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Envelopes of slant lines in the hyperbolic plane

Takashi, ASHINO
Department of Mathematics, Hokkaido University

Hisayoshi ICHIWARA
Department of Mathematics, Hokkaido University

Shyuichi IZUMIYA
Department of Mathematics, Hokkaido University
izumiya@math.sci.hokudai.ac.jp

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Abstract. In this paper we consider envelopes of families of equidistant curves and horocycles in the hyperbolic plane. As a special case, we consider a kind of evolutes as the envelope of normal equidistant families of a curve. The hyperbolic evolute of a curve is a special case. Moreover, a new notion of horocyclic evolutes of curves is induced. We investigate the singularities of such envelopes and introduce new invariants in the Lie algebra of the Lorentz group.

Keywords: slant geometry, Hyperbolic plane, horocycles, equidistant curves

MSC 2010 classification: primary 53B30, secondary 58K99, 53A35, 58C25

1 Introduction

We consider the Poincaré disk model D of the hyperbolic plane which is conformally equivalent to the Euclidean plane, so that a circle or a line in the Poincaré disk is also a circle or a line in the Euclidean plane. A geodesic in the Poincaré disk is a Euclidean circle or a line which is perpendicular to the ideal boundary (i.e., the unit circle). If we adopt geodesics as lines in the Poincaré disk, we have the model of the hyperbolic geometry. A *horocycle* is an Euclidean circle which is tangent to the ideal boundary. If we adopt horocycles as lines, we call this geometry a *horocyclic geometry* (a *horospherical geometry* for the higher dimensional case) [4, 6, 7, 8, 9]. We also have another kind of curves with the properties similar to those of Euclidean lines. A curve in the Poincaré disk is called an *equidistant curve* if it is a Euclidean circle or a Euclidean line whose intersection with the ideal boundary consists of two points. We define an equidistant curve depends on $\phi \in [0, \pi/2]$ whose angles with the ideal boundary at the intersection points are ϕ (cf., [10]). A geodesic is the special case with $\phi = \pi/2$ and a horocycle is the case with $\phi = 0$. Therefore, a geodesic is called a *vertical pseudo-line* and a horocycle a *horizontal pseudo-line*. For $\phi \in (0, \pi/2]$, the corresponding pseudo-line is an equidistant curve, which we call a *ϕ-slant pseudo-line*. If we consider a ϕ-slant pseudo-line as a line, we call this geometry a *slant geometry*(cf., [1]).

In this paper we consider envelopes of families of ϕ-slant pseudo-lines in the general setting. We investigate the singularities of such envelopes. Throughout the remainder of the paper, we adopt the Lorentz-Minkowski space model of the hyperbolic plane. For a 3×3-matrix A, we say that A is a member of the Lorentz group $SO_0(1,2)$ if $\det A > 0$ and the induced linear mapping preserves the Lorentz-Minkowski scalar product. The Lorentz group $SO_0(1,2)$ canonically acts on the hyperbolic plane. It is well known that this action is transitive, so that the hyperbolic space is canonically identified with the homogeneous space $SO_0(1,2)/SO(2)$. It follows that any point of the hyperbolic space can be identified with a matrix $A \in SO_0(1,2)$ (cf., §3). Therefore, a one parameter family of ϕ-slant pseudo-lines can be parametrized by using a curve in $SO_0(1,2)$ (cf., §3 and 4). Then we apply the theory of unfoldings of function germs (cf., [2]) and obtain a classification of singularities of the envelopes of the families of ϕ-slant pseudo-lines (cf., Theorem 5.6). The singularities of the envelopes are characterized by using invariants represented by the elements of Lie algebra $\mathfrak{so}(1,2)$ of $SO_0(1,2)$. In §6 we introduce the notion of ϕ-slant evolutes of unit speed curves in the hyperbolic plane. If $\phi = \pi/2$, then the ϕ-slant evolute is a hyperbolic evolutes defined in [5]. Moreover, if $\phi = 0$, then the ϕ-slant evolute is called a *horocyclic evolute*. It means that the ϕ-slant evolutes depending on ϕ connects the hyperbolic evolute and the horocyclic evolute of the curve in the hyperbolic plane.

In [3] families of equal-angle envelopes in the Euclidean plane is investigated.

2 Basic concepts

We now present basic notions on Lorentz-Minkowski 3-space. Let $\mathbb{R}^3 = \{(x_0, x_1, x_2) | x_i \in \mathbb{R},\ i = 0, 1, 2\}$ be a 3-dimensional vector space. For any vectors $\boldsymbol{x} = (x_0, x_1, x_2), \boldsymbol{y} = (y_0, y_1, y_2) \in \mathbb{R}^3$, the *pseudo scalar product* (or, the *Lorentz-Minkoski scalar product*) of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2$. The space $(\mathbb{R}^3, \langle , \rangle)$ is called *Lorentz-Minkowski* 3-*space* which is denoted by $\mathbb{R}^3_1$. We assume that $\mathbb{R}^3_1$ is time-oriented and choose $\boldsymbol{e}_0 = (1, 0, 0)$ as the *future timelike vector*.

We say that a non-zero vector $\boldsymbol{x}$ in $\mathbb{R}^3_1$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0 respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^3_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. Given a non-zero vector $\boldsymbol{n} \in \mathbb{R}^3_1$ and a real number c, the plane with pseudo normal $\boldsymbol{n}$ is given by

$$P(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^3_1 | \langle \boldsymbol{x}, \boldsymbol{n} \rangle = c\}.$$

We say that $P(\boldsymbol{n}, c)$ is *spacelike* , *timelike* or *lightlike* if $\boldsymbol{n}$ is timelike, spacelike or lightlike respectively.

For any vectors $\boldsymbol{x} = (x_0, x_1, x_2)$, $\boldsymbol{y} = (y_0, y_1, y_2) \in \mathbb{R}^3_1$, *pseudo exterior product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined to be

$$\boldsymbol{x} \wedge \boldsymbol{y} = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{vmatrix} = (-(x_1y_2 - x_2y_1), x_2y_0 - x_0y_2, x_0y_1 - x_1y_0),$$

where $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ is the canonical basis of $\mathbb{R}^3_1$. We also define *Hyperbolic plane* by

$$H^2_+(-1) = \{\boldsymbol{x} \in \mathbb{R}^3_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 \geq 1\},$$

de Sitter 2*-space* by

$$S^2_1 = \{\boldsymbol{x} \in \mathbb{R}^3_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \}$$

and *the* (*open*) *lightcone* at the origin by

$$LC^* = \{\boldsymbol{x} = (x_0, x_1, x_2) \in \mathbb{R}^3_1 \ | x_0 \neq 0, \ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

We remark that $H^2_+(-1)$ is a Riemannian manifold if we consider the induced metric from $\mathbb{R}^3_1$.

We now consider the plane defined by $\mathbb{R}^2_0 = \{(x_0, x_1, x_2) \in \mathbb{R}^3_1 \mid x_0 = 0\}$. Since $\langle , \rangle|_{\mathbb{R}^2_0}$ is the canonical Euclidean scalar product, we call it *Euclidean plane.* We adopt coordinates (x_1, x_2) of $\mathbb{R}^2_0$ instead of $(0, x_1, x_2)$. On Euclidean plane $\mathbb{R}^2_0$, we have the *Poincaré disc model* of the hyperbolic plane. We consider a unit open disc $D = \{\boldsymbol{x} \in \mathbb{R}^2_0 \mid \|\boldsymbol{x}\| < 1\}$ and consider a Riemannian metric

$$ds^2 = \frac{4(dx_1^2 + dx_2^2)}{1 - x_1^2 - x_2^2}.$$

Define a mapping $\Psi : H^2_+ \longrightarrow D$ by

$$\Psi(x_0, x_1, x_2) = \left(\frac{x_1}{x_0 + 1}, \frac{x_2}{x_0 + 1} \right).$$

It is known that Ψ is an isometry. Moreover, the Poinaré disc model is conformally equivalent to the Euclidean plane.

