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Ingham type inequalities towards Parseval equality

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Abstract. We consider Trigonometric series with real exponents λ_k:

$$\sum_{k=1}^{+\infty} x_k e^{i\lambda_k t}.$$

Under an assumption on the gap γ_M between λ_k, we show the inequality

$$\frac{2\pi}{\gamma_M(2-c_M)} \sum_{n=1}^{M} |x_n|^2 \leq \int_{-\pi/\gamma_M}^{\pi/\gamma_M} |\sum_{k=1}^{M} x_k e^{i\lambda_k t}|^2 dt \leq \frac{2\pi}{c_M \gamma_M} \sum_{n=1}^{M} |x_n|^2$$

and we show for a class of problems that the limit as $M \to +\infty$ leads to the Parseval's equality. The role of constants c_M in the above formula is one of the key points of the paper.

Keywords: Trigonometric polynomials, inequalities, Parseval equality

MSC 2000 classification: 42A05,41A58

1 Introduction

In this paper we establish direct and inverse inequalities involving Trigonometric series $\sum_{k=1}^{+\infty} x_k e^{i\lambda_k t}$ under assumptions that ensure Parseval's equality as a limiting case.

The inequalities are here obtained for finite M- sums $\sum_{k=1}^{M} x_k e^{i\lambda_k t}$, then we investigate the behavior of the constants $c_{1,M}, c_{2,M}$ and of the gap γ_M appearing on them, as $M \to +\infty$.

$$c_{1,M} \sum_{k=-M}^{M} |x_k|^2 \leq \int_{-\pi/\gamma_M}^{\pi/\gamma_M} |\sum_{k=-M}^{+\infty} x_k e^{i\lambda_k t}|^2 dt \leq c_{2,M} \sum_{k=-M}^{M} |x_k|^2. \tag{1.1}$$

The method used here takes into account the arguments in [2]. In that paper the author is interested to obtain estimates for the single coefficient x_k in the critical time $T = \pi/\gamma$. However many of his arguments have to be adapted to the proofs of this paper. The role of the function k in (2.13) is also inspired by [2], as well as the choice of the polynomials P and Q: here we introduce a new couple of polynomials R and S helpful to define the constant c_M, that appears in the main result of the paper (see formula (2.2) in Theorem 2), and whose limit as $M \to +\infty$ gives the expected result. During the development of the theoretical part of the paper, we had in mind an application that we describe in Section 3. In the application we prove Parseval's equality in the limit.
For any fixed $M \in \mathbb{N}$ and any set $(x_k)_{k=-M}^{M} \in \mathbb{C}^{2M+1}$, we consider $g_M : \mathbb{R} \to \mathbb{C}$ defined by

$$g_M(t) = \sum_{k=-M}^{M} x_k e^{ikt}. \tag{1.2}$$

The orthogonality of the set $\{e^{ikt}\}_{k=-M}^{M}$ in $L_2(-\pi,\pi)$ leads to the identity

$$\int_{-\pi}^{\pi} |g_M(t)|^2 dt = 2\pi \sum_{k=-M}^{M} |x_k|^2. \tag{1.3}$$

For any $g \in L^2(-\pi,\pi)$, the limit as $M \to +\infty$ gives *Parseval's equality*,

$$\int_{-\pi}^{\pi} |g(t)|^2 dt = 2\pi \sum_{k=-\infty}^{\infty} |x_k|^2, \tag{1.4}$$

valid for any sequence $(x_k)_{k\in\mathbb{Z}}$ such that $\sum_{k=-\infty}^{\infty} |x_k|^2 < +\infty$ with

$$g(t) = \sum_{k=-\infty}^{\infty} x_k e^{ikt}. \tag{1.5}$$

Let $(\lambda_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers. We consider

$$f_M(t) = \sum_{k=-M}^{M} x_k e^{i\lambda_k t}, \qquad f(t) = \sum_{k=-\infty}^{\infty} x_k e^{i\lambda_k t}, \tag{1.6}$$

