Characterization of ultradifferentiable test functions defined by weight matrices in terms of their Fourier Transform

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Abstract. We prove that functions with compact support in non-quasianalytic classes $\mathcal{E}_{\{\mathcal{M}\}}$ of Roumieu-type and $\mathcal{E}_{(\mathcal{M})}$ of Beurling-type defined by a weight matrix $\mathcal{M}$ with some mild regularity conditions can be characterized by the decay properties of their Fourier transform. For this we introduce the abstract technique of constructing from $\mathcal{M}$ multi-index matrices and associated function spaces. We study the behaviour of this construction in detail and characterize its stability. Moreover non-quasianalyticity of the classes $\mathcal{E}_{\{\mathcal{M}\}}$ and $\mathcal{E}_{(\mathcal{M})}$ is characterized.

Keywords: Ultradifferentiable functions, non-quasianalyticity, Fourier transform

MSC 2000 classification: 26E10, 30D60, 46E10, 46A13

1 Introduction

Spaces of ultradifferentiable functions are sub-classes of smooth functions with certain growth conditions on all their derivatives. In the literature two different approaches are considered to introduce these classes, either using a weight sequence $M = (M_k)_k$ or using a weight function ω. Given a compact set K the classes

$$\left\{ \frac{f^{(k)}(x)}{h^k M_k} : x \in K, k \in \mathbb{N} \right\},$$

respectively

$$\left\{ \frac{f^{(k)}(x)}{\exp(1/l\varphi_\omega^*(lk))} : x \in K, k \in \mathbb{N} \right\},$$

should be bounded, where the positive real number h or l is subject to either a universal or an existential quantifier and φ_ω^* denotes the Young-conjugate of

$\varphi_\omega = \omega \circ \exp$. In the case of a universal quantifier we call the class of *Beurling type*, denoted by $\mathcal{E}_{(M)}$ or $\mathcal{E}_{(\omega)}$. In the case of an existential quantifier we call the class of *Roumieu type*, denoted by $\mathcal{E}_{\{M\}}$ or $\mathcal{E}_{\{\omega\}}$. In the following we write $\mathcal{E}_{[\star]}$ if either $\mathcal{E}_{\{\star\}}$ or $\mathcal{E}_{(\star)}$ is considered.

The classes $\mathcal{E}_{[M]}$ were considered earlier than $\mathcal{E}_{[\omega]}$. For the weight sequence approach see e.g. [7] and [6], for $\mathcal{E}_{[\omega]}$ we refer to [2]. In [1] both methods were compared and it was shown that in general a class $\mathcal{E}_{[M]}$ cannot be obtained by a weight function ω and vice versa. At the beginning, ultradifferentiable classes were studied using the growth of the derivatives and later with the Fourier transform. Finally, Braun, Meise and Taylor in [2] have unified both theories. For a detailed survey we refer to the introductions in [2] and [1].

In [9] we have considered classes $\mathcal{E}_{[\mathcal{M}]}$ defined by (one-parameter) weight matrices $\mathcal{M} := \{M^x : x \in \Lambda\}$. The spaces $\mathcal{E}_{[M]}$ and $\mathcal{E}_{[\omega]}$ were identified as particular cases of $\mathcal{E}_{[\mathcal{M}]}$ but one is able to describe more classes, e.g. the class defined by the *Gevrey-matrix* $\mathcal{G} := \{(p!^{s+1})_{p \in \mathbb{N}} : s > 0\}$, see [9, 5.19]. Using this new method one is able to transfer results from one setting into the other one and to prove results for $\mathcal{E}_{[M]}$ and $\mathcal{E}_{[\omega]}$ simultaneously, e.g. see [9] and [10].

The main aim of this work is to show that assuming some mild properties for $\mathcal{M}$ the functions with compact support $\mathcal{D}_{[\mathcal{M}]} \subseteq \mathcal{E}_{[\mathcal{M}]}$ can be characterized in terms of the decay properties of their Fourier transform.

First, we generalize in Section 3 a central new idea in [9]. We have shown that to each ω we can associate a weight matrix $\Omega := \{(\Omega_j^l)_{j \geq 0} : l > 0\}$, defined by $\Omega_j^l := \exp(1/l\varphi_\omega^*(lj))$, such that $\mathcal{E}_{[\omega]} = \mathcal{E}_{[\Omega]}$ holds as locally convex vector spaces.

In this work we start with an abstractly given weight matrix $\mathcal{M} = \{M^x : x \in \Lambda\}$ satisfying some standard assumptions. To $\mathcal{M}$ we associate another matrix $\omega_{\mathcal{M}} := \{\omega_{M^x} : x \in \Lambda\}$ consisting of *associated functions* ω_{M^x}. Applying again the idea of [9] we obtain a matrix $\{M^{x;l} : x \in \Lambda, l > 0\}$ and iterating this procedure we get a sequence of multi-index weight matrices consisting of weight sequences and weight functions. In Section 3 this technique is studied in detail.

First, in 3.3, we will characterize the case where all multi-index weight matrices of weight sequences are equivalent. Thus $\mathcal{E}_{[\mathcal{M}]}$ is stable as locally convex vector space under adjoining indices, see Theorem 3.4. It will turn out that only in the first step a non-stable effect can occur, see Corollary 3.8.

The spaces associated to the matrices of weight functions in this construction are always stable. Using results from 3.10 and Theorem 3.4 we can prove the first main result Theorem 3.2: As locally convex vector spaces the equality $\mathcal{E}_{[\mathcal{M}]} = \mathcal{E}_{[\omega_{\mathcal{M}}]}$ is valid.

In the next step, in Section 4, we characterize the non-quasianalyticity of $\mathcal{E}_{[\mathcal{M}]}$, see Theorem 4.1. Thus the cases where the spaces $\mathcal{D}_{[\mathcal{M}]}$ are non-trivial

are classified. The Roumieu case is quite clear and for the Beurling case we generalize [14, Lemma 5.1], where stronger conditions for the matrix $\mathcal{M}$ were assumed.

In Section 5 we combine Theorem 3.2 and Theorem 4.1. Using and generalizing the methods and estimates introduced in [2] we are able to characterize functions in $\mathcal{D}_{[\mathcal{M}]}$ in terms of the decay properties of their Fourier transform, see Theorem 5.1. As special case this holds for the Gevrey-matrix $\mathcal{G}$.

Finally, in Section 6, we apply the technique of associating a weight matrix to prove some variations of comparison results due to [1] concerning the classes $\mathcal{E}_{[M]}$ and $\mathcal{E}_{[\omega]}$.

This work contains some results of the author PhD Thesis, see [12]. The author thanks his advisors A. Kriegl, P.W. Michor and A. Rainer for the supervision and their helpful ideas.

1.1 Basic notation

We denote by $\mathcal{E}$ the class of smooth functions, $\mathcal{C}^\omega$ is the class of all real analytic functions. We will write $\mathbb{N}_{>0} = \{1, 2, \dots\}$ and $\mathbb{N} = \mathbb{N}_{>0} \cup \{0\}$. Moreover we put $\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$, i.e. the set of all positive real numbers. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we use the usual multi-index notation, write $\alpha! := \alpha_1! \dots \alpha_n!$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We also put $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and for a given function $f : U \subseteq \mathbb{R}^r \to \mathbb{R}^s$ defined on a non-empty open set $U \subseteq \mathbb{R}^r$ we denote by $f^{(k)}$ the k-th order *Fréchet derivative* of f. Let $E_1, \dots, E_k$ and F be topological vector spaces, then $L(E_1, \dots, E_k, F)$ is the space of all bounded k-linear mappings $E_1 \times \cdots \times E_k \to F$. If $E = E_i$ for $i = 1, \dots, k$, then we write $L^k(E, F)$. With $\| \cdot \|_{\mathbb{R}^n}$ we denote the Euclidian norm on $\mathbb{R}^n$.

Let $K \subset\subset \mathbb{R}^r$ be a compact set with smooth boundary, then $\mathcal{E}(K, \mathbb{R}^s)$ denotes the space of all smooth functions on the interior K° such that each derivative of f can be continuously extended to K.

Convention: Let $\star \in \{M, \omega, \mathcal{M}\}$, then we write $\mathcal{E}_{[\star]}$ if either $\mathcal{E}_{\{\star\}}$ or $\mathcal{E}_{(\star)}$ is considered with the following restriction: Statements that involve more than one $\mathcal{E}_{[\star]}$ symbol must not be interpreted by mixing $\mathcal{E}_{\{\star\}}$ and $\mathcal{E}_{(\star)}$. The same notation resp. convention will be used for the conditions, so write $(\mathcal{M}_{[\star]})$ for either $(\mathcal{M}_{\{\star\}})$ or $(\mathcal{M}_{(\star)})$.

2 Basic definitions

2.1 Weight sequences and classes of ultradifferentiable functions $\mathcal{E}_{[M]}$

$M = (M_k)_k \in \mathbb{R}_{>0}^{\mathbb{N}}$ is called a *weight sequence*. We introduce also $m = (m_k)_k$ defined by $m_k := \frac{M_k}{k!}$ and $\mu = (\mu_k)_k$ by $\mu_k := \frac{M_k}{M_{k-1}}$, $\mu_0 := 1$. M is called *normalized* if $1 = M_0 \leq M_1$ holds (w.l.o.g.).

(1) M is called *log-convex* if

$$(\text{lc}) :\Leftrightarrow \ \forall \, j \in \mathbb{N} : \ M_j^2 \leq M_{j-1}M_{j+1}.$$

M is log-convex if and only if $(\mu_k)_k$ is increasing. If M is log-convex and normalized, then M and $k \mapsto (M_k)^{1/k}$ are both increasing, see e.g. [11, Lemma 2.0.4].

