 f_{48}} + 2q^2 \frac{f_6 f_8^2 f_{12}^3 f_{48}}{f_2^2 f_{16} f_{24}} \pmod{4}. \end{aligned} \quad (170)$$

Extracting the terms involving  $q^{2n+1}$  from (170), dividing by  $q$  and then replacing  $q^2$  by  $q$ , we have

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+17)q^n \equiv 2 \frac{f_2^3 f_3^3 f_4^2 f_{24}}{f_1^4 f_6 f_8 f_{12}} + 2 \frac{f_3 f_6^2 f_8 f_{12}^2}{f_1^2 f_4 f_{24}} \pmod{4}. \quad (171)$$

Invoking (112) into (171), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+17)q^n \equiv 2f_2 f_3 f_{12} + 2f_2 f_3 f_{12} \pmod{4}, \quad (172)$$

which implies (139).  $\square$

**Theorem 9.** For  $n \geq 0, \alpha \geq 0$

$$PD_{2,3}(648n+459) \equiv 0 \pmod{4}, \quad (173)$$

$$PD_{2,3}(8 \cdot 9^{\alpha+3}n + 51 \cdot 9^{\alpha+2}) \equiv 0 \pmod{4}. \quad (174)$$

*Proof.* Employing (18) into (142), we get

$$PD_{2,3}(216n+27)q^n \equiv 2f(q^3, q^6) + 2q\psi(q^9) \pmod{4}. \quad (175)$$

Congruences (173) follows extracting the terms involving  $q^{3n+2}$  from (175).

Extracting the terms involving  $q^{3n+1}$  from (175), dividing by  $q$  and then replacing  $q^3$  by  $q$ , we have

$$PD_{2,3}(648n+243)q^n \equiv 2\psi(q^3) \pmod{4}. \quad (176)$$

Extracting the terms involving  $q^{3n}$  from (176) and replacing  $q^3$  by  $q$ , we obtain

$$PD_{2,3}(1944n + 243)q^n \equiv 2\psi(q) \pmod{4}. \quad (177)$$

In view of congruences (142) and (177), we have

$$PD_{2,3}(1944n + 243) \equiv PD_{2,3}(216n + 27) \pmod{4}. \quad (178)$$

Utilizing (178) and by mathematical induction on  $\alpha$ , we get

$$PD_{2,3}(24 \cdot 9^{\alpha+2}n + 3 \cdot 9^{\alpha+2}) \equiv PD_{2,3}(216n + 27) \pmod{4}. \quad (179)$$

Using (173) into (179), we obtain (174).  $\square$

## 4 Congruences Modulo 3.

**Theorem 10.** *For  $n \geq 0$  and  $\alpha \geq 0$ , then*

$$PD_{2,3}(6n + 3) \equiv 0 \pmod{3}, \quad (180)$$

$$PD_{2,3}(6n + 5) \equiv 0 \pmod{3}, \quad (181)$$

$$PD_{2,3}(36n + 30) \equiv 0 \pmod{3}, \quad (182)$$

$$PD_{2,3}(4 \cdot 3^{\alpha+3}n + 10 \cdot 3^{\alpha+2}) \equiv 0 \pmod{3}. \quad (183)$$

*Proof.* Substituting (15) into (35), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(n)q^n = \frac{f_6^2 f_9 f_{18}^2}{f_3^3 f_{12} f_{36}} + q \frac{f_6^4 f_9 f_{36}^2}{f_3^4 f_{12}^2 f_{18}^4} + 2q^2 \frac{f_6^3 f_9 f_{36}^2}{f_3^3 f_{12}^2 f_{18}}. \quad (184)$$

Extracting the terms involving  $q^{3n}$  from (184) and replacing  $q^3$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(3n)q^n = \frac{f_2^2 f_3 f_6^2}{f_1^3 f_4 f_{12}}. \quad (185)$$

By the binomial theorem, it is easy to see that for positive integers  $k$  and  $m$ ,

$$f_{3k}^m \equiv f_k^{3m} \pmod{3}. \quad (186)$$

Invoking (186) into (185), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(3n)q^n \equiv \frac{f_2^8}{f_4^4} \pmod{3}. \quad (187)$$

Extracting the terms involving  $q^{2n}$  from (187) and replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n \equiv \frac{f_1^8}{f_2^4} \pmod{3}. \quad (188)$$

But

$$\frac{f_1^8}{f_2^4} = \frac{f_1^2 f_3^2}{f_2^4}. \quad (189)$$

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n \equiv \frac{f_1^2 f_3^2}{f_2^4} \pmod{3}. \quad (190)$$

Substituting (30) into (190), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n \equiv \frac{f_8^4 f_{12}^8}{f_2^2 f_4^4 f_6^2 f_{24}^4} + q^2 \frac{f_4^8 f_6^2 f_{24}^4}{f_2^6 f_8^4 f_{12}^4} - 2q \frac{f_4^2 f_{12}^2}{f_2^4} \pmod{3}. \quad (191)$$

Extracting the terms involving  $q^{2n+1}$  from (191), dividing by  $q$  and then replacing  $q^2$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+6)q^n \equiv \frac{f_2^2 f_6^2}{f_1^4} \pmod{3}. \quad (192)$$

Invoking (186) into (192), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+6)q^n \equiv \frac{f_2^2 f_6^2}{f_1 f_3} \pmod{3}, \quad (193)$$

which implies

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+6)q^n \equiv \psi(q)\psi(q^3) \pmod{3}. \quad (194)$$

Employing (18) into (194), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+6)q^n \equiv \psi(q^3)f(q^3, q^6) + q\psi(q^3)\psi(q^9) \pmod{3}. \quad (195)$$

Congruences (182) follows by extracting the terms involving  $q^{3n+2}$  from (195).

Extracting the terms involving  $q^{3n+1}$  from (195), dividing by  $q$  and then replacing  $q^3$  by  $q$ , we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+18)q^n \equiv \psi(q)\psi(q^3) \pmod{3}. \quad (196)$$

In view of congruences (194) and (196), we obtain

$$PD_{2,3}(36n + 18)q^n \equiv PD_{2,3}(12n + 6) \pmod{3}. \quad (197)$$

Utilizing (197) and by mathematical induction on  $\alpha$ , we get

$$PD_{2,3}(4 \cdot 3^{\alpha+2}n + 2 \cdot 3^{\alpha+2}) \equiv PD_{2,3}(12n + 6) \pmod{3}. \quad (198)$$

Using (182) into (198), we get (183).

Invoking (186) into (145), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(2n + 1)q^n \equiv \frac{f_6^4}{f_3^4} \pmod{3}. \quad (199)$$

Congruences (180) and (181) follows by extracting the terms involving  $q^{3n+1}$  and  $q^{3n+2}$  from (199).  $\square$

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