and we investigate the problem to find inverse and direct inequalities, that is

$$c_{1,M} \sum_{k=-M}^{M} |x_k|^2 \le \int_{-\pi/\gamma_M}^{\pi/\gamma_M} |f_M(t)|^2 dt \le c_{2,M} \sum_{k=-M}^{M} |x_k|^2, \tag{1.7}$$

where $\gamma_M, c_{1,M}, c_{2,M}$ are real positive constants. The novelty of this note is to establish inverse and direct inequalities as (1.7) with accurate estimates of the constants $c_{1,M}$, $c_{2,M}$. The estimates narrow, as $M \to +\infty$, to the equality

$$\int_{-\pi}^{\pi} |f(t)|^2 dt = 2\pi \sum_{k=-\infty}^{\infty} |x_k|^2. \tag{1.8}$$

However we are unable to give general conditions under which the theorem holds. We give an application of general interest to support the validity of the estimates we obtain.

As a reference in this study we recall Ingham's theorem with explicit constants on non-harmonic Fourier series (see [1], see also the Young's textbook [6], and [3].)

Here we recall Ingham's theorem. The result is largely used in control theory and it is the basis of the Fourier series method in observability problems. The explicit constant on the right hand side of formula (1.9) may be deduced from [3] pg 63 by a change of variables. For the explicit constant on the left hand side of formula (1.9) we refer to [1] pg. 369.

Theorem 1. *Assume the following* gap condition*: there exists* $\gamma > 0$ *such that*

$$\lambda_{k+1} - \lambda_k \geq \gamma.$$

Then for any fixed $T > 0$ *the following inequality holds for all square summable sequences* $(x_k)_{k\in\mathbb{Z}} \in \mathbb{C}$ *:*

$$\frac{4T}{\pi}\left(1 - \frac{\pi^2}{T^2\gamma^2}\right) \sum_{k=-\infty}^{\infty} |x_k|^2 \leq \int_{-T}^{T} |\sum_{k=-\infty}^{\infty} x_k e^{i\lambda_k t}|^2 \, dt \leq \left(4T + \frac{4\pi}{\gamma}\right) \sum_{k=-\infty}^{\infty} |x_k|^2 \tag{1.9}$$

Remark 1. To get positive constants in (1.9) we need

$$1 - \frac{\pi^2}{T^2\gamma^2} > 0 \qquad \Longleftrightarrow \qquad T > \frac{\pi}{\gamma}$$

Ingham's theorem does not generalize Parseval's equality since if $T \to \pi$ we obtain

$$4\left(1 - \frac{1}{\gamma^2}\right) \sum_{k=-\infty}^{\infty} |x_k|^2 \leq \int_{-\pi}^{\pi} |\sum_{k=-\infty}^{\infty} x_k e^{i\lambda_k t}|^2 \, dt \leq 4\pi\left(1 + \frac{1}{\gamma}\right) \sum_{k=-\infty}^{\infty} |x_k|^2$$

As $\gamma = 1$

$$0 \leq \int_{-\pi}^{\pi} |\sum_{k=-\infty}^{\infty} x_k e^{i\lambda_k t}|^2 \, dt \leq 8\pi \sum_{k=-\infty}^{\infty} |x_k|^2.$$

2 The result

To simplify, we assume that $k \in \mathbb{N}$, the arguments used here can be adapted to $k \in \mathbb{Z}$. Let

$$f_M(t) = \sum_{k=1}^{M} x_k e^{i\lambda_k t}. \tag{2.1}$$

The main result of the paper is the following

Theorem 2. *We assume that the sequence $(\lambda_k)_{k\in\mathbb{N}}$ is increasing and it satisfies a gap condition*

$$\exists \gamma_M > 1 \ \text{ such that } \ \lambda_{k+1} - \lambda_k > \gamma_M, \quad \forall k \in \{1, 2, \ldots, M-1\}$$