(2) M has *moderate growth* if

$$(\text{mg}) :\Leftrightarrow \exists \, C \geq 1 \ \forall \, j, k \in \mathbb{N} : \ M_{j+k} \leq C^{j+k} M_j M_k.$$

(3) M is called *non-quasianalytic* if

$$(\text{nq}) :\Leftrightarrow \ \sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < +\infty.$$

Using *Carleman's inequality* one can show that if M has (lc), then

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < +\infty \Leftrightarrow \sum_{p=1}^{\infty} \frac{1}{(M_p)^{1/p}} < +\infty.$$

(4) M has (β_3) if

$$\exists \, Q \in \mathbb{N}_{>0} : \ \liminf_{p \to \infty} \frac{\mu_{Qp}}{\mu_p} > 1.$$

(5) For $M = (M_p)_p$ and $N = (N_p)_p$ we write $M \leq N$ if and only if $M_p \leq N_p$ holds for all $p \in \mathbb{N}$. Moreover we define

$$M \preceq N :\Leftrightarrow \ \exists \, C_1, C_2 \geq 1 \ \forall \, p \in \mathbb{N} : \ M_p \leq C_2 C_1^p N_p \Longleftrightarrow \sup_{p \in \mathbb{N}_{>0}} \left(\frac{M_p}{N_p} \right)^{1/p} < +\infty$$

and call the sequences equivalent if

$$M \approx N :\Leftrightarrow \ M \preceq N \text{ and } N \preceq M.$$

(mg) and (nq) are stable w.r.t. $\approx$. Furthermore we will write

$$M \lhd N :\Leftrightarrow \forall\, h > 0 \,\exists\, C_h \geq 1 \,\forall\, j \in \mathbb{N}: \ M_j \leq C_h h^j N_j \iff \lim_{p \to \infty} \left(\frac{M_p}{N_p} \right)^{1/p} = 0.$$

For convenience we introduce the set

$$\mathcal{LC} := \{M \in \mathbb{R}_{>0}^{\mathbb{N}} : \ M \text{ is normalized, log-convex, } \lim_{k \to \infty} (M_k)^{1/k} = +\infty\}.$$

Let $r, s \in \mathbb{N}_{>0}$ and $U \subseteq \mathbb{R}^r$ be a non-empty open set. We introduce the classes of ultradifferentiable functions of Roumieu type by

$$\mathcal{E}_{\{M\}}(U, \mathbb{R}^s) := \{f \in \mathcal{E}(U, \mathbb{R}^s) : \ \forall\, K \subset\subset U \,\exists\, h > 0: \ \|f\|_{M,K,h} < +\infty\},$$

and the classes of ultradifferentiable functions of Beurling type by

$$\mathcal{E}_{(M)}(U, \mathbb{R}^s) := \{f \in \mathcal{E}(U, \mathbb{R}^s) : \ \forall\, K \subset\subset U \,\forall\, h > 0: \ \|f\|_{M,K,h} < +\infty\},$$

where we denote

$$\|f\|_{M,K,h} := \sup_{k \in \mathbb{N}, x \in K} \frac{\|f^{(k)}(x)\|_{L^k(\mathbb{R}^r, \mathbb{R}^s)}}{h^k M_k} \tag{1}$$

and $\|f^{(k)}(x)\|_{L^k(\mathbb{R}^r, \mathbb{R}^s)} := \sup\{\|f^{(k)}(x)(v_1, \ldots, v_k)\|_{\mathbb{R}^s} : \|v_i\|_{\mathbb{R}^r} \leq 1 \,\forall\, 1 \leq i \leq k\}$.

For a compact set K with smooth boundary

$$\mathcal{E}_{M,h}(K, \mathbb{R}^s) := \{f \in \mathcal{E}(K, \mathbb{R}^s) : \|f\|_{M,K,h} < +\infty\}$$

is a Banach space and we define the following topological vector spaces

$$\mathcal{E}_{\{M\}}(U, \mathbb{R}^s) := \varprojlim_{K \subset\subset U} \varinjlim_{h > 0} \mathcal{E}_{M,h}(K, \mathbb{R}^s) = \varprojlim_{K \subseteq U} \mathcal{E}_{\{M\}}(K, \mathbb{R}^s) \tag{2}$$

and

$$\mathcal{E}_{(M)}(U, \mathbb{R}^s) := \varprojlim_{K \subset\subset U} \varprojlim_{h > 0} \mathcal{E}_{M,h}(K, \mathbb{R}^s) = \varprojlim_{K \subseteq U} \mathcal{E}_{(M)}(K, \mathbb{R}^s). \tag{3}$$

In $\mathcal{E}_{M,h}(K, \mathbb{R}^s)$ instead of compact sets K with smooth boundary one can also consider a relatively compact open subset K of U (see [15]) or one can work with *Whitney jets* on the compact set K (see [6] and also [1]).

We recall some facts for log-convex M:

(*i*) We write $\mathcal{E}_{\{M\}}^{\text{global}}(U, \mathbb{R}^s) := \{f \in \mathcal{E}(U, \mathbb{R}^s) : \ \exists\, h > 0: \ \|f\|_{M,U,h} < +\infty\}$. Then there exist *characteristic functions*

$$\theta_M \in \mathcal{E}_{\{M\}}^{\text{global}}(\mathbb{R}, \mathbb{R}) : \forall\, j \in \mathbb{N} : \left| \theta_M^{(j)}(0) \right| \geq M_j, \tag{4}$$

see [9, Lemma 2.9] and [15, Theorem 1]. Note that the Beurling class $\mathcal{E}_{(M)}^{\text{global}}(\mathbb{R}, \mathbb{R})$ cannot contain such θ_M, see [11, Proposition 3.1.2].

(*ii*) If N is arbitrary, then $M \preceq N \iff \mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{\{N\}}$ and $M \vartriangleleft N \iff \mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{(N)}$. If $M \in \mathcal{LC}$, then $M \preceq N \iff \mathcal{E}_{[M]} \subseteq \mathcal{E}_{[N]}$.

(*iii*) For any non-empty open set $U \subseteq \mathbb{R}^r$ both classes $\mathcal{E}_{\{M\}}(U, \mathbb{R})$ and $\mathcal{E}_{(M)}(U, \mathbb{R})$ are closed under pointwise multiplication, see e.g. [11, Proposition 2.0.8].

2.2   Classes of ultradifferentiable functions defined by weight matrices

Definition 2.3. Let $(\Lambda, \leq)$ be a partially ordered set which is both up- and downward directed, $\Lambda = \mathbb{R}_{>0}$ is the most important example. A *weight matrix* $\mathcal{M}$ associated to Λ is a family of weight sequences $\mathcal{M} := \{ M^x \in \mathbb{R}_{>0}^{\mathbb{N}} : x \in \Lambda \}$ such that

$$(\mathcal{M}) :\Leftrightarrow \ \forall \, x \in \Lambda : \ M^x \text{ is normalized, increasing, } M^x \leq M^y \text{ for } x \leq y.$$

We call $\mathcal{M}$ *standard log-convex*, if

$$(\mathcal{M}_{\mathrm{sc}}) :\Leftrightarrow (\mathcal{M}) \text{ and } \forall \, x \in \Lambda : \ M^x \in \mathcal{LC}.$$

Also the sequences $m_k^x := \frac{M_k^x}{k!}$ and $\mu_k^x := \frac{M_k^x}{M_{k-1}^x}$, $\mu_0^x := 1$, will be used.

We introduce spaces of vector-valued ultradifferentiable functions classes defined by a weight matrices of Roumieu type $\mathcal{E}_{\{\mathcal{M}\}}$ and Beurling type $\mathcal{E}_{(\mathcal{M})}$ as follows, see also [9, 4.2].

Let $r, s \in \mathbb{N}_{>0}$, let $U \subseteq \mathbb{R}^r$ be a non-empty open set. For all compact sets $K \subset\subset U$ we put

$$\mathcal{E}_{\{\mathcal{M}\}}(K, \mathbb{R}^s) := \bigcup_{x \in \Lambda} \mathcal{E}_{\{M^x\}}(K, \mathbb{R}^s) \qquad \mathcal{E}_{\{\mathcal{M}\}}(U, \mathbb{R}^s) := \bigcap_{K \subset\subset U} \bigcup_{x \in \Lambda} \mathcal{E}_{\{M^x\}}(K, \mathbb{R}^s) \tag{5}$$

and

$$\mathcal{E}_{(\mathcal{M})}(K, \mathbb{R}^s) := \bigcap_{x \in \Lambda} \mathcal{E}_{(M^x)}(K, \mathbb{R}^s) \qquad \mathcal{E}_{(\mathcal{M})}(U, \mathbb{R}^s) := \bigcap_{x \in \Lambda} \mathcal{E}_{(M^x)}(U, \mathbb{R}^s). \tag{6}$$

For a compact set $K \subset\subset \mathbb{R}^r$ one has the representations

$$\mathcal{E}_{\{\mathcal{M}\}}(K, \mathbb{R}^s) := \varinjlim_{x \in \Lambda} \varinjlim_{h > 0} \mathcal{E}_{M^x, h}(K, \mathbb{R}^s)$$

and so for $U \subseteq \mathbb{R}^r$ non-empty open

$$\mathcal{E}_{\{\mathcal{M}\}}(U, \mathbb{R}^s) := \varprojlim_{K \subset\subset U} \varinjlim_{x \in \Lambda} \varinjlim_{h > 0} \mathcal{E}_{M^x, h}(K, \mathbb{R}^s). \tag{7}$$

Similarly we get for the Beurling case

$$\mathcal{E}_{(\mathcal{M})}(U, \mathbb{R}^s) := \varprojlim_{K \subset\subset U} \varprojlim_{x \in \Lambda} \varprojlim_{h > 0} \mathcal{E}_{M^x, h}(K, \mathbb{R}^s). \tag{8}$$

If $\Lambda = \mathbb{R}_{>0}$ we can assume that all occurring limits are countable and restrict to $\Lambda = \mathbb{N}_{>0}$ in the Roumieu case. Thus $\mathcal{E}_{(\mathcal{M})}(U, \mathbb{R}^s)$ is a *Fréchet space* and $\varinjlim_{x \in \Lambda} \varinjlim_{h > 0} \mathcal{E}_{M^x, h}(K, \mathbb{R}^s) = \varinjlim_{n \in \mathbb{N}_{>0}} \mathcal{E}_{M^n, n}(K, \mathbb{R}^s)$ is a *Silva space*, i.e. a countable inductive limit of Banach spaces with compact connecting mappings. For more details concerning the locally convex topology we refer to [9, 4.2-4.4]. In the appendix in Proposition 7.2 we will show that for some weight matrices the connecting mappings are even nuclear.