3 Pseudo-lines in the hyperbolic plane

We consider a curve defined by the intersection of the hyperbolic plane with a plane in Lorentz-Minkowski 3-space, which is called a *pseudo-circle* if it is non-empty. The image of a pseudo-circle by the isometry Ψ is a part of a Euclidean circle in the Poincaré disc D. Let $P(\boldsymbol{n}, c)$ be a plane with a unit pseudo-normal $\boldsymbol{n}$. We call $H^2_+(-1) \cap P(\boldsymbol{n}, c)$ a *circle*, an *equidistant curve* and a *horocyle* if $\boldsymbol{n}$ is timelike, spacelike or lightlike respectively. Moreover, if $\boldsymbol{n}$ is spacelike and $c = 0$, then we call it a *hyperbolic line* (or, a *geodesic*). We remark that circles are compact and other pseudo-circles are non-compact. Therefore, equidistant curves or horocycles are called *pseudo-lines*.

We now consider a hyperbolic line

$$HL(\boldsymbol{n}) = \{\boldsymbol{x} \in H^2_+(-1) \mid \langle \boldsymbol{x}, \boldsymbol{n} \rangle = 0\}$$

and a horocycle

$$HC(\boldsymbol{\ell}, -1) = \{\boldsymbol{x} \in H^2_+(-1) \mid \langle \boldsymbol{x}, \boldsymbol{\ell} \rangle = -1\},$$

where $\boldsymbol{\ell}$ is a lightlike vector. In general, a horocycle is defined by $\langle \boldsymbol{x}, \boldsymbol{\ell} \rangle = c$ for a lightlike vector $\boldsymbol{\ell}$ and $c \neq 0$. However, if we choose $-\boldsymbol{\ell}/c$ instead of $\boldsymbol{\ell}$, then we have the above equation. We now consider parametrizations of a horocycle and a hyperbolic line respectively. For any $\boldsymbol{a}_0 \in HC(\boldsymbol{\ell}, -1)$, let $\boldsymbol{a}_1$ be a unit tangent vector of $HC(\boldsymbol{\ell}, -1)$ at $\boldsymbol{a}_0$, so that $\langle \boldsymbol{a}_1, \boldsymbol{\ell} \rangle = 0$. We define $\boldsymbol{a}_2 = \boldsymbol{a}_0 \wedge \boldsymbol{a}_1$. Then we have a pseudo orthonormal basis $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2\}$ of $\mathbb{R}^3_1$ such that $\langle \boldsymbol{a}_0, \boldsymbol{a}_0 \rangle = -1$. We remark that $\boldsymbol{a}_0$ is timelike and $\boldsymbol{a}_1, \boldsymbol{a}_2$ are spacelike. Since $\langle \boldsymbol{\ell} - \boldsymbol{a}_0, \boldsymbol{a}_0 \rangle = \langle \boldsymbol{\ell}, \boldsymbol{a}_1 \rangle = 0$, we have $\pm \boldsymbol{a}_2 = \boldsymbol{\ell} - \boldsymbol{a}_0$. We choose the direction of $\boldsymbol{a}_1$ such that $\boldsymbol{a}_2 = \boldsymbol{\ell} - \boldsymbol{a}_0$. It follows that $A = ({}^t\boldsymbol{a}_0\ {}^t\boldsymbol{a}_1\ {}^t\boldsymbol{a}_2) \in SO_0(1,2)$, where

$$SO_0(1,2) = \left\{ A = \begin{pmatrix} a_0^0 & a_0^1 & a_0^2 \\ a_1^0 & a_1^1 & a_1^2 \\ a_2^0 & a_2^1 & a_2^2 \end{pmatrix} \;\middle|\; {}^tAI_{1,2}A = I_{1,2},\ a_0^0 \geq 1 \right\}$$

is the *Lorentz group*, where

$$I_{1,2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For any $A = ({}^t\boldsymbol{a}_0\ {}^t\boldsymbol{a}_1\ {}^t\boldsymbol{a}_2) \in SO_0(1,2)$, $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2\}$ is a pseudo orthonormal basis of $\mathbb{R}^3_1$. Then $\boldsymbol{\ell} = \boldsymbol{a}_0 + \boldsymbol{a}_2$ is lightlike. It follows that we have $HC(\boldsymbol{\ell}, -1) = HC(\boldsymbol{a}_0 + \boldsymbol{a}_2, -1)$ such that $\boldsymbol{a}_0 \in HC(\boldsymbol{a}_0 + \boldsymbol{a}_2, -1)$ and $\boldsymbol{a}_1$ is tangent to $HC(\boldsymbol{a}_0 + \boldsymbol{a}_2, -1)$ at $\boldsymbol{a}_0$. Moreover, we have $\boldsymbol{a}_0 \in HL(\boldsymbol{a}_2)$ and $\boldsymbol{a}_1$ is tangent to $HL(\boldsymbol{a}_2)$ at $\boldsymbol{a}_0$. Then we have the following lemma.

Lemma 3.1. With the above notation, we have

(1) $HC(\boldsymbol{\ell},-1)=\left\{\boldsymbol{x}=\boldsymbol{a}_0+r\boldsymbol{a}_1+\dfrac{1}{2}r^2(\boldsymbol{a}_0+\boldsymbol{a}_2)\mid r\in\mathbb{R}\right\}$.

(2) $HL(\boldsymbol{a}_2)=\{\sqrt{r^2+1}\boldsymbol{a}_0+r\boldsymbol{a}_1\ |r\in\mathbb{R}\}$.

Proof. (1) For any $\boldsymbol{x}\in HC(\boldsymbol{\ell},-1)$, there exist $\alpha,\beta,\gamma\in\mathbb{R}$ such that

$$\boldsymbol{x}=\alpha\boldsymbol{a}_0+\beta\boldsymbol{a}_1+\gamma\boldsymbol{a}_2\ \ (\alpha\geq 1).$$

We put $\beta=r$. Since $\langle\boldsymbol{x},\boldsymbol{\ell}\rangle=-\alpha+\gamma=-1$, we have $\alpha=\gamma+1$. Moreover, we also have $\langle\boldsymbol{x},\boldsymbol{x}\rangle=-\alpha^2+\beta^2+\gamma^2=-(\gamma+1)^2+r^2+\gamma^2=-1$, so that $\gamma=\frac{1}{2}r^2$. Thus,

$$\boldsymbol{x}=\boldsymbol{a}_0+r\boldsymbol{a}_1+\frac{1}{2}r^2(\boldsymbol{a}_0+\boldsymbol{a}_2)$$

holds. For the converse, we can easily show that $\langle\boldsymbol{x},\boldsymbol{x}\rangle=-1$ and $\langle\boldsymbol{x},\boldsymbol{\ell}\rangle=-1$ for the above vector.