Then there exists a positive constant $c_M < 1$ such that

$$\frac{2\pi}{\gamma_M(2-c_M)} \sum_{n=1}^{M} |x_n|^2 \leq \int_{-\pi/\gamma_M}^{\pi/\gamma_M} |f_M(t)|^2 dt \leq \frac{2\pi}{c_M \gamma_M} \sum_{n=1}^{M} |x_n|^2 \tag{2.2}$$

Proposition 1. *Assume that*

$$\lim_{M\to+\infty} c_M = 1 \quad \text{and} \quad \lim_{M\to+\infty} \gamma_M = 1$$

then Parseval's equality holds

$$\int_{-\pi}^{\pi} |f(t)|^2 dt = 2\pi \sum_{n=1}^{\infty} |x_n|^2.$$

The proof of the proposition follows from formula (2.2).

Proof of the theorem 2. Let $j, q \in \{1, 2, \ldots, M\}$, and set

$$\mu_{j,q} = \lambda_j - \lambda_q$$

with $\mu_{q,j} = -\mu_{j,q}$.

We consider the integers $m_{j,q}$ uniquely defined by the relations

$$m_{j,q} = \begin{cases} \lfloor \frac{\mu_{j,q}}{\gamma_M} \rfloor & \text{if} \quad q < j \leq M \\ -m_{q,j} & \text{if} \quad j < q, \\ 0 & j = q, \ q \in \{1, \ldots, M\}. \end{cases} \tag{2.3}$$

The sequence of integer numbers $m_{1,q}, m_{2,q}, \ldots, m_{M,q}$ is strictly increasing, for any fixed $q \in \{1, \ldots, M\}$. It is useful to observe that, by the definition (2.3), the numbers $m_{j,q}$ and $\mu_{j,q}$ have the same sign for $j \neq q$, and for $j > q$ we have

$$\frac{\gamma_M}{\mu_{j,q}} \leq \frac{\gamma_M m_{j,q}}{\mu_{j,q}} \leq 1, \tag{2.4}$$

We denote by $\mathcal{I}_M$ the set $\{(j,q) \in \{1,2,\dots,M\}^2 : j \neq q\}$ and we consider the polynomials

$$R(u) = \prod_{(j,q)\in\mathcal{I}_M} \Big(\frac{u}{\mu_{j,q}} - 1\Big) \qquad S(u) = u \prod_{(j,q)\in\mathcal{I}_M} \Big(\frac{u}{\gamma_M m_{j,q}} - 1\Big). \tag{2.5}$$

We set

$$c_M = \lim_{|u|\to\infty} \frac{uR(u)}{S(u)} = \prod_{(j,q)\in\mathcal{I}_M} \frac{\gamma_M m_{j,q}}{\mu_{j,q}} \tag{2.6}$$

We see that

$$\frac{R(u)}{S(u)} = c_M \frac{P(u)}{Q(u)}, \tag{2.7}$$

with

$$P(u) = \prod_{(j,q)\in\mathcal{I}_M} \big(u - \mu_{j,q}\big) \qquad Q(u) = u \prod_{(j,q)\in\mathcal{I}_M} \big(u - \gamma_M m_{j,q}\big). \tag{2.8}$$

Assuming simple roots, we look for the decomposition

$$c_M \frac{P(u)}{Q(u)} = \frac{B_0}{u} + \sum_{(j,q)\in\mathcal{I}_M} \frac{B_{l.q}}{u - \gamma_M m_{l,q}}. \tag{2.9}$$

we find the coefficients $B_0 = 1$ and

$$B_{l,r} = c_M \frac{P(\gamma_M m_{l,r})}{Q'(\gamma_M m_{l,r})} = c_M \frac{\gamma_M m_{l,r} - \mu_{l,r}}{\gamma_M m_{l,r}} \prod_{(j,q)\in\mathcal{I}_M,(j,q)\neq(l,r)} \frac{\gamma_M m_{l,r} - \mu_{j,q}}{\gamma_M (m_{l,r} - m_{j,q})} \tag{2.10}$$