2.4   Conditions for a weight matrix $\mathcal{M}$

We are going to introduce several conditions on $\mathcal{M}$, see also [9, 4.1]. First consider the following conditions *of Roumieu type*.

$(\mathcal{M}_{\{dc\}})$ $\forall x \in \Lambda \,\exists\, C > 0 \,\exists\, y \in \Lambda \,\forall j \in \mathbb{N} : M_{j+1}^x \leq C^{j+1} M_j^y$

$(\mathcal{M}_{\{mg\}})$ $\forall x \in \Lambda \,\exists\, C > 0 \,\exists\, y_1, y_2 \in \Lambda \,\forall j, k \in \mathbb{N} : M_{j+k}^x \leq C^{j+k} M_j^{y_1} M_k^{y_2}$

$(\mathcal{M}_{\{L\}})$ $\forall C > 0 \,\forall x \in \Lambda \,\exists\, D > 0 \,\exists\, y \in \Lambda \,\forall k \in \mathbb{N} : C^k M_k^x \leq D M_k^y$

$(\mathcal{M}_{\{strict\}})$ $\forall x \in \Lambda \,\exists\, y \in \Lambda \,:\, \sup_{k \in \mathbb{N}_{>0}} \left(\frac{M_k^y}{M_k^x} \right)^{1/k} = +\infty$

$(\mathcal{M}_{\{BR\}})$ $\forall x \in \Lambda \,\exists\, y \in \Lambda : M^x \lhd M^y$

Analogously we introduce the *Beurling type conditions*.

$(\mathcal{M}_{(dc)})$ $\forall x \in \Lambda \,\exists\, C > 0 \,\exists\, y \in \Lambda \,\forall j \in \mathbb{N} : M_{j+1}^y \leq C^{j+1} M_j^x$

$(\mathcal{M}_{(mg)})$ $\forall x_1, x_2 \in \Lambda \,\exists\, C > 0 \,\exists\, y \in \Lambda \,\forall j, k \in \mathbb{N} : M_{j+k}^y \leq C^{j+k} M_j^{x_1} M_k^{x_2}$

$(\mathcal{M}_{(L)})$ $\forall C > 0 \,\forall x \in \Lambda \,\exists\, D > 0 \,\exists\, y \in \Lambda \,\forall k \in \mathbb{N} : C^k M_k^y \leq D M_k^x$

$(\mathcal{M}_{(strict)})$ $\forall x \in \Lambda \,\exists\, y \in \Lambda \,:\, \sup_{k \in \mathbb{N}_{>0}} \left(\frac{M_k^x}{M_k^y} \right)^{1/k} = +\infty$

$(\mathcal{M}_{(BR)})$ $\forall x \in \Lambda \,\exists\, y \in \Lambda : M^y \lhd M^x$

2.5 Inclusion relations

Given two matrices $\mathcal{M} = \{M^x : x \in \Lambda\}$ and $\mathcal{N} = \{N^y : y \in \Lambda'\}$ we introduce

$$\mathcal{M}\{\preceq\}\mathcal{N} \ :\Leftrightarrow \forall\, x \in \Lambda \,\exists\, y \in \Lambda' : \ M^x \preceq N^y$$

and

$$\mathcal{M}(\preceq)\mathcal{N} \ :\Leftrightarrow \forall\, y \in \Lambda' \,\exists\, x \in \Lambda : \ M^x \preceq N^y.$$

By definition $\mathcal{M}[\preceq]\mathcal{N}$ implies $\mathcal{E}_{[\mathcal{M}]} \subseteq \mathcal{E}_{[\mathcal{N}]}$ and write

$$\mathcal{M}\{\approx\}\mathcal{N} \ :\Leftrightarrow \mathcal{M}\{\preceq\}\mathcal{N} \text{ and } \mathcal{N}\{\preceq\}\mathcal{M}$$

and

$$\mathcal{M}(\approx)\mathcal{N} \ :\Leftrightarrow \mathcal{M}(\preceq)\mathcal{N} \text{ and } \mathcal{N}(\preceq)\mathcal{M}.$$

Moreover, we introduce

$$\mathcal{M} \lhd \mathcal{N} \ :\Leftrightarrow \forall\, x \in \Lambda \,\forall\, y \in \Lambda' : M^x \lhd N^y,$$

so $\mathcal{M} \lhd \mathcal{N}$ implies $\mathcal{E}_{\{\mathcal{M}\}} \subseteq \mathcal{E}_{(\mathcal{N})}$. In [9, Proposition 4.6] the relations above were characterized for $(\mathcal{M}_{\mathrm{sc}})$ matrices with $\Lambda = \Lambda' = \mathbb{R}_{>0}$. In this context we introduce also

$(\mathcal{M}_{\{\mathcal{C}^\omega\}})$ $\exists\, x \in \Lambda : \ \liminf_{k\to\infty}(m_k^x)^{1/k} > 0$,

$(\mathcal{M}_{\mathcal{H}})$ $\forall\, x \in \Lambda : \ \liminf_{k\to\infty}(m_k^x)^{1/k} > 0$,

$(\mathcal{M}_{(\mathcal{C}^\omega)})$ $\forall\, x \in \Lambda : \ \lim_{k\to\infty}(m_k^x)^{1/k} = +\infty$.

Recall [9, Proposition 4.6]: If $(\mathcal{M}_{\{\mathcal{C}^\omega\}})$ holds then the class of real-analytic-functions is contained in $\mathcal{E}_{\{\mathcal{M}\}}$, if $(\mathcal{M}_{(\mathcal{C}^\omega)})$ then the real-analytic functions are contained in $\mathcal{E}_{(\mathcal{M})}$. If $(\mathcal{M}_{\mathcal{H}})$ is satisfied, then the restrictions of entire functions are contained in $\mathcal{E}_{(\mathcal{M})}$.

Convention: If $\Lambda = \mathbb{R}_{>0}$ or $\mathbb{N}_{>0}$, then $\mathbb{R}_{>0}$ or $\mathbb{N}_{>0}$ are always regarded with its natural order $\leq$. We will call $\mathcal{M}$ *constant* if $\mathcal{M} = \{M\}$ or more generally if $M^x \approx M^y$ for all $x, y \in \Lambda$, which violates both $(\mathcal{M}_{\{\mathrm{strict}\}})$ and $(\mathcal{M}_{(\mathrm{strict})})$. Otherwise it will be called *non-constant*.

2.6 Classes of ultradifferentiable functions $\mathcal{E}_{[\omega]}$

A function $\omega : [0, \infty) \to [0, \infty)$ (sometimes ω is extended to $\mathbb{C}$, by $\omega(x) := \omega(|x|)$) is called a *weight function* if

(i) ω is continuous,

(ii) ω is increasing,

(iii) $\omega(x) = 0$ for all $x \in [0, 1]$ (normalization, w.l.o.g.),

(iv) $\lim_{x\to\infty} \omega(x) = +\infty$.

For convenience we will write that ω has (ω_0) if it satisfies $(i)-(iv)$.
   Moreover we consider the following conditions:

(ω_1) $\omega(2t) = O(\omega(t))$ as $t \to +\infty$.

(ω_2) $\omega(t) = O(t)$ as $t \to \infty$.

(ω_3) $\log(t) = o(\omega(t))$ as $t \to +\infty$ $(\Leftrightarrow \lim_{t\to+\infty} \frac{t}{\varphi_\omega(t)} = 0)$.

(ω_4) $\varphi_\omega : t \mapsto \omega(e^t)$ is a convex function on $\mathbb{R}$.

(ω_5) $\omega(t) = o(t)$ as $t \to +\infty$.

(ω_6) $\exists\, H \geq 1\ \forall\, t \geq 0 :\ 2\omega(t) \leq \omega(Ht) + H$.

(ω_7) $\exists\, H > 0\ \exists\, C > 0\ \forall\, t \geq 0 :\ \omega(t^2) \leq C\omega(Ht) + C$.

(ω_{nq}) $\int_1^\infty \frac{\omega(t)}{t^2} dt < \infty$.

An interesting example is $\omega_s(t) := \max\{0, \log(t)^s\}$, $s > 1$, which satisfies all
listed properties *except* (ω_6). For convenience we define the sets

$$\mathcal{W}_0 := \{\omega : [0,\infty) \to [0,\infty) : \omega \text{ has } (\omega_0), (\omega_3), (\omega_4)\},$$

$$\mathcal{W} := \{\omega \in \mathcal{W}_0 : \omega \text{ has } (\omega_1)\}.$$

For $\omega \in \mathcal{W}_0$ we can define the *Legendre-Fenchel-Young-conjugate* φ_ω^* by

$$\varphi_\omega^*(x) := \sup\{xy - \varphi_\omega(y) : y \geq 0\}, \quad x \geq 0$$

with the following properties, e.g. see [2, Remark 1.3, Lemma 1.5]: It is convex
and increasing, $\varphi_\omega^*(0) = 0$, $\varphi_\omega^{**} = \varphi_\omega$, $\lim_{x\to\infty} \frac{x}{\varphi_\omega^*(x)} = 0$ and finally $x \mapsto \frac{\varphi_\omega(x)}{x}$
and $x \mapsto \frac{\varphi_\omega^*(x)}{x}$ are increasing on $[0, +\infty)$.
   For two weights $\sigma, \tau \in \mathcal{W}_0$ we write

$$\sigma \preceq \tau :\Leftrightarrow \tau(t) = O(\sigma(t)) \text{ as } t \to +\infty$$

and call them equivalent if

$$\sigma \sim \tau :\Leftrightarrow \sigma \preceq \tau \text{ and } \tau \preceq \sigma.$$

Moreover introduce

$$\sigma \lhd \tau :\Leftrightarrow \tau(t) = o(\sigma(t)) \text{ as } t \to +\infty.$$