(2) For any $\boldsymbol{x}\in HL(\boldsymbol{a}_2)$, there exist $\alpha,\beta,\gamma\in\mathbb{R}$ such that

$$\boldsymbol{x}=\alpha\boldsymbol{a}_0+\beta\boldsymbol{a}_1+\gamma\boldsymbol{a}_2\ \ (\alpha\geq 1).$$

Since $\langle\boldsymbol{x},\boldsymbol{a}_2\rangle=0$, $\gamma=0$. If we put $\beta=r$, then we have $\langle\boldsymbol{x},\boldsymbol{x}\rangle=-\alpha^2+r^2=-1$, so that $\alpha=\pm\sqrt{r^2+1}$. Since $\alpha\geq 1$, we have $\alpha=\sqrt{r^2+1}$. By a straightforward calculation, the converse holds. □

It is known that a horocycle $\Psi(HC(\boldsymbol{a}_0+\boldsymbol{a}_2,-1))$ in the Poincaré disc D is a Euclidean circle tangent to the ideal boundary $S^1=\{\boldsymbol{x}\in\mathbb{R}^2_0\mid\|\boldsymbol{x}\|=1\}$. It is also known that a hyperbolic line $\Psi(HL(\boldsymbol{a}_2))$ is a Euclidean circle or a Euclidean line orthogonal to the ideal boundary (cf., [11]). By these reasons, a horocycle is called a *horizontal pseudo-line* and a hyperbolic-line is called an *orthogonal pseudo-line* respectively. We now define a *ϕ-slant pseudo-line* by

$$SL(\boldsymbol{n}_\phi,-\cos\phi)=\left\{\boldsymbol{x}\in H^2_+(-1)\mid\langle\boldsymbol{x},\boldsymbol{n}_\phi\rangle=-\cos\phi\right\},$$

where $\boldsymbol{n}_\phi(t)=\cos\phi\boldsymbol{a}_0+\boldsymbol{a}_2,\ \phi\in[0,\ \pi/2]$. Since $\langle\boldsymbol{n}_\phi,\boldsymbol{n}_\phi\rangle=\sin^2\phi>0$, $\boldsymbol{n}_\phi$ is spacelike. Thus, $SL(\boldsymbol{n}_\phi,-\cos\phi)=H^2_+(-1)\cap P(\boldsymbol{n}_\phi,-\cos\phi)$ is an equidistant curve. Moreover, $\boldsymbol{a}_0\in SL(\boldsymbol{n}_\phi,-\cos\phi)$ and $\boldsymbol{a}_1$ is tangent to $SL(\boldsymbol{n}_\phi,-\cos\phi)$ at $\boldsymbol{a}_{\boldsymbol{0}}$. Then $SL(\boldsymbol{n}_{\pi/2},-\cos(\pi/2))=HL(\boldsymbol{a}_2)$ and $SL(\boldsymbol{n}_0,-\cos 0)=HC(\boldsymbol{a}_0+\boldsymbol{a}_2,-1)$. We have the following parametrization of a ϕ-slant pseudo-line.

Lemma 3.2. With the same notations as those in Lemma 3.1, we have

$$SL(\boldsymbol{n}_\phi,-\cos\phi)=\left\{\boldsymbol{a}_0+r\boldsymbol{a}_1+\frac{\sqrt{r^2\sin^2\phi+1}-1}{\sin^2\phi}(\boldsymbol{a}_0+\cos\phi\boldsymbol{a}_2)\;\middle|\;r\in\mathbb{R}\right\}.$$

Proof. We consider a point $\boldsymbol{x} \in SL(\boldsymbol{n}_\phi(t), -\cos\phi)$. Since $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2\}$ is a pseudo-orthonormal basis of $\mathbb{R}^3_1$, There exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\boldsymbol{x} = \alpha\boldsymbol{a}_0 + \beta\boldsymbol{a}_1 + \gamma\boldsymbol{a}_2$, $(\alpha \geq 1)$. Therefore, we have

$$\begin{aligned}\langle \boldsymbol{x}, \boldsymbol{n}_\phi\rangle &= \langle \alpha\boldsymbol{a}_0 + \beta\boldsymbol{a}_1 + \gamma\boldsymbol{a}_2, \cos\phi\boldsymbol{a}_0 + \boldsymbol{a}_2\rangle \\ &= -\cos\phi\alpha + \gamma = -\cos\phi\end{aligned}$$

Thus, we have $\gamma = \cos\phi(\alpha - 1)$. Moreover, $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = -\alpha^2 + \beta^2 + \gamma^2 = -\alpha^2 + \beta^2 + \cos^2\phi(\alpha - 1)^2 = -1$. It follows that

$$\alpha = \frac{1}{\sin^2\phi}(\pm\sqrt{\beta^2\sin^2\phi + 1} - \cos^2\phi).$$

If we choose $\alpha = -\dfrac{1}{\sin^2\phi}(\sqrt{\beta^2\sin^2\phi + 1} + \cos^2\phi)$, then $\alpha < 0$. It contradicts to $\alpha \geq 1$. Hence, we have

$$\alpha = \frac{1}{\sin^2\phi}(\sqrt{\beta^2\sin^2\phi + 1} - \cos^2\phi),\ \gamma = \frac{\cos\phi}{\sin^2\phi}(\sqrt{\beta^2\sin^2\phi + 1} - 1).$$

We put $\beta = r$. Then

$$\begin{aligned}\boldsymbol{x} &= \frac{1}{\sin^2\phi}(\sqrt{r^2\sin^2\phi + 1} - \cos^2\phi)\boldsymbol{a}_0 + r\boldsymbol{a}_1 + \frac{\cos\phi}{\sin^2\phi}(\sqrt{r^2\sin^2\phi + 1} - 1)\boldsymbol{a}_2 \\ &= \boldsymbol{a}_0 + r\boldsymbol{a}_1 + \frac{1}{\sin^2\phi}(\sqrt{r^2\sin^2\phi + 1} - 1)(\boldsymbol{a}_0 + \cos\phi\boldsymbol{a}_2)\end{aligned}.$$

For the converse, we have $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = -1$, $\langle \boldsymbol{x}, \boldsymbol{n}_\phi\rangle = -\cos\phi$ and $(\sqrt{r^2\sin^2\phi + 1} - \cos^2\phi)/\sin^2\phi \geq 1$. Then $\boldsymbol{x} \in SL(\boldsymbol{n}_\phi(t), -\cos\phi)$. □

Remark 3.3. We can show $\lim_{\phi\to 0}(\sqrt{r^2\sin^2\phi + 1} - 1)/\sin^2\phi = r^2/2$. In [10] the third author showed that the angle between $\Psi(SL(\boldsymbol{n}_\phi))$ and the ideal boundary S^1 of the Poincaré disc D at an intersection point is equal to ϕ. This is the reason why we call $SL(\boldsymbol{n}_\phi)$ the ϕ-slant pseudo line.

4 One-parameter families of pseudo-lines

In this section we consider one-parameter families of pseudo-lines. By Lemmas 3.1 and 3.2, we consider a one-parameter family of pseudo-orthonormal bases of $\mathbb{R}^3_1$. Let $A : J \longrightarrow SO_0(1, 2)$ be a C^∞-mapping. If we write $A(t) = ({}^t\boldsymbol{a}_0(t)\ {}^t\boldsymbol{a}_1(t)\ {}^t\boldsymbol{a}_2(t))$, then $\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t)\}$ is a one-parameter family of pseudo-orthonormal bases of $\mathbb{R}^3_1$. We call it a *pseudo-orthonormal moving frame* of $\mathbb{R}^3_1$. By the standard arguments, we can show the following Frenet-Serret type formulae for the pseudo-orthonormal moving frame $\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t)\}$:

$$\begin{pmatrix} \boldsymbol{a}_0'(t) \\ \boldsymbol{a}_1'(t) \\ \boldsymbol{a}_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & c_1(t) & c_2(t) \\ c_1(t) & 0 & c_3(t) \\ c_2(t) & -c_3(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_0(t) \\ \boldsymbol{a}_1(t) \\ \boldsymbol{a}_2(t) \end{pmatrix}.$$

Here,

$$\begin{cases} c_1(t) = \langle \boldsymbol{a}_0'(t), \boldsymbol{a}_1(t) \rangle \\ c_2(t) = \langle \boldsymbol{a}_0'(t), \boldsymbol{a}_2(t) \rangle \\ c_3(t) = \langle \boldsymbol{a}_1'(t), \boldsymbol{a}_2(t) \rangle \end{cases}$$

Then, the matrix $C(t) = \begin{pmatrix} 0 & c_1(t) & c_2(t) \\ c_1(t) & 0 & c_3(t) \\ c_2(t) & -c_3(t) & 0 \end{pmatrix}$ is an element of Lie algebra $\mathfrak{so}(1,2)$ of the Lorentz group $SO_0(1,2)$. The above Frenet-Serret type formulae are written by $A'(t)A^{-1}(t) = C(t)$. For any C^∞-mapping $C : J \longrightarrow \mathfrak{so}(1,2)$ and $A_0 \in SO_0(1,2)$, we can apply the unique existence theorem for systems of linear ordinary differential equations, so that there exists a unique $A(t) \in SO_0(1,2)$ such that $A(0) = A_0$ and $A'(t)A^{-1}(t) = C(t)$.