By the previous computation we get

$$1 + \sum_{(j,q)\in\mathcal{I}_M} B_{l,q} = c_M \tag{2.11}$$

To end the proof we observe that for every $(l,r) \in \mathcal{I}_M$ each $B_{l,r}$ is negative. Indeed since

$$\gamma_M m_{l,r} < \mu_{j,q} \iff \gamma_M m_{l,r} < \gamma_M m_{j,q}$$

all the factors $\frac{\gamma_M m_{l,r} - \mu_{j,q}}{\gamma_M m_{l,r} - \gamma_M m_{j,q}}$ are positive, on the contrary $\frac{\gamma_M m_{l,r} - \mu_{l,r}}{\gamma_M m_{l,r}}$ is negative, then for every $(l,r) \in \mathcal{I}_M$ each $B_{l,r}$ is negative.

Hence by (2.11)

$$1 - c_M = - \sum_{(j,q)\in\mathcal{I}_M} B_{l,q} = \sum_{(j,q)\in\mathcal{I}} |B_{l,q}| \tag{2.12}$$

We are going to define the functions k, whose Fourier transform verifies the properties:

$$\hat{k}(\lambda_n - \lambda_q) = \hat{k}(\mu_{n,q}) = 0, \quad \forall n \neq q, \quad \hat{k}(0) = \hat{k}(\mu_{q,q}) > 0.$$

Inspired by the Ingham paper [2], we can set as k function the following

$$k(\xi) = \begin{cases} 1 + \sum_{(j,q)\in\mathcal{I}_M} (-1)^{m_{j,q}} B_{j,q} e^{im_{j,q}\xi} & \text{if } |\xi| \leq \pi/\gamma. \\ 0 & \text{if } |\xi| > \pi/\gamma. \end{cases} \tag{2.13}$$

The Fourier transform of the function k is

$$\hat{k}(u) = \frac{1}{2\pi} \int_{-\pi/\gamma_M}^{\pi/\gamma_M} \big(1 + \sum_{(l,q)\in\mathcal{I}_M} (-1)^{m_{l,q}} B_{l,q} e^{im_{l,q}\xi}\big) e^{-i\xi u} d\xi =$$

$$\frac{1}{\pi}\left[\frac{\sin \frac{\pi u}{\gamma_M}}{u} + \sum_{(l,q)\in\mathcal{I}_M} (-1)^{m_{l,q}} B_{l,q} \frac{\sin(\frac{\pi}{\gamma_M}(u - \gamma_M m_{l,q}))}{u - \gamma_M m_{l,q}}\right] =$$

$$\frac{1}{\pi}\left[\frac{\sin \frac{\pi u}{\gamma_M}}{u} + \sum_{(l,q)\in\mathcal{I}_M} \frac{B_{l,q}}{u - \gamma_M m_{l,q}} \sin(\frac{\pi}{\gamma_M} u)\right] =$$

$$\frac{1}{\pi}\left[\frac{1}{u} + \sum_{(l,q)\in\mathcal{I}_M} \frac{B_{l,q}}{u - \gamma_M m_{l,q}}\right] \sin \frac{\pi}{\gamma_M} u =$$

$$= \frac{1}{\pi} c_M \frac{P(u)}{Q(u)} \sin \frac{\pi}{\gamma_M} u = \frac{1}{\pi} \frac{R(u)}{S(u)} \sin \frac{\pi}{\gamma_M} u$$

Then

$$\lim_{u\to 0} \hat{k}(u) = \frac{1}{\gamma_M}.$$

Moreover, since the function $\frac{\sin \frac{\pi}{\gamma} u}{Q(u)}$ is regular in all the zeros of the polinomial $Q(u)$, and $P(\lambda_j - \lambda_q) = 0$ for $(j, q) \in \mathcal{I}_M$, it follows

$$\begin{cases} \hat{k}(\lambda_j - \lambda_q) = 0 & \forall (j, q) \in \mathcal{I}_M \\ \hat{k}(0) = \frac{1}{\gamma_M}. \end{cases} \tag{2.14}$$