Let $r, s \in \mathbb{N}_{>0}$, $U \subseteq \mathbb{R}^r$ be a non-empty open set and $\omega \in \mathcal{W}_0$. The space of vector-valued ultradifferentiable functions of Roumieu type is defined by

$$\mathcal{E}_{\{\omega\}}(U, \mathbb{R}^s) := \{f \in \mathcal{E}(U, \mathbb{R}^s) : \forall\, K \subset\subset U \,\exists\, l > 0 : \|f\|_{\omega, K, l} < +\infty\}$$

and the space of vector-valued ultradifferentiable functions of Beurling type by

$$\mathcal{E}_{(\omega)}(U, \mathbb{R}^s) := \{f \in \mathcal{E}(U, \mathbb{R}^s) : \forall\, K \subset\subset U \,\forall\, l > 0 : \|f\|_{\omega, K, l} < +\infty\},$$

where

$$\|f\|_{\omega, K, l} := \sup_{k \in \mathbb{N}, x \in K} \frac{\|f^{(k)}(x)\|_{L^k(\mathbb{R}^r, \mathbb{R}^s)}}{\exp(\frac{1}{l}\varphi_\omega^*(lk))}. \tag{9}$$

For compact sets K with smooth boundary

$$\mathcal{E}_{\omega, l}(K, \mathbb{R}^s) := \{f \in \mathcal{E}(K, \mathbb{R}^s) : \|f\|_{\omega, K, l} < +\infty\}$$

is a Banach space and we consider the following topological vector spaces

$$\mathcal{E}_{\{\omega\}}(U, \mathbb{R}^s) := \varprojlim_{K \subset\subset U} \varinjlim_{l > 0} \mathcal{E}_{\omega, l}(K, \mathbb{R}^s) = \varprojlim_{K \subset\subset U} \mathcal{E}_{\{\omega\}}(K, \mathbb{R}^s) \tag{10}$$

and

$$\mathcal{E}_{(\omega)}(U, \mathbb{R}^s) := \varprojlim_{K \subset\subset U} \varprojlim_{l > 0} \mathcal{E}_{\omega, l}(K, \mathbb{R}^s) = \varprojlim_{K \subset\subset U} \mathcal{E}_{(\omega)}(K, \mathbb{R}^s). \tag{11}$$

For $\sigma, \tau \in \mathcal{W}$ we get $\sigma \preceq \tau \Leftrightarrow \mathcal{E}_{[\sigma]} \subseteq \mathcal{E}_{[\tau]}$ and $\tau \lhd \sigma \Leftrightarrow \mathcal{E}_{\{\tau\}} \subseteq \mathcal{E}_{(\sigma)}$, see [9, Corollary 5.17].

We summarize some facts which are shown in [9, Section 5].

(i) A central new idea was that to each $\omega \in \mathcal{W}$ we can associate a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix $\Omega := \{\Omega^l = (\Omega^l_j)_{j \in \mathbb{N}} : l > 0\}$ by

$$\boxed{\Omega^l_j := \exp\left(\tfrac{1}{l}\varphi_\omega^*(lj)\right).}$$

(ii) $\mathcal{E}_{[\omega]} = \mathcal{E}_{[\Omega]}$ holds as locally convex vector spaces and Ω satisfies $(\mathcal{M}_{\{\mathrm{mg}\}})$, $(\mathcal{M}_{(\mathrm{mg})})$ and $(\mathcal{M}_{\{\mathrm{L}\}})$, $(\mathcal{M}_{(\mathrm{L})})$.

(iii) Equivalent weight functions ω yield equivalent weight matrices w.r.t. both ($\approx$) and $\{\approx\}$. Note that $(\mathcal{M}_{[\mathrm{mg}]})$ is stable w.r.t. $[\approx]$, whereas $(\mathcal{M}_{[\mathrm{L}]})$ not.

(iv) Defining classes of ultradifferentiable functions by weight matrices as in (5) and in (6) is a common generalization of defining them by using a (single) weight sequence M, i.e. a constant weight matrix, or by a weight function $\omega \in \mathcal{W}$. But one is able to describe also other classes, e.g. the class defined by the Gevrey-matrix $\mathcal{G} := \{(p!^{s+1})_{p \in \mathbb{N}} : s > 0\}$.

2.7 Classes of ultra-differentiable functions defined by a weight matrix of associated functions

Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$, the *associated function* $\omega_M : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\omega_M(t) := \sup_{p \in \mathbb{N}} \log \left(\frac{t^p M_0}{M_p} \right) \quad \text{for } t > 0, \qquad \omega_M(0) := 0. \tag{12}$$

Lemma 2.8. If $M \in \mathcal{LC}$, then ω_M belongs to $\mathcal{W}_0$.

Moreover $\liminf_{p \to \infty} (m_p)^{1/p} > 0$ implies (ω_2), $\lim_{p \to \infty} (m_p)^{1/p} = +\infty$ implies (ω_5) for ω_M.

We refer to [6, Definition 3.1] and [1, Lemma 12 $(iv) \Rightarrow (v)$]. That $\lim(m_p)^{1/p} = +\infty$ implies (ω_5) for ω_M follows analogously as $\liminf(m_p)^{1/p} > 0$ implies (ω_2) for ω_M as shown in [1, Lemma 12 $(iv) \Rightarrow (v)$]. Note that by *Stirling's formula* $\liminf(m_p)^{1/p} > 0$ is precisely $(M0)$ in [1].

Remark 2.9. Let $\omega \in \mathcal{W}_0$ be given, then

(1) $\Omega^l \in \mathcal{LC}$ for each $l > 0$ by [9, 5.5],

(2) $\omega \sim \omega_{\Omega^l}$ for each $l > 0$ by [9, Lemma 5.7],

(3) ω satisfies

    (a) (ω_{nq}) if and only if some/each Ω^l satisfies (nq),

    (b) (ω_6) if and only if some/each Ω^l satisfies (mg) if and only if $\Omega^l \approx \Omega^n$ for each $l, n > 0$,

by [9, Corollary 5.8, Theorem 5.14].

Let $\mathcal{M} = \{M^x : x \in \Lambda\}$ be $(\mathcal{M}_{\mathrm{sc}})$, then we introduce the new weight matrix $\omega_{\mathcal{M}} := \{\omega_{M^x} : x \in \Lambda\}$. Let $U \subseteq \mathbb{R}^r$ be non-empty open and put

$$\mathcal{E}_{\{\omega_{\mathcal{M}}\}}(U, \mathbb{R}^s) := \{f \in \mathcal{E}(U, \mathbb{R}^s) : \forall K \subset\subset U \ \exists x \in \Lambda \ \exists l > 0 : \|f\|_{\omega_{M^x},K,l} < +\infty\}$$

and

$$\mathcal{E}_{(\omega_{\mathcal{M}})}(U, \mathbb{R}^s) := \{f \in \mathcal{E}(U, \mathbb{R}^s) : \forall K \subset\subset U \ \forall x \in \Lambda \ \forall l > 0 : \|f\|_{\omega_{M^x},K,l} < +\infty\}.$$

Thus we obtain the topological vector spaces representations

$$\mathcal{E}_{\{\omega_{\mathcal{M}}\}}(U, \mathbb{R}^s) := \varprojlim_{K \subset\subset U} \varinjlim_{x \in \Lambda, l > 0} \mathcal{E}_{\omega_{M^x},l}(K, \mathbb{R}^s) \tag{13}$$

and

$$\mathcal{E}_{(\omega_{\mathcal{M}})}(U, \mathbb{R}^s) := \varprojlim_{K \subset\subset U} \varprojlim_{x \in \Lambda, l > 0} \mathcal{E}_{\omega_{M^x},l}(K, \mathbb{R}^s) \tag{14}$$

3 Stability of constructing multi-index weight matrices

3.1 Introduction

Let $\mathcal{M} := \{M^x : x \in \Lambda\}$ be $(\mathcal{M}_{sc})$. By Lemma 2.8 we get $\omega_{M^x} \in \mathcal{W}_0$ for each $x \in \Lambda$. On the other hand by [9, 5.5] to each $\omega \in \mathcal{W}_0$ we can associate a $(\mathcal{M}_{sc})$ weight matrix $\Omega := \{(\Omega_j^l)_{j \in \mathbb{N}} : l > 0\}$ by putting $\Omega_j^l := \exp\left(\frac{1}{l}\varphi_\omega^*(lj)\right)$.

So one can consider the construction

$$M^x \mapsto \omega_{M^x} \mapsto M^{x;l_1} \mapsto \omega_{M^{x;l_1}} \mapsto M^{x;l_1,l_2} \mapsto \dots, \qquad (15)$$

where for $x \in \Lambda$, $l_j \in \mathbb{R}_{>0}$, $j \in \mathbb{N}_{>0}$, and $i \in \mathbb{N}$ we put

$$M_i^{x;l_1,\dots,l_{j+1}} := \exp\left(\frac{1}{l_{j+1}}\varphi_{\omega_{M^{x;l_1,\dots,l_j}}}^*(l_{j+1}i)\right), \quad M_i^{x;l_1} := \exp\left(\frac{1}{l_1}\varphi_{\omega_{M^x}}^*(l_1 i)\right)$$

respectively

$$\omega_{M^{x;l_1,\dots,l_j}}(t) := \sup_{p \in \mathbb{N}} \log\left(\frac{t^p}{M_p^{x;l_1,\dots,l_j}}\right) \text{ for } t > 0, \qquad \omega_{M^{x;l_1,\dots,l_j}}(0) := 0.$$

On the one hand we obtain a sequence of matrices of weight functions. [9, Lemma 5.7] implies

$$\forall\, x \in \Lambda\ \forall\, j \in \mathbb{N}_{>0}\ \forall\, l_1,\dots,l_j > 0 : \ \omega_{M^{x;l_1,\dots,l_{j+1}}} \sim \omega_{M^{x;l_1,\dots,l_j}} \sim \dots \sim \omega_{M^x}, \quad (16)$$

hence this construction is always stable. So for each non-empty open $U \subseteq \mathbb{R}^r$ we get

$$\mathcal{E}_{\{\omega_{\mathcal{M}}\}}(U, \mathbb{R}^s) = \varprojlim_{K \subset\subset U}\ \varinjlim_{x \in \Lambda, l, h > 0} \mathcal{E}_{\omega_{M^{x;l}}, h}(K, \mathbb{R}^s) \qquad (17)$$

and

$$\mathcal{E}_{(\omega_{\mathcal{M}})}(U, \mathbb{R}^s) = \varprojlim_{K \subset\subset U}\ \varprojlim_{x \in \Lambda, l, h > 0} \mathcal{E}_{\omega_{M^{x;l}}, h}(K, \mathbb{R}^s). \qquad (18)$$

On the other hand we get a sequence of matrices of weight sequences. In Theorem 3.4 we are going to characterize the stability of this construction and we will see that only in the first step of (15) there can occur a non-stable effect (see Corollary 3.8).