We now consider a mapping $g_\phi : I \times J \longrightarrow H^2_+$, where $g_\phi(r,t)$ is defined by

$$\begin{cases} \boldsymbol{a}_0(t) + r\boldsymbol{a}_1(t) + \dfrac{\sqrt{r^2 \sin^2\phi + 1} - 1}{\sin^2\phi}(\boldsymbol{a}_0(t) + \cos\phi \boldsymbol{a}_2(t)) & \text{if } \phi \neq 0, \\ \boldsymbol{a}_0(t) + r\boldsymbol{a}_1(t) + \dfrac{r^2}{2}(\boldsymbol{a}_0(t) + \boldsymbol{a}_2(t)) \quad \text{if } \phi = 0, \end{cases}$$

where $I, J \subset \mathbb{R}$ are intervals. Then we have $SL(\boldsymbol{n}_\phi(t), -\cos\phi) = \{g_\phi(I \times \{t\}) \mid t \in J\}$, for $\boldsymbol{n}_\phi(t) = \cos\phi \boldsymbol{a}_0(t) + \boldsymbol{a}_2(t)$. Thus g_ϕ is a one-parameter family of ϕ-slant pseudo-lines. Moreover, g_0 is a one-parameter family of horocycles and $g_{\pi/2}$ is a one-parameter family of hyperbolic lines.

5 Height functions

For a one parameter family of ϕ-slant pseudo-lines g_ϕ, we define a *family of height functions* $H : J \times H^2_+ \longrightarrow \mathbb{R}$ by $H(t, \boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{n}_\phi(t) \rangle + \cos\phi$. Then we have the following proposition.

Proposition 5.1. For $g_\phi : I \times J \longrightarrow H^2_+$, we have the following:
(1) $H(t, \boldsymbol{x}) = 0$ if and only if there exists $r \in I$ such that $\boldsymbol{x} = g_\phi(r,t)$,
(2) $H(t, \boldsymbol{x}) = \dfrac{\partial H}{\partial t}(t, \boldsymbol{x}) = 0$ if and only if there exists $r \in I$ such that $\boldsymbol{x} = g_\phi(r,t)$ and

$$-\sqrt{r^2 \sin^2\phi + 1} c_2(t) + r(\cos\phi c_1(t) - c_3(t)) = 0.$$

Proof. (1) If $H(t,\boldsymbol{x})=0$, then $\langle \boldsymbol{x}, \boldsymbol{n}_\phi(t)\rangle = -\cos\phi\ ,\ \boldsymbol{x}\in H^2_+(-1)$. Thus, there exists $r\in I$ such that $\boldsymbol{x}=g_\phi(r,t)$. The converse also holds.
(2)Since $H(t,\boldsymbol{x})=0$, there exists $r\in I$ such that $\boldsymbol{x}=g_\phi(r,t)$. Suppose that $\phi\neq 0$. Since $\boldsymbol{n}'_\phi(t)=c_2(t)\boldsymbol{a}_0(t)+(\cos\phi c_1(t)-c_3(t))\,\boldsymbol{a}_1(t)+\cos\phi c_2(t)\boldsymbol{a}_2(t)$,

$$
\begin{aligned}
\frac{\partial H}{\partial t}(t,\boldsymbol{x}) &= \langle \boldsymbol{x}, \boldsymbol{n}'_\phi(t)\rangle \\
&= \Bigg\langle \frac{\sqrt{r^2\sin^2\phi+1}-\cos^2\phi}{\sin^2\phi}\boldsymbol{a}_0(t)+r\boldsymbol{a}_1(t) \\
&\qquad +\frac{\cos\phi(\sqrt{r^2\sin^2\phi+1}-1)}{\sin^2\phi}\boldsymbol{a}_2(t),\ \boldsymbol{n}'_\phi(t)\Bigg\rangle \\
&= -\sqrt{r^2\sin^2\phi+1}c_2(t)+r\left(\cos\phi c_1(t)-c_3(t)\right).
\end{aligned}
$$

If $\phi=0$, then

$$
\frac{\partial H}{\partial t}(t,\boldsymbol{x})=\langle \boldsymbol{x}, \boldsymbol{n}'_0(t)\rangle = -c_2(t)+r(c_1(t)-c_3(t)).
$$

This completes the proof. □

We now review some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book[2]. Let $F:(\mathbb{R}\times\mathbb{R}^r,(s_0,x_0))\to\mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f, where $f(s)=F_{x_0}(s,x_0)$. We say that f has an *A_k-singularity* at s_0 if $f^{(p)}(s_0)=0$ for all $1\le p\le k$, and $f^{(k+1)}(s_0)\neq 0$. We also say that f has an *$A_{\ge k}$-singularity* at s_0 if $f^{(p)}(s_0)=0$ for all $1\le p\le k$. Let F be an unfolding of f and $f(s)$ has an A_k-singularity ($k\ge 1$) at s_0. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s,x_0))(s_0)=\sum_{j=0}^{k-1}\alpha_{ji}(s-s_0)^j$ for $i=1,\dots,r$. Then F is called an *$\mathcal{R}$-versal unfolding* if the $k\times r$ matrix of coefficients $(\alpha_{ji})_{j=0,\dots,k-1;i=1,\dots,r}$ has rank k ($k\le r$). We introduce an important set concerning the unfoldings relative to the above notions. The *discriminant set* of F is the set

$$
\mathcal{D}_F=\left\{x\in\mathbb{R}^r \middle| \text{there exists } s \text{ suchthat } F(s,x)=\frac{\partial F}{\partial s}(s,x)=0\right\}.
$$

Then we have the following classification (cf., [2]).

Theorem 5.2. Let $F:(\mathbb{R}\times\mathbb{R}^r,(s_0,x_0))\to\mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has an A_k singularity at s_0 ($k=1,2$). Suppose that F is an $\mathcal{R}$-versal unfolding.
(1) If $k=1$, then $\mathcal{D}_F$ is locally diffeomorphic to $\mathbb{R}^{r-1}$.

(2) If $k = 2$, then $\mathcal{D}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.
Here, $C = \{(x_1, x_2) \in (\mathbb{R}^2, 0) \mid x_1 = t^2, x_2 = t^3,\ t \in (\mathbb{R}, 0)\ \}$ is the *ordinary cusp*.