Moreover we have the estimate

$$k(t) \geq 1 - \sum_{(j,q)\in\mathcal{I}_M} |B_{j,q}| = c_M \tag{2.15}$$

$$k(t) \leq 1 + \sum_{(j,q)\in\mathcal{I}_M} |B_{l,q}| = 2 - c_M \tag{2.16}$$

Therefore

$$\int_{-\pi/\gamma_M}^{\pi/\gamma_M} k(t)|f_M(t)|^2 dt \geq \Big(1 - \sum_{(j,q)\in\mathcal{I}_M} |B_{j,q}|\Big) \int_{-\pi/\gamma_M}^{\pi/\gamma_M} |f_M(t)|^2 dt = c_M \int_{-\pi/\gamma_M}^{\pi/\gamma_M} |f_M(t)|^2 dt \tag{2.17}$$

and

$$\int_{-\pi/\gamma_M}^{\pi/\gamma_M} k(t)|f_M(t)|^2 dt \leq \Big(1 + \sum_{(j,q)\in\mathcal{I}_M} |B_{j,q}|\Big) \int_{-\pi/\gamma_M}^{\pi/\gamma_M} |f_M(t)|^2 dt = (2 - c_M) \int_{-\pi/\gamma_M}^{\pi/\gamma_M} |f_M(t)|^2 dt \tag{2.18}$$

By the properties of $\hat{k}$, see (2.14)

$$\int_{-\pi/\gamma_M}^{\pi/\gamma_M} k(t)|f_M(t)|^2 dt = 2\pi\hat{k}(0) \sum_{n=1}^{M} |x_n|^2$$

we get the estimate

$$\frac{2\pi}{\gamma_M(2 - c_M)} \sum_{n=1}^{M} |x_n|^2 \leq \int_{-\pi/\gamma_M}^{\pi/\gamma_M} |f_M(t)|^2 dt \leq \frac{2\pi}{c_M \gamma_M} \sum_{n=1}^{M} |x_n|^2. \tag{2.19}$$

3 Application

3.1 Perturbed wave equation

At first, consider the one dimensional interval, $(0, T)$, $T \geq 2\pi$, and the wave equation with Dirichlet boundary conditions:

$$\begin{cases} f_{tt}(x,t) - f_{xx}(x,t) = 0 \quad \text{in } (0,\pi) \times (0,T), \\ f(x,0) = f_0(x) \quad \text{in } (0,\pi), \\ f_t(x,0) = f_1(x) \quad \text{in } (0,\pi), \\ f(0,t) = f(\pi,t) = 0 \quad \text{in } (0,T) \end{cases} \tag{3.1}$$

The solution of (3.1) is given in terms of Fourier series by

$$f(x,t) = \sum_{k\in\mathbb{Z}^*} x_k e^{ikt} \sin kx.$$

The system (3.1) is the dual observability problem of the exact controllability boundary control problem for the wave equation, the observation functions are f_x in $x = 0$, or/and in $x = \pi$. We have

$$f_x(0,t) = \sum_{k\in\mathbb{Z}^*} kx_k e^{i\lambda_k t} \qquad f_x(\pi,t) = \sum_{k\in\mathbb{Z}^*} (-1)^k kx_k e^{i\lambda_k t}.$$

We write $g \asymp h$ if there exist two positive constants α, β such that $\alpha g \leq h \leq \beta g$. With this notation we have

$$\|f_x(0,t)\|^2_{L^2(0,T)} \asymp \sum_{k\in\mathbb{Z}^*} |kx_k|^2 \qquad \|f_x(\pi,t)\|^2_{L^2(0,T)} \asymp \sum_{k\in\mathbb{Z}^*} |kx_k|^2.$$

Due to the orthogonality of trigonometric polynomials, $T = 2\pi$ gives

$$\|f_x(0,t)\|^2_{L^2(0,T)} = 2\pi \sum_{k\in\mathbb{Z}^*} |kx_k|^2.$$

However Ingham's theorem does not include the critical time $T = 2\pi$ as $\gamma = 1$.