Finally the aim of this Section is to prove the following result:

Theorem 3.2. Let $\mathcal{M} := \{M^x \in \mathbb{R}_{>0}^{\mathbb{N}} : x \in \Lambda\}$ be $(\mathcal{M}_{sc})$, let $r, s \in \mathbb{N}_{>0}$ and U be a non-empty open set in $\mathbb{R}^r$. If $\mathcal{M}$ has $(\mathcal{M}_{[L]})$ and $(\mathcal{M}_{[mg]})$, then we get as locally convex vector spaces

$$\mathcal{E}_{[\mathcal{M}]}(U, \mathbb{R}^s) = \mathcal{E}_{[\omega_{\mathcal{M}}]}(U, \mathbb{R}^s).$$

3.3 Stability of constructing multi-index matrices consisting of weight sequences

In this section we show the following result which is the first step to prove Theorem 3.2.

Theorem 3.4. Let $\mathcal{M} = \{M^x : x \in \Lambda\}$ be $(\mathcal{M}_{\mathrm{sc}})$. Then $\mathcal{M}[\approx]\{M^{x;l} : x \in \Lambda, l > 0\}$ if and only if

(1) in the Roumieu-case $(\mathcal{M}_{\{\mathrm{mg}\}})$ holds,

(2) in the Beurling-case $(\mathcal{M}_{(\mathrm{mg})})$ holds, provided $\Lambda = \mathbb{R}_{>0}$.

First we prove

Lemma 3.5. For each $x \in \Lambda$, $l \in \mathbb{N}_{>0}$ and $j \in \mathbb{N}$ we get

$$M_j^{x;l} = (M_{jl}^x)^{1/l}. \tag{19}$$

PROOF. We use [6, Proposition 3.2] and get

$$M_j^{x;l} := \exp\left(\frac{1}{l}\varphi_{\omega_{M^x}}^*(lj)\right) = \exp\left(\frac{1}{l}\sup_{y\geq 0}\{y(lj) - \varphi_{\omega_{M^x}}(y)\}\right)$$

$$= \exp\left(\sup_{y\geq 0}\left\{(yj) - \frac{1}{l}\varphi_{\omega_{M^x}}(y)\right\}\right) = \sup_{y\geq 0}\frac{\exp(yj)}{\exp\left(\frac{1}{l}\varphi_{\omega_{M^x}}(y)\right)}$$

$$= \sup_{s\geq 1}\frac{s^j}{\exp\left(\frac{1}{l}\omega_{M^x}(s)\right)} = \left(\sup_{s\geq 0}\frac{s^{jl}}{\exp(\omega_{M^x}(s))}\right)^{1/l} = (M_{jl}^x)^{1/l}.$$

All steps except the last one hold also for $l > 0$ instead of $l \in \mathbb{N}_{>0}$. $\boxed{QED}$

The next result generalizes [6, Proposition 3.6].

Proposition 3.6. Let $\mathcal{M}$ be $(\mathcal{M}_{\mathrm{sc}})$, then

$$(\mathcal{M}_{\{\mathrm{mg}\}}) \iff \forall\, x \in \Lambda \,\exists\, H \geq 1 \,\exists\, y \in \Lambda \,\forall\, t \geq 0 : 2\omega_{M^y}(t) \leq \omega_{M^x}(Ht) + H, \tag{20}$$

$$(\mathcal{M}_{(\mathrm{mg})}) \iff \forall\, x \in \Lambda \,\exists\, H \geq 1 \,\exists\, y \in \Lambda \,\forall\, t \geq 0 : 2\omega_{M^x}(t) \leq \omega_{M^y}(Ht) + H. \tag{21}$$

Even if $\omega_{M^x} \sim \omega_{M^y}$ for all $x, y \in \Lambda$, (20) or (21) does not imply necessarily (ω_6) for each ω_{M^x}.

PROOF. We follow [6, Lemma 3.5, Proposition 3.6] and consider the Roumieu case. $(\mathcal{M}_{\{\mathrm{mg}\}})$ is equivalent to

$$\forall\, x \in \Lambda \,\exists\, H \geq 1 \,\exists\, y \in \Lambda \,\forall\, p \in \mathbb{N} : M_p^x \leq H^p \min_{0\leq q\leq p} M_q^y M_{p-q}^y =: H^p N_p^y.$$

By [6, Lemma 3.5] we have $\omega_{N^y} = 2\omega_{M^y}$ and proceed as in [6, Proposition 3.6] to get

$$2\omega_{M^y}(t) = \sup_{p \in \mathbb{N}} \log\left(\frac{t^p}{N_p^y}\right) = \sup_{p \in \mathbb{N}} \log\left(\frac{t^p}{\min_{0 \le q \le p} M_q^y M_{p-q}^y}\right)$$
$$\le \sup_{p \in \mathbb{N}} \log\left(\frac{t^p H^p}{M_p^x}\right) = \omega_{M^x}(Ht).$$

Conversely, again as in [6, Proposition 3.6]

$$N_p^y = \sup_{t \ge 0} \frac{t^p}{\exp(\omega_{N^y}(t))} = \sup_{t \ge 0} \frac{t^p}{\exp(2\omega_{M^y}(t))}$$
$$\ge \sup_{t \ge 0} \frac{t^p}{\exp(\omega_{M^x}(Ht) + H)} = \frac{1}{H^p \exp(H)} M_p^x.$$

$$\boxed{QED}$$

Now we are able to prove the first part of Theorem 3.4.

Theorem 3.7. Let $\mathcal{M} := \{M^x \in \mathbb{R}_{>0}^{\mathbb{N}} : x \in \Lambda\}$ be $(\mathcal{M}_{\text{sc}})$, $r, s \in \mathbb{N}_{>0}$. If $(\mathcal{M}_{\{\text{mg}\}})$ holds then for each non-empty open set $U \subseteq \mathbb{R}^r$ we get as locally convex vector spaces

$$\mathcal{E}_{\{\mathcal{M}\}}(U, \mathbb{R}^s) = \varprojlim_{K \subset\subset U} \varinjlim_{x \in \Lambda, l, h > 0} \mathcal{E}_{M^{x;l}, h}(K, \mathbb{R}^s).$$

If $(\mathcal{M}_{(\text{mg})})$ holds then we get as locally convex vector spaces

$$\mathcal{E}_{(\mathcal{M})}(U, \mathbb{R}^s) = \varprojlim_{K \subset\subset U} \varprojlim_{x \in \Lambda, l, h > 0} \mathcal{E}_{M^{x;l}, h}(K, \mathbb{R}^s).$$

PROOF. *Roumieu case.* By (19) implication ($\subseteq$) holds in any case since $M^{x;1} = M^x \le M^y$ for $x \le y$. We show ($\supseteq$) and by (19) it suffices to prove

$$\forall\, x \in \Lambda\; \forall\, l \in \mathbb{N}_{>0}\; \exists\, y \in \Lambda\; \exists\, C \ge 1\; \forall\, j \in \mathbb{N}:$$
$$(M_{jl}^x)^{1/l} \le C^j M_j^y \Leftrightarrow M_{jl}^x \le C^{jl}(M_j^y)^l, \quad (22)$$

which implies $\mathcal{E}_{M^{x;l}, h}(K, \mathbb{R}^s) \subseteq \mathcal{E}_{M^y, Ch}(K, \mathbb{R}^s)$. Now for each $x \in \Lambda$ there exists $D \ge 1$ and $y \in \Lambda$ such that $M_{2j}^x \le D^{2j}(M_j^y)^2$ for all $j \in \mathbb{N}$ by $(\mathcal{M}_{\{\text{mg}\}})$ and so (22) follows by iterating this estimate l-times.

Beurling case. ($\supseteq$) is valid in any case since $M^{x;1} = M^x$ for each $x \in \Lambda$. Let us prove ($\subseteq$), more precisely we show

$$\forall\, x \in \Lambda\; \forall\, l > 0\; \exists\, y \in \Lambda\; \exists\, C \ge 1\; \forall\, j \in \mathbb{N}: \quad M_j^y \le C^j M_j^{x;l}, \quad (23)$$

which implies $\mathcal{E}_{M^y,h}(K,\mathbb{R}^s) \subseteq \mathcal{E}_{M^{x;l},Ch}(K,\mathbb{R}^s)$. Iterating (21) gives

$$\forall\, x \in \Lambda\; \forall\, k \in \mathbb{N}_{>0}\; \exists\, y \in \Lambda\; \exists\, H \geq 1\; \forall\, t \geq 0: \; 2^k\omega_{M^x}(t) \leq \omega_{M^y}(H^k t) + (2^k-1)H.$$
$$(24)$$

Let $l \in \mathbb{N}_{>0}$ be given (large) and $k \in \mathbb{N}_{>0}$ be chosen minimal with $l \leq 2^k$. For all $x \in \Lambda$ and $j \in \mathbb{N}$ we have as in the proof of (19)

$$M_j^{x;1/l} = \sup_{t \geq 0} \frac{t^j}{\exp(l\omega_{M^x}(t))} \geq \sup_{t \geq 0} \frac{t^j}{\exp(\omega_{M^y}(H^k t) + (2^k-1)H)}$$
$$= \frac{1}{\exp((2^k-1)H)} \left(\frac{1}{H^k}\right)^j M_j^y.$$

Consequently for arbitrary $x \in \Lambda$ and $l \in \mathbb{N}_{>0}$ we find $y \in \Lambda$ such that $M^y \preceq M^{x;1/l}$. $\boxed{QED}$

An immediate consequence of Theorem 3.7 is

Corollary 3.8. Let $\mathcal{M}$ be $(\mathcal{M}_{\mathrm{sc}})$, then after the first step in (15) the construction yields always equivalent weight matrices of weight sequences w.r.t. to both $\{\approx\}$ and $(\approx)$.