By Proposition 5.1, the discriminant set $\mathcal{D}_H$ of H is

$$D_H = \left\{ g_\phi(r,t) \;\middle|\; -\sqrt{r^2\sin^2\phi + 1}c_2(t) + r\left(\cos\phi c_1(t) - c_3(t)\right) = 0 \right\}.$$

Suppose $c_2(t) \neq 0, \cos\phi c_1(t) - c_3(t) \neq 0$. If

$$-\sqrt{r^2\sin^2\phi + 1}c_2(t) + r\left(\cos\phi c_1(t) - c_3(t)\right) = 0,$$

then we have

$$r = \pm\frac{c_2(t)}{\sqrt{(\cos\phi c_1(t) - c_3(t))^2 - (\sin\phi c_2(t))^2}}.$$

If $r = -c_2(t)/\sqrt{(\cos\phi c_1(t) - c_3(t))^2 - (\sin\phi c_2(t))^2}$, then

$$-\sqrt{r^2\sin^2\phi + 1}c_2(t) + r\left(\cos\phi c_1(t) - c_3(t)\right) \neq 0,$$

so that

$$r = \frac{c_2(t)}{\sqrt{(\cos\phi c_1(t) - c_3(t))^2 - (\sin\phi c_2(t))^2}}.$$

For $\phi = 0$, we can also choose $r = c_2(t)/(c_1(t) - c_3(t))$. Therefore, if $\phi \neq 0$, then

$$D_H = \left\{ g_\phi(r,t) \;\middle|\; r = \frac{c_2(t)}{\sqrt{(\cos\phi c_1(t) - c_3(t))^2 - (\sin\phi c_2(t))^2}}, c_2(t) \neq 0, \cos\phi c_1(t) - c_3(t) \neq 0 \right\}.$$

Under the assumptions that $c_2(t) \neq 0$ and $\cos\phi c_1(t) - c_3(t) \neq 0$, we have a $g[\phi] : J \longrightarrow H^2_+$, where $g[\phi](t)$ is defined by

$$\begin{cases} \boldsymbol{a}_0(t) + r(t)\boldsymbol{a}_1(t) + \dfrac{\sqrt{r(t)^2\sin^2\phi + 1} - 1}{\sin^2\phi}(\boldsymbol{a}_0(t) + \cos\phi\boldsymbol{a}_2(t)) & \text{if } \phi \neq 0, \\ \boldsymbol{a}_0(t) + r(t)\boldsymbol{a}_1(t) + \dfrac{r(t)^2}{2}(\boldsymbol{a}_0(t) + \boldsymbol{a}_2(t)) & \text{if } \phi = 0. \end{cases}$$

Here

$$r(t) = \begin{cases} \dfrac{c_2(t)}{\sqrt{(\cos\phi c_1(t) - c_3(t))^2 - (\sin\phi c_2(t))^2}} & \text{if } \phi \neq 0, \\ \dfrac{c_2(t)}{c_1(t) - c_3(t)} & \text{if } \phi = 0. \end{cases}$$

Then $g[\phi](t)$ is a parametrization of D_H and it is the envelope of the family of ϕ-slant pseudo lines $\{SL(\boldsymbol{n}_\phi(t), -\cos\phi)\}_{t\in J}$.

In order to classify the singularities of $g[\phi]$, we apply the theory of unfoldings to H. For any $(r_0, t_0) \in I \times J$, we put $\boldsymbol{x}_0 = g_\phi(r_0, t_0)$ and consider the function germ $h_{\boldsymbol{x}_0} : (J, t_0) \longrightarrow (\mathbb{R}, 0)$ defined by

$$h_{\boldsymbol{x}_0}(t_0) = H(t_0, \boldsymbol{x}_0) = \langle \boldsymbol{x}_0, \boldsymbol{n_\phi}(t_0)\rangle + \cos\phi.$$

Then the germ of H at $(t_0, \boldsymbol{x}_0)$ is a two-dimensional unfolding of $h_{\boldsymbol{x}_0}$. We now try to search the conditions for $h_{\boldsymbol{x}_0}$ has the A_k-singularity, $(k = 1, 2)$. If $\phi \neq 0$, then we define two invariants

$$\begin{cases} \delta[\phi]_1(t) = -\sqrt{r^2\sin^2\phi + 1}c_2' + r(\cos\phi c_1' - c_3') + rc_2(c_1 - \cos\phi c_3) \\ \qquad -c_1(\cos\phi c_1 - c_3) - \cos\phi c_2^2 \\ \qquad +\dfrac{\sqrt{r^2\sin^2\phi+1}-1}{\sin^2\phi}(\cos\phi c_1 - c_3)(\cos\phi c_3 - c_1), \\ \delta[\phi]_2(t) = -\sqrt{r^2\sin^2\phi + 1}c_2'' \\ \qquad +r(\cos\phi c_1'' - c_3'') + r\left(c_2(c_1' - \cos\phi c_3') + 2c_2'(c_1 - \cos\phi c_3)\right) \\ \qquad -2c_1(\cos\phi c_1' - c_3') - c_1'(\cos\phi c_1 - c_3) - 3\cos\phi c_2 c_2' \\ \qquad +\dfrac{\sqrt{r^2\sin^2\phi+1}-1}{\sin^2\phi}\big((\cos\phi c_1 - c_3)(\cos\phi c_3' - c_1') \\ \qquad +2(\cos\phi c_1' - c_3')(\cos\phi c_3 - c_1)\big), \end{cases}$$

where $c_i = c_i(t)$, $(i = 1, 2, 3)$ and $r = r(t)$. For $\phi = 0$, we also define

$$\begin{cases} \delta[0]_1(t) = -c_2' + r(c_1' - c_3') - c_1(c_1 - c_3) - \dfrac{1}{2}c_2^2, \\ \delta[0]_2(t) = -c_2'' + r(c_1'' - c_3'') + rc_2(c_1' - c_3') - 2c_1(c_1' - c_3') \\ \qquad -c_1'(c_1 - c_3) - c_2c_2' + \dfrac{3}{2}rc_2(c_3' - c_1'). \end{cases}$$

We remark that $\lim_{\phi\to 0}\delta[\phi]_1(t) = \delta[0]_1(t)$ and $\lim_{\phi\to 0}\delta[\phi]_2(t) = \delta[0]_2(t)$. We expect that $\delta[\phi]_1(t) = \delta[\phi]_2(t) = 0$ if and only if $\delta[\phi]_1(t) = \delta[\phi]_1'(t) = 0$. However, we can only show this relation for a special case (cf., §6).

Proposition 5.3. We have the following assertions:
(1) $h_{\boldsymbol{x}_0}'(t_0) = 0$ always holds,
(2) $h_{\boldsymbol{x}_0}''(t_0) = 0$ if and only if $\delta[\phi]_1(t_0) = 0$,
(3) $h_{\boldsymbol{x}_0}''(t_0) = h_{\boldsymbol{x}_0}'''(t_0) = 0$ if and only if $\delta[\phi]_1(t_0) = \delta[\phi]_2(t_0) = 0$.

Proof. Assertion (1) holds by Proposition 5.1, (2).

(2) Since

$$\begin{aligned}\boldsymbol{n}''_\phi &= (\boldsymbol{n}'_\phi)' \\ &= c'_2\boldsymbol{a}_0 + (\cos\phi c'_1 - c'_3)\boldsymbol{a}_1 + \cos\phi c'_2\boldsymbol{a}_2 + (c_1(\cos\phi c_1 - c_3) + \cos\phi c_2^2)\boldsymbol{a}_0 \\ &\quad + c_2(c_1 - \cos\phi c_3)\boldsymbol{a}_1 + \left(c_3(\cos\phi c_1 - c_3) + c_2^2\right)\boldsymbol{a}_2,\end{aligned}$$

we have

$$\begin{aligned}&\frac{\partial^2 H}{\partial t^2}(t,\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{n}''_\phi\rangle \\ &= -\sqrt{s^2\sin^2\phi + 1}c'_2 + s(\cos\phi c'_1 - c'_3) \\ &+ \frac{(\cos^2\phi - \sqrt{s^2\sin^2\phi+1})\left(c_1(\cos\phi c_1 - c_3) + \cos\phi c_2^2\right)}{\sin^2\phi} \\ &+ sc_2(c_1 - \cos\phi c_3) + \frac{\cos\phi(\sqrt{s^2\sin^2\phi+1} - 1)\left(c_3(\cos\phi c_1 - c_3) + c_2^2\right)}{\sin^2\phi} \\ &= -\sqrt{s^2\sin^2\phi + 1}c'_2 + s(\cos\phi c'_1 - c'_3) - c_1(\cos\phi c_1 - c_3) - \cos\phi c_2^2 \\ &+ \frac{(1 - \sqrt{s^2\sin^2\phi+1})\left(c_1(\cos\phi c_1 - c_3) + \cos\phi c_2^2\right)}{\sin^2\phi} + sc_2(c_1 - \cos\phi c_3) \\ &+ \frac{\cos\phi(\sqrt{s^2\sin^2\phi+1} - 1)\left(c_3(\cos\phi c_1 - c_3) + c_2^2\right)}{\sin^2\phi} \\ &= -\sqrt{s^2\sin^2\phi + 1}c'_2 + s(\cos\phi c'_1 - c'_3) + sc_2(c_1 - \cos\phi c_3) \\ &- c_1(\cos\phi c_1 - c_3) - \cos\phi c_2^2 + \frac{(\sqrt{s^2\sin^2\phi+1} - 1)(\cos\phi c_1 - c_3)(\cos\phi c_3 - c_1)}{\sin^2\phi} \\ &= \delta[\phi]_1(t).\end{aligned}$$