We consider the wave equation with a perturbation of zero-order.
For any c such that $|c| < 1$, we consider

$$\begin{cases} f_{tt}(x,t) - f_{xx}(x,t) - c^2 f(x,t) = 0 & \text{in } (0,\pi)\times(0,T), \\ f(x,0) = f_0(x) & \text{in } (0,\pi), \\ f_t(x,0) = f_1(x) & \text{in } (0,\pi), \\ f(0,t) = f(\pi,t) = 0 & \text{in } (0,T). \end{cases} \tag{3.2}$$

The solution of (3.2) is given in terms of Fourier series by

$$f(x,t) = \sum_{n\in\mathbb{Z}^*} x_n e^{i\lambda_n t} \sin nx$$

with

$$\lambda_n = sgn(n)\sqrt{n^2 - c^2} = n\sqrt{1 - \frac{c^2}{n^2}}.$$

For large n we have

$$\lambda_n \approx n - \frac{c^2}{2n}.$$

3.2 An application of the main result

For any real positive number a we consider the set of real numbers $\lambda_n(a) = \lambda_n \quad n = 1 \ldots M$, with

$$\lambda_n = n - \frac{1}{an} \qquad \lambda_{n+1} - \lambda_n = 1 + \frac{1}{an(n+1)} \tag{3.3}$$

Then

$$\lambda_{n+1} - \lambda_n > \gamma_M, \qquad \forall n \in \{1, 2, \ldots, M-1\}$$

with

$$\gamma_M = 1 + \frac{1}{aM(M+1)} \tag{3.4}$$

$$\mathcal{I}_M = \Big\{(j,q) \in \{1, 2, \ldots, M\}^2 \, \text{such that} \, j > q\Big\}$$

and

$$\mathcal{I} = \bigcup_{M=1}^{+\infty} \mathcal{I}_M$$

Then, with the same meaning of $\mu_{j,q}$ and $m_{j,q}$ of previous section we set as previously

$$R(u) = \prod_{(j,q)\in\mathcal{I}_M} \Big(\frac{u}{\mu_{j,q}} - 1\Big). \tag{3.5}$$

$$S(u) = u \prod_{(j,q)\in\mathcal{I}_M} \Big(\frac{u}{\gamma_M m_{j,q}} - 1\Big) \tag{3.6}$$

and can be determined as

$$c_M = \lim_{u\to\infty} \frac{uR(u)}{S(u)} = \lim_{u\to\infty} \prod_{(j,q)\in\mathcal{I}_M} \frac{\Big(\frac{u}{\mu_{j,q}} - 1\Big)}{\Big(\frac{u}{\gamma_M m_{j,q}} - 1\Big)} = \prod_{(j,q)\in\mathcal{I}_M} \frac{\gamma_M m_{j,q}}{\mu_{j,q}}. \tag{3.7}$$

The aim of this section is to show

Proposition 2. *Let c_M be as in* (3.7). *Then*

$$\lim_{M\to\infty} c_M = 1.$$

To give the proof of the above proposition we recall some basic facts on the notion of limit for generalized sequences

3.2.1 Limits for generalized sequences

For any (j,q) and (r,s) belonging to $\mathcal{I}$ we write

$$(j,q) \succeq (r,s) \qquad \Longleftrightarrow \qquad j \geq r \,\text{and}\, q \geq s.$$

The set $\mathcal{I}$ endowed with $\succeq$ is a partial ordered set, the set $\mathcal{I}_M$ is a direct set and the generalized sequence

$$(j,q) \in \mathcal{I} \to \frac{ajq}{ajq+1} \in \mathbb{R},$$

is increasing and bounded in $\mathcal{I}_M$, with maximum value $\frac{aM(M-1)}{aM(M-1)+1}$.