PROOF. Let $x \in \Lambda$ be arbitrary but fixed. By Lemma 2.8 we have $\omega_{M^x} \in \mathcal{W}_0$ and so [9, 5.5] implies that each matrix $\mathcal{M}^x := \{M^{x;l} : l > 0\}$, $x \in \Lambda$, satisfies both $(\mathcal{M}_{\{\mathrm{mg}\}})$ and $(\mathcal{M}_{(\mathrm{mg})})$. $\boxed{QED}$

Now we prove the converse implication for Theorem 3.4. Here, the assumption $\Lambda = \mathbb{R}_{>0}$ for the Beurling case is necessary.

Proposition 3.9. Let $\mathcal{M} := \{M^x \in \mathbb{R}_{>0}^{\mathbb{N}} : x \in \Lambda\}$ be $(\mathcal{M}_{\mathrm{sc}})$.

(i) The equality

$$\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R}) = \varprojlim_{K \subset\subset \mathbb{R}}\; \varinjlim_{x \in \Lambda, l, h > 0} \mathcal{E}_{M^{x;l},h}(K,\mathbb{R})$$

implies $(\mathcal{M}_{\{\mathrm{mg}\}})$ for $\mathcal{M}$.

(ii) Assume that $\Lambda = \mathbb{R}_{>0}$, then

$$\mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{R}) = \varprojlim_{K \subset\subset \mathbb{R}}\; \varprojlim_{x \in \Lambda, l, h > 0} \mathcal{E}_{M^{x;l},h}(K,\mathbb{R})$$

implies $(\mathcal{M}_{(\mathrm{mg})})$ for $\mathcal{M}$.

PROOF. We generalize the technique in the proof of [9, Lemma 5.9 (5.11)].

Roumieu case. For each $x \in \Lambda$ and $l > 0$ there exists a characteristic function $\theta_{x,l} \in \mathcal{E}^{\text{global}}_{\{M^{x;l}\}}(\mathbb{R}, \mathbb{R})$, see (4). So the inclusion ($\supseteq$) implies

$$\forall\, x \in \Lambda \,\forall\, l > 0 \,\exists\, y \in \Lambda : \; M^{x;l} \preceq M^y = M^{y;1},$$

equivalently

$$\forall\, x \in \Lambda \,\forall\, l > 0 \,\exists\, y \in \Lambda \,\exists\, C \geq 1 \,\forall\, j \in \mathbb{N} : \; \frac{1}{l}\varphi^*_{\omega_{M^x}}(lj) \leq j \log(C) + \varphi^*_{\omega_{M^y}}(j). \tag{25}$$

Consider (25) for all $t \geq 0$ instead of all $j \in \mathbb{N}$. Then

$$\left(\frac{1}{l}\varphi^*_{\omega_{M^x}}(l\cdot) \right)^* (s) = \sup_{t \geq 0}\left\{ st - \frac{1}{l}\varphi^*_{\omega_{M^x}}(lt) \right\} = \frac{1}{l}\sup_{t' \geq 0}\left\{ st' - \varphi^*_{\omega_{M^x}}(t') \right\}$$

$$= \frac{1}{l}\varphi^{**}_{\omega_{M^x}}(s) = \frac{1}{l}\varphi_{\omega_{M^x}}(s) = \frac{1}{l}\omega_{M^x}(\exp(s)),$$

which holds since $\omega_{M^x} \in \mathcal{W}_0$ and so $\varphi^{**}_{\omega_{M^x}}(s) = \varphi_{\omega_{M^x}}(s)$. The right hand side gives

$$(\cdot D + \varphi^*_{\omega_{M^y}}(\cdot))^*(s) = \sup_{t \geq 0}\{(s - D)t - \varphi^*_{\omega_{M^y}}(t)\} = \varphi^{**}_{\omega_{M^y}}(s - D)$$

$$= \varphi_{\omega_{M^y}}(s - D) = \omega_{M^y}\left(\frac{\exp(s)}{C} \right).$$

Then we use [9, Lemma 5.7] (since $\omega_{M^x} \in \mathcal{W}_0$ we can replace ω by $\omega_{M^y} = \omega_{M^{y;1}}$ there) and get for $s \geq 0$ sufficiently large:

$$\sup_{t \geq 0}\left\{ st - \frac{1}{l}\varphi^*_{\omega_{M^x}}(lt) \right\} \geq \sup_{j \in \mathbb{N}}\left\{ sj - \frac{1}{l}\varphi^*_{\omega_{M^x}}(lj) \right\}$$

$$\underbrace{\geq}_{(25)} \sup_{j \in \mathbb{N}}\{sj - jD - \varphi^*_{\omega_{M^y}}(j)\} \geq \frac{1}{2}\sup_{t \geq 0}\{st - tD - \varphi^*_{\omega_{M^y}}(t)\}$$

$$= \frac{1}{2}\varphi^{**}_{\omega_{M^y}}(s - D) = \frac{1}{2}\omega_{M^y}\left(\frac{\exp(s)}{C} \right).$$

Thus for all t sufficiently large $\frac{1}{l}\omega_{M^x}(t) \geq \frac{1}{2}\omega_{M^y}\left(\frac{t}{C} \right)$ holds. Put $l = 4$ and by (20) we have shown $(\mathcal{M}_{\{\text{mg}\}})$.

Beurling case. We follow the second Section in [3], see also [9, Proposition 4.6 (1)]. By assumption $\bigcap_{x \in \Lambda} \mathcal{E}_{(M^x)}(\mathbb{R}, \mathbb{R}) \subseteq \bigcap_{x \in \Lambda, l > 0} \mathcal{E}_{(M^{x;l})}(\mathbb{R}, \mathbb{R})$ and both are Fréchet spaces. Using the closed graph theorem the inclusion is continuous. Hence for each compact set $K_1 \subseteq \mathbb{R}$, $x \in \Lambda$, $l > 0$ and $h > 0$, there exist $C, h_1 >$

$0, y \in \Lambda$ and a compact set $K_2 \subseteq \mathbb{R}$ such that for each $f \in \bigcap_{x \in \Lambda} \mathcal{E}_{(M^x)}(\mathbb{R}, \mathbb{R})$ we obtain

$$\|f\|_{M^{x;l},K_1,h} = \sup_{t \in K_1, j \in \mathbb{N}} \frac{|f^{(j)}(t)|}{h^j M_j^{x;l}} \leq C \sup_{t \in K_2, j \in \mathbb{N}} \frac{|f^{(j)}(t)|}{h_1^j M_j^y} = C\|f\|_{M^y,K_2,h_1}.$$

Let K_1 be a compact interval containing 0, put $h = 1$ and take $f_s(t) := \sin(st) + \cos(st)$ for $t \in \mathbb{R}$ and $s \geq 0$. Note that $f_s \in \bigcap_{x \in \Lambda} \mathcal{E}_{(M^x)}^{\text{global}}(\mathbb{R}, \mathbb{R})$ for any $s \geq 0$ since $\lim_{k \to \infty} (M_k^x)^{1/k} = +\infty$ for each $x \in \Lambda$. Then

$$\sup_{j \in \mathbb{N}} \frac{s^j}{M_j^{x;l}} = \sup_{j \in \mathbb{N}} \frac{|f_s^{(j)}(0)|}{M_j^{x;l}} \leq \sup_{t \in K_1, j \in \mathbb{N}} \frac{|f_s^{(j)}(t)|}{M_j^{x;l}}$$

$$\leq C \sup_{t \in K_2, j \in \mathbb{N}} \frac{|f_s^{(j)}(t)|}{h_1^j M_j^y} \leq C \sup_{j \in \mathbb{N}} \frac{2s^j}{h_1^j M_j^y},$$

which implies $\exp(\omega_{M^{x;l}}(s)) \leq 2C \exp\left(\omega_{M^y}\left(\frac{s}{h_1}\right)\right)$. Using [6, Proposition 3.2] we get for all $j \in \mathbb{N}$

$$M_j^{x;l} = \sup_{t \geq 0} \frac{t^j}{\exp(\omega_{M^{x;l}}(t))} \geq \sup_{t \geq 0} \frac{t^j}{2C \exp\left(\omega_{M^y}\left(\frac{t}{h_1}\right)\right)} = \frac{h_1^j}{2C} M_j^y,$$

hence $M^y \preceq M^{x;l}$. We summarize:

$$\forall\, x \in \Lambda\, \forall\, l > 0\, \exists\, y \in \Lambda\, \exists\, D \geq 1\, \forall\, j \in \mathbb{N}: \quad \varphi_{\omega_{M^y}}^*(j) \leq j \log(D) + \frac{1}{l} \varphi_{\omega_{M^x}}^*(lj).$$
(26)

Now use the proof of the Roumieu case to get $\omega_{M^y}(t) \geq \frac{1}{2l} \omega_{M^x}\left(\frac{t}{D}\right)$ for t sufficiently large. The choice $l = \frac{1}{4}$ and (21) imply $(\mathcal{M}_{(\text{mg})})$. $\boxed{QED}$

3.10 Classes $\mathcal{E}_{[\omega_{\mathcal{M}}]}$ defined by a weight matrix of associated functions

The goal of this section is to prove

Theorem 3.11. Let $\mathcal{M} := \{M^x \in \mathbb{R}_{>0}^{\mathbb{N}} : x \in \Lambda\}$ be $(\mathcal{M}_{\text{sc}})$, let $r, s \in \mathbb{N}_{>0}$ and U be a non-empty open set in $\mathbb{R}^r$.

(i) $(\mathcal{M}_{\{L\}})$ for $\mathcal{M}$ implies

$$\mathcal{E}_{\{\omega_{\mathcal{M}}\}}(U, \mathbb{R}^s) = \varprojlim_{K \subset\subset U} \varinjlim_{x \in \Lambda, l, h > 0} \mathcal{E}_{M^{x;l},h}(K, \mathbb{R}^s),$$

(ii) ($\mathcal{M}_{(\mathrm{L})}$) for $\mathcal{M}$ implies

$$\mathcal{E}_{(\omega_{\mathcal{M}})}(U, \mathbb{R}^s) = \varprojlim_{K \subset\subset U} \ \varprojlim_{x \in \Lambda, l, h > 0} \mathcal{E}_{M^{x;l,h}}(K, \mathbb{R}^s)$$

as locally convex vector spaces.

The main Theorem 3.2 follows then by combining Theorem 3.7 and Theorem 3.11.