(3) We have

$$\begin{aligned}&\boldsymbol{n}'''_\phi = c''_2\boldsymbol{a}_0 + (\cos\phi c''_1 - c''_3)\boldsymbol{a}_1 + \cos\phi c''_2\boldsymbol{a}_2 \\ &+ 2\left(c'_2\boldsymbol{a}'_0 + (\cos\phi c'_1 - c'_3)\boldsymbol{a}'_1 + \cos\phi c'_2\boldsymbol{a}'_2\right) + c_2\boldsymbol{a}''_0 + (\cos\phi c_1 - c_3)\boldsymbol{a}''_1 + \cos\phi c_2\boldsymbol{a}''_2.\end{aligned}$$

Here, $\boldsymbol{a}'_0 = c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2$, $\boldsymbol{a}'_1 = c_1\boldsymbol{a}_0 + c_3\boldsymbol{a}_2$, $\boldsymbol{a}'_2 = c_2\boldsymbol{a}_0 - c_3\boldsymbol{a}_1$. Then

$$\begin{cases}\boldsymbol{a}''_0 = (c_1^2 + c_2^2)\boldsymbol{a}_0 + (c'_1 - c_2c_3)\boldsymbol{a}_1 + (c'_2 + c_1c_3)\boldsymbol{a}_2, \\ \boldsymbol{a}''_1 = (c'_1 + c_2c_3)\boldsymbol{a}_0 + (c_1^2 - c_3^2)\boldsymbol{a}_1 + (c'_3 + c_1c_2)\boldsymbol{a}_2, \\ \boldsymbol{a}''_2 = (c'_2 - c_1c_3)\boldsymbol{a}_0 + (c_1c_2 - c'_3)\boldsymbol{a}_1 + (c_2^2 - c_3^2)\boldsymbol{a}_2.\end{cases}$$

Therefore,

$$\begin{aligned}\boldsymbol{n}_\phi''' &= c_2''\boldsymbol{a}_0 + (\cos\phi c_1'' - c_3'')\boldsymbol{a}_1 + \cos\phi c_2''\boldsymbol{a}_2 \\ &+ \left(c_2(c_1^2+c_2^2-c_3^2) + 2c_1(\cos\phi c_1' - c_3') + c_1'(\cos\phi c_1 - c_3) + 3\cos\phi c_2 c_2'\right)\boldsymbol{a}_0 \\ &+ \left((\cos\phi c_1 - c_3)(c_1^2+c_2^2-c_3^2) + c_2(c_1' - \cos\phi c_3') + 2c_2'(c_1 - \cos\phi c_3)\right)\boldsymbol{a}_1 \\ &+ \left(\cos\phi c_2(c_1^2+c_2^2-c_3^2) + 2c_3(\cos\phi c_1' - c_3') + c_3'(\cos\phi c_1 - c_3) + 3c_2c_2'\right)\boldsymbol{a}_2.\end{aligned}$$

By the calculation similar to case (2), we have

$$\frac{\partial^3 H}{\partial t^3}(t, \boldsymbol{x}) = \delta[\phi]_2(t).$$

For the case $\phi = 0$, we also have the similar arguments to the above case. This completes the proof. □

We have the following corollary.

Corollary 5.4. For $h_{\boldsymbol{x}_0}$ as the above proposition, we have the following:
(1) $h_{\boldsymbol{x}_0}$ has the A_1-singularity at $t = t_0$ if and only if $\delta[\phi]_1(t_0) \neq 0$.
(2) $h_{\boldsymbol{x}_0}$ has the A_2-singularity at $t = t_0$ if and only if $\delta[\phi]_1(t_0) = 0,\ \delta[\phi]_2(t_0) \neq 0$.

Then we have the following proposition.

Proposition 5.5. For $h_{\boldsymbol{x}_0}$ as the above proposition, we have the following:
(1) If $h_{\boldsymbol{x}_0}$ has the A_1-singularity, then H is a $\mathcal{R}$-versal unfolding of $h_{\boldsymbol{x}_0}$,
(2) If $h_{\boldsymbol{x}_0}$ has the A_2-singularity, then H is a $\mathcal{R}$-versal unfolding of $h_{\boldsymbol{x}_0}$.

Proof. We consider a parametrization of H^2_+ defined by

$$\psi(x_2, x_3) = (\sqrt{x_2^2 + x_3^2 + 1}, x_2, x_3).$$

Then we have

$$H(t, x_2, x_3) = H(t, \psi(x_2, x_3)) = \langle \psi(x_2, x_3), \boldsymbol{n}_\phi(t)\rangle + \cos\phi.$$

We write $\boldsymbol{n}_\phi(t) = (n_{\phi 1}(t), n_{\phi 2}(t), n_{\phi 3}(t))$ and have

$$\frac{\partial \tilde{H}}{\partial x_i}(t, x_2, x_3) = n_{\phi i}(t) - \frac{x_i}{\sqrt{x_2^2 + x_3^2 + 1}} n_{\phi 1}(t) \quad (i = 2, 3).$$

Moreover, we have

$$\frac{\partial}{\partial t}\frac{\partial \tilde{H}}{\partial x_i}(t, x_2, x_3) = n'_{\phi i}(t) - \frac{x_i}{\sqrt{x_2^2 + x_3^2 + 1}} n'_{\phi 1}(t).$$

We write that $\boldsymbol{x}_0 = (x_{01}, x_{02}, x_{03})$. Then the 1-jet of $(\partial \tilde{H}/\partial x_i)(t, x_{02}, x_{03})$ at $t = t_0$ is

$$\frac{\partial \tilde{H}}{\partial x_i}(t, x_{02}, x_{03}) = \frac{\partial \tilde{H}}{\partial x_i}(t_0, x_{02}, x_{03}) + \frac{1}{2}\frac{\partial}{\partial t}\frac{\partial \tilde{H}}{\partial x_i}(t_0, x_{02}, x_{03})(t - t_0).$$

From now on, we remove (t_0) for abbreviation.

(1) Since $h_{\boldsymbol{x_0}}$ has the A_1-singularity, we show that the rank of the matrix

$$\begin{pmatrix} n_{\phi 2} - \dfrac{x_{02}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1} & n_{\phi 3} - \dfrac{x_{03}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1} \end{pmatrix}$$

Is equal to one. If the rank is zero, then

$$n_{\phi 2} - \frac{x_{02}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1} = -\frac{\cos\phi x_{02}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} = 0$$

$$n_{\phi 3} - \frac{x_{03}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1} = 1 - \frac{\cos\phi x_{03}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} = 0$$

Thus, we have the sum of the power of the both equations

$$\begin{aligned} 0 &= \frac{\cos^2\phi(x_{02}^2 + x_{03}^2) - (x_{02}^2 + x_{03}^2 + 1)}{x_{02}^2 + x_{03}^2 + 1} \\ &= -\frac{\sin^2\phi(x_{02}^2 + x_{03}^2) + 1}{x_{02}^2 + x_{03}^2 + 1} \neq 0. \end{aligned}$$

This is a contradiction.