Lemma 1. *With the previous notations*

$$\lim_{M\to\infty} \prod_{(j,q)\in\mathcal{I}_M} \left(1+\frac{1}{aM(M+1)}\right)\frac{ajq}{ajq+1} = 1$$

Proof. We consider

$$\prod_{(j,q)\in\mathcal{I}_M} \left(1+\frac{1}{aM(M+1)}\right)\frac{ajq}{ajq+1} = \left(1+\frac{1}{aM(M+1)}\right)^{\frac{M(M-1)}{2}} \prod_{(j,q)\in\mathcal{I}_M} \frac{ajq}{ajq+1}$$

Since

$$\lim_{M\to\infty} \left(1+\frac{1}{aM(M+1)}\right)^{\frac{M(M-1)}{2}} = e^{\frac{1}{2a}}.$$

The geometric mean theorem implies

$$\lim_{M\to\infty} \left(\prod_{(j,q)\in\mathcal{I}_M} \frac{ajq}{ajq+1}\right)^{\frac{2}{M(M-1)}} = \lim_{M\to\infty} \frac{aM(M-1)}{aM(M-1)+1}$$

hence

$$\lim_{M\to\infty} \prod_{(j,q)\in\mathcal{I}_M} \frac{ajq}{ajq+1} = \lim_{M\to\infty} \left(\left(\prod_{(j,q)\in\mathcal{I}_M} \frac{ajq}{ajq+1}\right)^{\frac{2}{M(M-1)}}\right)^{\frac{M(M-1)}{2}} =$$

$$\lim_{M\to\infty} \left(\frac{aM(M-1)}{aM(M-1)+1}\right)^{\frac{M(M-1)}{2}} = \frac{1}{e^{\frac{1}{2a}}}$$

Plug the limits in the formula

$$\lim_{M\to\infty} \prod_{(j,q)\in\mathcal{I}_M} \left(1+\frac{1}{aM(M+1)}\right)\frac{ajq}{ajq+1} = 1$$

QED

Proof of the Proposition.

We recall that

$$c_M = \prod_{(j,q)\in\mathcal{I}_M} \frac{\gamma_M m_{j,q}}{\mu_{j,q}}.$$

Then we compute

$$\frac{\gamma_M m_{j,q}}{\mu_{j,q}} = \frac{\gamma_M[\lambda_n - \lambda_m]}{\lambda_n - \lambda_m} \ge \gamma_M \frac{j-q}{(j-q)\big(1+\frac{1}{ajq}\big)} = \left(1 + \frac{1}{aM(M+1)}\right)\frac{ajq}{ajq+1}.$$

It follows

$$1 \ge c_M = \prod_{(j,q)\in\mathcal{I}_M} \frac{\gamma_M m_{j,q}}{\mu_{j,q}} \ge \prod_{(j,q)\in\mathcal{I}_M} \left(1 + \frac{1}{aM(M+1)}\right)\frac{ajq}{ajq+1},$$

the result will follow by the lemma 1

$$\lim_{M\to\infty} \prod_{(j,q)\in\mathcal{I}_M} \left(1 + \frac{1}{aM(M+1)}\right)\frac{ajq}{ajq+1} = 1. \tag{3.8}$$

References

[1] A. E. Ingham: *Some trigonometrical inequalities with applications in the theory of series*, Math. Z. **41** (1936), 367–379.

[2] A. E. Ingham: *A further note on trigonometrical inequalities*, Mathematical Proceedings of the Cambridge Philosophical Society, Volume**46**, Issue 04, (1950), 535–537.

[3] V. Komornik and P. Loreti: *Fourier Series in Control Theory*, Springer-Verlag, New York, 2005.

[4] J.-L. Lions: *Exact controllability, stabilizability, and perturbations for distributed systems*, Siam Rev. 30 (1988), 1–68.

[5] P. Loreti: *On some gap theorems*, European women in mathematics—Marseille 2003, 39–45, CWI Tract, 135, Centrum Wisk. Inform., Amsterdam, 2005.

[6] Robert M. Young: *An Introduction to Non-Harmonic Fourier Series*, Revised Edition, Cambridge University Press, 1959.

[7] A. Zygmund,: *Trigonometric Series I-II*, Cambridge University Press, 1959.