We start with the following result:

Proposition 3.12. Let $\mathcal{M} := \{M^x \in \mathbb{R}_{>0}^{\mathbb{N}} : x \in \Lambda\}$ be ($\mathcal{M}_{\mathrm{sc}}$).

(i) ($\mathcal{M}_{\{\mathrm{L}\}}$) implies

$$\forall \, x \in \Lambda \, \exists \, y \in \Lambda : \quad \omega_{M^y}(2t) = O(\omega_{M^x}(t)) \ \text{ as } t \to \infty. \qquad (27)$$

(ii) ($\mathcal{M}_{(\mathrm{L})}$) implies

$$\forall \, x \in \Lambda \, \exists \, y \in \Lambda : \quad \omega_{M^x}(2t) = O(\omega_{M^y}(t)) \ \text{ as } t \to \infty. \qquad (28)$$

If all associated functions are equivalent w.r.t. $\sim$, then each/some ω_{M^x} satisfies (ω_1).

PROOF. By ($\mathcal{M}_{\{\mathrm{L}\}}$) for each $x \in \Lambda$ and each $h > 0$ there exists $y \in \Lambda$ and $D > 0$ such that $M_k^x h^k \leq D M_k^y$ holds for all $k \in \mathbb{N}$. Multiplying with t^k for arbitrary $t > 0$ we get $\frac{(ht)^k}{M_k^y} \leq D \frac{t^k}{M_k^x}$ and finally $\log\left(\frac{(ht)^k}{M_k^y}\right) \leq \log\left(\frac{t^k}{M_k^x}\right) + D_1$, which holds for all $k \in \mathbb{N}$. So by definition $\omega_{M^y}(ht) \leq \omega_{M^x}(t) + D_1$ holds and it is enough to take $h = 2$.

The Beurling case is completely analogous, use ($\mathcal{M}_{(\mathrm{L})}$) instead of ($\mathcal{M}_{\{\mathrm{L}\}}$). $\boxed{QED}$

The next result generalizes [9, Lemma 5.9 (5.10)].

Proposition 3.13. Let $\{\sigma_x \in \mathcal{W}_0 : x \in \Lambda\}$ be given and assume the Roumieu type condition (see Proposition 3.12 above):

$$\forall \, x \in \Lambda \, \exists \, y \in \Lambda : \sigma_y(2t) = O(\sigma_x(t)) \ \text{ as } t \to \infty.$$

Then

$$\forall \, x \in \Lambda \, \forall \, s \in \mathbb{N} \, \exists \, y \in \Lambda \, \exists \, L \geq 1 \, \forall \, a > 0 \, \forall \, j \in \mathbb{N} :$$
$$\exp\left(\frac{1}{a}\varphi^*_{\sigma_x}(aj)\right) \exp(s)^j \leq \exp\left(\frac{\sum_{i=1}^s L^i}{L^s a}\right) \exp\left(\frac{1}{L^s a}\varphi^*_{\sigma_y}(L^s aj)\right).$$

If the Beurling type condition

$$\forall\, x \in \Lambda \,\exists\, y \in \Lambda : \; \sigma_x(2t) = O(\sigma_y(t)) \;\text{ as } t \to \infty$$

holds, then

$$\forall\, x \in \Lambda \,\forall\, s \in \mathbb{N} \,\exists\, y \in \Lambda \,\exists\, L \geq 1 \,\forall\, a > 0 \,\forall\, j \in \mathbb{N} :$$

$$\exp\left(\frac{1}{a}\varphi^*_{\sigma_y}(aj)\right)\exp(s)^j \leq \exp\left(\frac{\sum_{i=1}^{s} L^i}{L^s a}\right)\exp\left(\frac{1}{L^s a}\varphi^*_{\sigma_x}(L^s a j)\right).$$

If each ω_{M^x} has (ω_1), then the Roumieu and the Beurling case is satisfied with $x = y$.

PROOF. We consider the Roumieu case. For all $x \in \Lambda$ there exist $y \in \Lambda$ and $L \geq 1$ with $\sigma_y(4t) \leq L\sigma_x(t) + L$ for all $t \geq 0$, hence $\varphi_{\sigma_y}(t+1) = \sigma_y(\exp(t+1)) \leq L\sigma_x(\exp(t)) + L$. First we have

$$\varphi^*_{\sigma_y}(Ls) = L\sup\left\{st - \frac{1}{L}\varphi_{\sigma_y}(t) : t \geq 0\right\} \geq L\sup\{st - (1 + \varphi_{\sigma_x}(t-1)) : t \geq 0\}$$

$$\geq L\sup\{s(t-1) + s - 1 - \varphi_{\sigma_x}(t-1) : t \geq 1\} = Ls - L + L\varphi^*_{\sigma_x}(s),$$

and so

$$\forall\, x \in \Lambda \,\exists\, y \in \Lambda \,\exists\, L \geq 1 \,\forall\, t \geq 0 : \;\; L\varphi^*_{\sigma_x}(t) + Lt \leq L + \varphi^*_{\sigma_y}(Lt).$$

Using induction on this inequality we get

$$\forall\, x \in \Lambda \,\forall\, s \in \mathbb{N} \,\exists\, y \in \Lambda \,\exists\, L \geq 1 \,\forall\, t \geq 0 :$$

$$L^s\varphi^*_{\sigma_x}(t) + sL^s t \leq \varphi^*_{\sigma_y}(L^s t) + \sum_{i=1}^{s} L^i.$$

Now put $t = aj$ for $j \in \mathbb{N}$ and $a > 0$, divide by $L^s a$ and finally apply exp. $\boxed{QED}$

Propositions 3.12 and 3.13 imply

Corollary 3.14. Let $\mathcal{M} := \{M^x \in \mathbb{R}^{\mathbb{N}}_{>0} : x \in \Lambda\}$ be $(\mathcal{M}_{\mathrm{sc}})$.

(i) If $\mathcal{M}$ has $(\mathcal{M}_{\{L\}})$, then

$$\forall\, x \in \Lambda \,\forall\, h > 0 \,\exists\, y \in \Lambda \,\forall\, a > 0 \,\exists\, D > 0 \,\exists\, b > 0 \,\forall\, j \in \mathbb{N} :$$

$$M_j^{x;a} h^j \leq D M_j^{y;b}. \quad (29)$$

(ii) If $\mathcal{M}$ has $(\mathcal{M}_{(L)})$, then

$$\forall\, x \in \Lambda \,\forall\, h > 0 \,\exists\, y \in \Lambda \,\forall\, b > 0 \,\exists\, D > 0 \,\exists\, a > 0 \,\forall\, j \in \mathbb{N} :$$

$$M_j^{y;a} h^j \leq D M_j^{x;b}. \quad (30)$$

Using (29) in the Roumieu and (30) in the Beurling case we get Theorem 3.11 and are done.

We can also prove:

Corollary 3.15. Let $\mathcal{M} := \{M^x \in \mathbb{R}_{>0}^{\mathbb{N}} : x \in \Lambda\}$ be $(\mathcal{M}_{\mathrm{sc}})$, then $(27) \Longleftrightarrow (29)$ and $(28) \Longleftrightarrow (30)$.

PROOF. It remains to show $(\Longleftarrow)$. In (29) let $h = 2$, $a = 1$, multiply with t^j for arbitrary $t > 0$ and apply log. Thus $\omega_{M^{y;b}}(2t) = O(\omega_{M^x}(t))$ holds as $t \to \infty$. Finally [9, Lemma 5.7] implies $\omega_{M^{y;b}} \sim \omega_{M^y}$ for each $b > 0$. The case for (30) is analogous. $\boxed{QED}$

3.16   Applications of Theorem 3.2

If $\mathcal{M} = \Omega$ for some $\omega \in \mathcal{W}$, then by Theorem 3.2 and [9, Theorem 5.14] we get $\mathcal{E}_{[\omega]} = \mathcal{E}_{[\Omega]} = \mathcal{E}_{[\omega_\Omega]} = \mathcal{E}_{[\omega_{\Omega^l}]}$ for each $l > 0$. More generally we can prove

Corollary 3.17. Let $\mathcal{M} = \{M^x : x > 0\}$ have $(\mathcal{M}_{\mathrm{sc}})$. Then the following are equivalent:

(i) There exists $\omega \in \mathcal{W}$ with $\mathcal{E}_{[\mathcal{M}]} = \mathcal{E}_{[\omega]}$.

(ii) There exists a $(\mathcal{M}_{\mathrm{sc}})$-matrix $\mathcal{N} = \{N^x : x > 0\}$ with $\mathcal{M}[\approx]\mathcal{N}$, such that $\omega_{N^x} \sim \omega_{N^y}$ for each $x, y > 0$ and $\mathcal{N}$ has $(\mathcal{M}_{[\mathrm{mg}]})$ and $(\mathcal{M}_{[\mathrm{L}]})$.

PROOF. $(i) \Rightarrow (ii)$ We can take $\mathcal{N} = \Omega$, see [9, Proposition 4.6, Lemma 5.7] and [9, Theorem 5.14, Corollary 5.15].

$(ii) \Rightarrow (i)$ Combining Theorem 3.2 and [9, Theorem 5.14] we get

$$\forall\, x > 0:\quad \mathcal{E}_{[\mathcal{M}]} = \mathcal{E}_{[\mathcal{N}]} = \mathcal{E}_{[\omega_{\mathcal{N}}]} = \mathcal{E}_{[\omega_{N^x}]} = \mathcal{E}_{[\mathcal{N}^x]} \tag{31}$$

with $\mathcal{N}^x := \{N^{x;l} : l > 0\}$. Note that $\omega_{M^x} \in \mathcal{W}$ for each $x > 0$, see Proposition 3.12. So we can take $\omega = \omega_{N^x}$ and $\Omega = \mathcal{N}^x$ for some arbitrary $x > 0$, i.e. $\Omega^l = N^{x;l}$.