(2) Since $h_{\boldsymbol{x_0}}$ has the A_2-singularity, we show that the rank of the matrix

$$B = \begin{pmatrix} n_{\phi 2} - \dfrac{x_{02}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1} & n_{\phi 3} - \dfrac{x_{03}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1} \\ n'_{\phi 2} - \dfrac{x_{02}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n'_{\phi 1} & n'_{\phi 03} - \dfrac{x_{03}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n'_{\phi 1} \end{pmatrix}$$

is equal to two. Since $\boldsymbol{n}'_\phi = c_2\boldsymbol{a}_0 + (\cos\phi c_1 - c_3)\,\boldsymbol{a}_1 + \cos\phi c_2\boldsymbol{a}_2$,

$$\begin{aligned} \det B &= \begin{vmatrix} n_{\phi 2} - \dfrac{x_{02}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1} & n_{\phi 3} - \dfrac{x_{03}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n_{\phi 1} \\ n'_{\phi 2} - \dfrac{x_{02}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n'_{\phi 1} & n'_{\phi 3} - \dfrac{x_{03}}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} n'_{\phi 1} \end{vmatrix} \\ &= \frac{1}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} |\,\boldsymbol{x}_0,\ \boldsymbol{n}_\phi,\ \boldsymbol{n}'_\phi\,| \\ &= -\frac{(\cos\phi c_1 - c_3)\sqrt{s^2\sin^2\phi + 1} + s\sin^2\phi c_2}{\sqrt{x_{02}^2 + x_{03}^2 + 1}} \neq 0 \end{aligned}.$$

This means that the rank of B is two. □

It follows from Theorem 5.2 and Proposition 5.5, we have shown the following theorem.

Theorem 5.6. Let $\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t)\}_{t\in J}$ be pseudo-orthonormal moving frame of $\mathbb{R}^3_1$. Suppose $c_2(t) \neq 0$ and $\cos\phi c_1(t) - c_3(t) \neq 0$. Then we have the following:
(1) The envelope $g[\phi]$ of the family of ϕ-slant pseudo-lines $SL(\boldsymbol{n}_\phi, -\cos\phi)$ is regular at a point $t = t_0$ if and only if $\delta[\phi]_1(t_0) \neq 0$,
(2) The envelope $g[\phi]$ of the family of ϕ-slant pseudo-lines $SL(\boldsymbol{n}_\phi, -\cos\phi)$ at a point $t = t_0$ is locally diffeomorphic to the cusp C if and only if $\delta[\phi]_1(t_0) = 0,\ \delta[\phi]_2(t_0) \neq 0$.

6 Slant evolutes of hyperbolic plane curves

There is the notion of hyperbolic evolutes of hyperbolic plane curves [5]. Let $\boldsymbol{\gamma} : J \longrightarrow H^2_+$ be a unit speed curve, where we use the parameter $s \in J$ instead of t. We call $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ a *unit tangent vector* of $\boldsymbol{\gamma}$ at s. Since $\langle\boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s)\rangle = -1$ we have $\langle\boldsymbol{\gamma}(s), \boldsymbol{t}(s)\rangle = 0$. We define $\boldsymbol{e}(s) = \boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s)$, which is called a *unit binormal vector* of $\boldsymbol{\gamma}$ at $s \in J$. Then we have $\langle\boldsymbol{e}(s), \boldsymbol{e}(s)\rangle = \langle\boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s), \boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s)\rangle = -\langle\boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s)\rangle\langle\boldsymbol{t}(s), \boldsymbol{t}(s)\rangle + \langle\boldsymbol{\gamma}(s), \boldsymbol{t}(s)\rangle^2 = 1$. Therefore, we have a pseudo-orthonormal moving frame $\{\boldsymbol{\gamma}(s),\ -\boldsymbol{e}(s),\ \boldsymbol{t}(s)\}$ of $\mathbb{R}^3_1$, which is called a *hyperbolic Sabban frame* along $\boldsymbol{\gamma}$.

$$\boldsymbol{a}_0(s) = \boldsymbol{\gamma}(s)\ ,\ \boldsymbol{a}_1(s) = -\boldsymbol{e}(s)\ ,\ \boldsymbol{a}_2(s) = \boldsymbol{t}(s)$$

Then we have the following Frenet-Serret type formulae:

$$\begin{cases} \boldsymbol{\gamma}'(s) = \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) = \boldsymbol{\gamma}(s) + \kappa_g(s)\boldsymbol{e}(s) \\ \boldsymbol{e}'(s) = -\kappa_g(s)\boldsymbol{t}(s), \end{cases}$$

where $\kappa_g(s) = |\,\boldsymbol{\gamma}(s),\ \boldsymbol{\gamma}'(s),\ \boldsymbol{\gamma}''(s)\,|$ is called the *geodesic curvature* of $\boldsymbol{\gamma}$. Since $\boldsymbol{a}_0(s) = \boldsymbol{\gamma}(s)\ ,\ \boldsymbol{a}_1(s) = -\boldsymbol{e}(s)\ ,\ \boldsymbol{a}_2(s) = \boldsymbol{t}(s)$, we have $c_1(s) = 0$, $c_2(s) = 1$ and $c_3(s) = -\langle\boldsymbol{a}_1(s), \boldsymbol{a}_2'(s)\rangle = \langle\boldsymbol{e}(s), \boldsymbol{t}'(s)\rangle = \langle\boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s), \boldsymbol{t}'(s)\rangle = |\,\boldsymbol{\gamma}(s),\ \boldsymbol{t}(s),\ \boldsymbol{t}'(s)\,| = |\,\boldsymbol{\gamma}(s),\ \boldsymbol{\gamma}'(s),\ \boldsymbol{\gamma}''(s)\,| = \kappa_g(s)$. In this case, the family of ϕ-slant pseudo-lines $g_\phi : I \times J \longrightarrow H^2_+(-1)$ is

$$g_\phi(r, s) = \begin{cases} \boldsymbol{\gamma}(s) - r\,\boldsymbol{e}(s) + \dfrac{\sqrt{r^2\sin^2\phi + 1} - 1}{\sin^2\phi}(\boldsymbol{\gamma}(s) + \cos\phi\boldsymbol{t}(s)) & \text{if } \phi \neq 0, \\ \boldsymbol{\gamma}(s) - r\,\boldsymbol{e}(s) + \dfrac{r^2}{2}(\boldsymbol{\gamma}(s) + \boldsymbol{t}(s)) & \text{if } \phi = 0. \end{cases}$$

Therefore, the envelope $g[\phi] : J \longrightarrow H^2_+$ of g_ϕ is

$$g[\phi](s) = \begin{cases} \boldsymbol{\gamma}(s) - r(s)\,\boldsymbol{e}(s) + \dfrac{\sqrt{r(s)^2\sin^2\phi + 1} - 1}{\sin^2\phi}(\boldsymbol{\gamma}(s) + \cos\phi\boldsymbol{t}(s)) \text{ if } \phi \neq 0, \\ \boldsymbol{\gamma}(s) + \dfrac{1}{\kappa_g(s)}\boldsymbol{e}(s) + \dfrac{1}{2\kappa_g^2(s)}(\boldsymbol{\gamma}(s) + \boldsymbol{t}(s)) \text{ if } \phi = 0, \end{cases}$$

where

$$r(s) = \frac{1}{\sqrt{\kappa_g^2(s) - \sin^2\phi}}.$$

We call $g[\pi/2]$ a *hyperbolic evolute* and $g[0]$ a *horocyclic evolute* of $\boldsymbol{\gamma}$, respectively. For $s_0 \in J$, we define

$$\begin{cases} \sigma[\phi]_1(s_0) = \kappa_g'(s_0) + \dfrac{\cos\phi(\kappa_g(s_0)^2 - \sin^2\phi)}{\sqrt{\kappa_g^2(s_0) - \sin^2\phi} - \kappa_g(s_0)}, \\ \sigma[\phi]_2(s_0) = \kappa_g''(s_0) + \cos\phi\kappa_g'(s_0)\left(\dfrac{\sqrt{\kappa_g^2(s_0) - \sin^2\phi} + 2\kappa_g(s_0)}{\sqrt{\kappa_g^2(s_0) - \sin^2\phi} - \kappa_g(s_0)}\right). \end{cases}$$

In this case, by a straightforward calculation, we can show that $\sigma[\phi]_1'(s) = \sigma[\phi]_2(s)$. Moreover, we can show that $\delta[\phi]_1(s) = \delta[\phi]_2(s) = 0$ if and only if $\sigma[\phi]_1(s) = \sigma[\phi]_1'(s) = 0$. As special cases, we have

$$\sigma[0]_1(s) = \kappa_g'(s) - \frac{1}{2}\kappa_g(s),\ \sigma[\pi/2]_1(s) = \kappa_g'(s).$$

As a corollary of Theorem 5.6, we have the following theorem.

Theorem 6.1. Let $\boldsymbol{\gamma} : J \longrightarrow H^2_+(-1)$ be a unit speed curve with $\kappa_g(s)^2 - \sin^2\phi > 0$. Then we have the following:
(1) $g[\phi]$ is a regular curve at $s = s_0$ if and only if $\sigma[\phi]_1(s_0) \neq 0$,
(2) $g[\phi]$ is locally diffeomorphic to the cusp C at $s = s_0$ if and only if

$$\sigma[\phi]_1(s_0) = 0 \text{ and } \sigma[\phi]_1'(s_0) \neq 0.$$

As a special case, we have the following corollary.

Corollary 6.2. Let $\boldsymbol{\gamma} : J \longrightarrow H^2_+(-1)$ be a unit speed curve.
(A) Suppose $\kappa_g^2 > 1$. Then we have the following (cf., [5]):

(1) The hyperbolic evolute $g[\pi/2]$ is a regular curve at $s = s_0$ if and only if $\kappa_g'(s) \neq 0$.

(2) The hyperbolic evolute $g[\pi/2]$ is locally diffeomorphic to the cusp C at $s = s_0$ if and only if

$$\kappa_g'(s_0) \neq 0 \text{ and } \kappa_g''(s_0) \neq 0.$$

(B) Suppose $\kappa_g \neq 0$. Then we have the following:

(1) The horocyclic evolute $g[0]$ is a regular curve at $s = s_0$ if and only if $\kappa_g'(s) - \frac{1}{2}\kappa_g(s) \neq 0$.

(2) The horocyclic evolute $g[0]$ is locally diffeomorphic to the cusp C at $s = s_0$ if and only if

$$\kappa_g'(s_0) - \frac{1}{2}\kappa_g(s_0) = 0 \text{ and } \kappa_g''(s_0) - \frac{1}{2}\kappa_g'(s_0) \neq 0.$$

The hyperbolic evolute is given by

$$g[\pi/2](s) = \begin{cases} \dfrac{-1}{\sqrt{\kappa_g^2(s) - 1}}(\kappa_g(s)\boldsymbol{\gamma}(s) + \boldsymbol{e}(s)) \text{ if } \kappa_g(s) < -1, \\ \dfrac{1}{\sqrt{\kappa_g^2(s) - 1}}(\kappa_g(s)\boldsymbol{\gamma}(s) - \boldsymbol{e}(s)) \text{ if } \kappa_g(s) > 1 \end{cases}$$

and the horocyclic evolute is

$$g[0](s) = \boldsymbol{\gamma}(s) + \frac{1}{\kappa_g(s)}\boldsymbol{e}(s) + \frac{1}{2\kappa_g^2(s)}(\boldsymbol{\gamma}(s) + \boldsymbol{t}(s)).$$

In [5] hyperbolic evolutes was introduced and the classified the singularities. Moreover, a *de Sitter evolute* of $\boldsymbol{\gamma}$ was introduced in [5], which is located in the de Sitter 2-space. It corresponds to points of $\boldsymbol{\gamma}(s)$ with $\kappa_g^2(s) < 1$. Here we only consider families of hyperbolic lines, so that we do not consider de Sitter evolutes. It is also shown in [5] that $g[\pi/2](s)$ is a constant point if and only if $\boldsymbol{\gamma}$ is a part of a circle. This condition is also equivalent to $\kappa_g'(s) \equiv 0$. We have a natural question what is $\boldsymbol{\gamma}$ when $g[0](s)$ is a constant point. Of course it is equivalent to

$$\kappa_g'(s) - \frac{1}{2}\kappa_g(s) \equiv 0.$$

The solution of the above differential equation is $\kappa_g(s) = ce^{s/2}$ for a constant real number c. The curvature tends to infinity, so that $\boldsymbol{\gamma}$ is a kind of spirals in $H_+^2(-1)$. If $c = 1/2$, the curve with the curvature $\dfrac{1}{2}c^{s/2}$ in the Euclidean plane is called a *Nielsen spiral*. So we call $\boldsymbol{\gamma}$ with $\kappa_g(s) = \frac{1}{2}c^{s/2}$ a *hyperbolic Nielsen spiral*. We have two open problems as follows:

(1) What is $\boldsymbol{\gamma}$ with $\sigma[\phi]_1(s) \equiv 0$?

(2) For a general one-parameter family of pseudo-lines, is it always true that $\delta[\phi]_1(t) = \delta[\phi]_2(t) = 0$ if and only if $\delta[\phi]_1(t) = \delta[\phi]_1'(t) = 0$?

References

[1] M. Asayama, S. Izumiya, A. Tamaoki and H. Yıldırım, *Slant geometry of spacelike hypersurfaces in Hyperbolic space and de Sitter space.* Revista Matematica Iberoamericana, **28** (2012), 371–400.

[2] J. W. Bruce and P. J. Giblin, *Curves and singularities* (*second edition*). Cambridge University Press (1992)

[3] P.J. Giblin and J. P. Warder, *Evolving evolutoids.* Amer. Math. Monthly **121** (2014), 871–889.

[4] R. Hayashi, S. Izumiya and T. Sato, *Duals of curves in Hyperbolic space.* Note Mat. **33** (2013), 97–106.

[5] S. Izumiya, D-H. Pei, T. Sano and E. Torii, *Evolutes of hyperbolic plane curves.* Acta Mathematica Sinica, English Series **20** (2004), 543–550.

[6] S. Izumiya, D.-H.Pei and T. Sano, *Horospherical surfaces of curves in hyperbolic space.* Publ. Math. Debrecen, **64** (2004), 1–13.

[7] S. Izumiya, D-H. Pei, M.C. Romero-Fuster and M. Takahashi, *On the horospherical ridges of submanifolds of codimension 2 in hyperbolic n-space*, Bull. Braz. Math. Soc. (N.S.), **35** (2004), 177-198.

[8] S. Izumiya, D-H. Pei, M.C. Romero Fuster and M. Takahashi, *The horospherical geometry of submanifolds in hyperbolic space*, J. London Math. Soc., **2** 71 (2005), 779-800.

[9] S. Izumiya, *Horospherical geometry in the hyperbolic space*, Advanced Studies in Pure Mathematics **55** (2009), 31–49.

[10] S, Izumiya, *Slant pseudo-lines in the hyperbolic plane.* International Journal of Geometry **4** (2015), 37–41.

[11] A. Ramsay and R. D. Richtmyer, *Introduction to hyperbolic geometry.* Springer-Verlag, New York Berlin Heidelberg (1994)