Finally by [9, Proposition 4.6] we get $\mathcal{M}[\approx]\mathcal{N}^x$ and any $\sigma \in \mathcal{W}$ with $\mathcal{E}_{[\sigma]} = \mathcal{E}_{[\mathcal{M}]}$ satisfies $\sigma \sim \omega_{N^x}$ by [9, Corollary 5.17]. $\boxed{QED}$

Let $\mathcal{M} = \{M^x : x \in \Lambda\}$ be $(\mathcal{M}_{\mathrm{sc}})$ given, then in general we will not have $\omega_{M^x} \sim \omega_{M^y}$ for any $x, y \in \Lambda$. On the one hand by definition $\omega_{M^y} \le \omega_{M^x}$ whenever $x \le y$ and on the other hand [7, 1.8 III] yields $\omega_{M^x}(t) = \sup_{p \in \mathbb{N}} p \log(t) - \log(M_p^x) = p_{t,x} \log(t) - \log(M_{p_{t,x}}^x)$, where $\mu_{p_{t,x}}^x \le t < \mu_{p_{t,x}+1}^x$. So if $\mathcal{M}$ satisfies

$$\forall\, x, y > 0 \; x \le y \; \exists\, C \ge 1 \; \exists\, t_0 \ge 1 \; \forall\, t \ge t_0 \; \exists\, q \in \mathbb{N}: \quad \frac{(M_q^y)^C}{M_{p_{t,x}}^x} \le t^{qC - p_{t,x}}, \tag{32}$$

then all associated functions are equivalent w.r.t. $\sim$. Moreover we can prove:

Lemma 3.18. Let $M, N \in \mathcal{LC}$.

(1) If ω_M satisfies (ω_1), then $M \preceq N \implies \omega_M \preceq \omega_N$.

(2) If N satisfies (mg), then $\omega_N \preceq \omega_M \implies N \preceq M$.

PROOF. (1) For all $t > 0$ we get

$$\omega_M(t) = \sup_{p \in \mathbb{N}}(p \log(t) - \log(M_p)) \underbrace{\geq}_{M \preceq N} \sup_{p \in \mathbb{N}}(p \log(t) - \log(D^p N_p)) = \omega_N\left(\frac{t}{D}\right)$$

for a constant $D > 0$ (large). Iterating (ω_1) we have $\omega_M(2^n t) \leq C^n \omega_M(t) + C$ for a constant $C \geq 1$ and all $t \geq 0$. Choose now $n \in \mathbb{N}$ minimal such that $D \leq 2^n$, hence $\omega_N(t) \leq \omega_M(Dt) \leq \omega_M(2^n t) \leq C^n \omega_M(t) + C$ for all $t \geq 0$ and so $\omega_N(t) = O(\omega_M(t))$ as $t \to \infty$.

(2) By [6, Proposition 3.6] condition (mg) for N implies (ω_6) for ω_N. Using [6, Proposition 3.2] we can estimate for all $p \in \mathbb{N}$:

$$M_p = \sup_{t>0} \frac{t^p}{\exp(\omega_M(t))} \underbrace{\geq}_{\omega_M \preceq \omega_N} \sup_{t>0} \frac{t^p}{\exp(C_1 \omega_N(t) + C_1)}$$

$$\geq C_2 \sup_{t>0} \frac{t^p}{\exp(\omega_N(H^n t) + (2^n - 1)H)} = C_3 \left(\frac{1}{H^n}\right)^p N_p,$$

where $n \in \mathbb{N}$ is chosen minimal such that $C_1 \leq 2^n$ (iterating (ω_6) as in (24)). Thus $N \preceq M$ follows. $\boxed{QED}$

3.19 Roumieu case versus Beurling case

For $\mathcal{E}_{[\mathcal{M}]}$ and $\mathcal{E}_{[\omega_{\mathcal{M}}]}$ it is also important to know whether one can replace in their definitions the Roumieu classes $\mathcal{E}_{\{M^x\}}$, $\mathcal{E}_{\{\omega_{M^x}\}}$ by the Beurling classes $\mathcal{E}_{(M^x)}$, $\mathcal{E}_{(\omega_{M^x})}$. In the case $\mathcal{E}_{[\mathcal{M}]}$ this can be done assuming $(\mathcal{M}_{[BR]})$, see [9, 4.2 (4.4)]. If $\mathcal{M} = \Omega$ for some $\omega \in \mathcal{W}$, then (ω_7) is sufficient to guarantee this property for the Roumieu case and the Beurling case, see [9, Theorem 5.14 (4)].

Proposition 3.20. Let $\mathcal{M} = \{M^x : x \in \Lambda\}$ be $(\mathcal{M}_{sc})$.

(i) If $\mathcal{M}$ has $(\mathcal{M}_{\{BR\}})$ and each M^x has (mg), then

$$\forall\, x \in \Lambda \,\exists\, y \in \Lambda: \quad \omega_{M^x} \lhd \omega_{M^y}, \tag{33}$$

which implies $\bigcup_{x \in \Lambda} \mathcal{E}_{\{\omega_{M^x}\}} = \bigcup_{x \in \Lambda} \mathcal{E}_{(\omega_{M^x})}$.

(*ii*) If $\mathcal{M}$ has $(\mathcal{M}_{(\mathrm{BR})})$ and each M^x has (mg), then

$$\forall \, x \in \Lambda \, \exists \, y \in \Lambda : \quad \omega_{M^y} \lhd \omega_{M^x}, \tag{34}$$

which implies $\bigcap_{x \in \Lambda} \mathcal{E}_{\{\omega_{M^x}\}} = \bigcap_{x \in \Lambda} \mathcal{E}_{(\omega_{M^x})}$.

(*iii*) If each ω_{M^x} has (ω_1) and (33) holds, then $\mathcal{M}$ has $(\mathcal{M}_{\{\mathrm{BR}\}})$.

(*iv*) If each ω_{M^x} has (ω_1) and (34) holds, then $\mathcal{M}$ has $(\mathcal{M}_{(\mathrm{BR})})$.

PROOF. We consider the Roumieu case (*i*) and (*iii*), the Beurling case (*ii*) and (*iv*) is completely analogous.

(*i*) (33) means

$$\forall \, x \in \Lambda \, \exists \, y \in \Lambda \, \forall \, C > 0 \, \exists \, D > 0 \, \forall \, t \geq 0 : \quad \omega_{M^y}(t) \leq C\omega_{M^x}(t) + D.$$

By assumption $(\mathcal{M}_{\{\mathrm{BR}\}})$ holds, i.e.

$$\forall \, x \in \Lambda \, \exists \, y \in \Lambda \, \forall \, h > 0 \, \exists \, C_h > 0 \, \forall \, j \in \mathbb{N} : \quad M_j^x \leq C_h \, h^j M_j^y.$$

Multiplying with t^j for arbitrary $t > 0$ and $j \in \mathbb{N}$ we get by definition $\log(C_h) + \omega_{M^x}(t) \geq \omega_{M^y}(t/h)$.

Now let $1 > C > 0$ be given, (mg) for M^y implies (ω_6) for ω_{M^y}. Iterating this condition (see (24)) we take $k \in \mathbb{N}$ minimal with $C^{-1} \leq 2^k$ and choose $h := \frac{1}{H^k}$. Then $C^{-1}\omega_{M^y}(t) \leq \omega_{M^y}(H^k t) + (2^k - 1)H = \omega_{M^y}(t/h) + H_1 \leq \omega_{M^x}(t) + H_2$.

(*iii*) Iterating (ω_1) for ω_{M^x} gives $\omega_{M^x}(2^n t) \leq L^n \omega_{M^x}(t) + \sum_{i=1}^n L^i$. So let $1 > h > 0$ be given and choose $n \in \mathbb{N}_{>0}$ minimal with $h^{-1} \leq 2^n$. Then $\omega_{M^x}(t/h) \leq \omega_{M^x}(2^n t) \leq L^n \omega_{M^x}(t) + \sum_{i=1}^n L^i$ and choose $C := L^{-n}$ which depends only on $x \in \Lambda$ and given h. According to $x \in \Lambda$ and C we use (33) and [6, Proposition 3.2] to obtain, for all $j \in \mathbb{N}$:

$$M_j^y = \sup_{t \geq 0} \frac{t^j}{\exp(\omega_{M^y}(t))} \geq \sup_{t \geq 0} \frac{t^j}{\exp(C\omega_{M^x}(t) + D)}$$

$$\geq \frac{1}{D_1} \sup_{t \geq 0} \frac{t^j}{\exp(\omega_{M^x}(ht))} = \frac{1}{D_1 h^j} M_j^x.$$

Note that the constant D_1 depends also only on x and h. $\boxed{QED}$

4   Characterization of the non-quasianalyticity of $\mathcal{E}_{[\mathcal{M}]}$

Let $\mathcal{M}$ be $(\mathcal{M})$, then $\mathcal{E}_{[\mathcal{M}]}$ is called *non-quasianalytic* if $\mathcal{E}_{[\mathcal{M}]}$ contains nontrivial functions with compact support.

The goal is to characterize this property in terms of the weight matrix $\mathcal{M}$ which gives answer to [9, Remark 4.8].

Theorem 4.1. Let $\mathcal{M} = \{M^x : x \in \Lambda\}$ be $(\mathcal{M})$.

(i) $\mathcal{E}_{\{\mathcal{M}\}}$ is non-quasianalytic if and only if there exists $x_0 \in \Lambda$ such that $\mathcal{E}_{[M^{x_0}]}$ is non-quasianalytic.

(ii) $\mathcal{E}_{(\mathcal{M})}$ is non-quasianalytic if and only if each $\mathcal{E}_{[M^x]}$ is non-quasianalytic, provided $\Lambda = \mathbb{R}_{>0}$.

Remark 4.2. The theorem above still holds if we assume that each $M^x \in \mathbb{R}_{>0}^{\mathbb{N}}$ is arbitrary with $M_0^x = 1$ and $M^x \leq M^y$ whenever $x \leq y$, i.e. the assumption that each M^x is increasing is not necessary. This holds by the definitions of $\mathcal{E}_{[\mathcal{M}]}$ given in 2.2 and since we work in the proofs of Propositions 4.4 and 4.7 below with the regularizations M^{lc} and M^I which will be defined in 4.3. Note that $M \leq N$ implies $M^{\mathrm{lc}} \leq N^{\mathrm{lc}}$ and $M^I \leq N^I$.

4.3 Non-quasianalyticity of $\mathcal{E}_{[M]}$

Before we start proving Theorem 4.1 we recall and summarize some facts for classical Denjoy-Carleman-classes $\mathcal{E}_{[M]}$. Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ with $M_0 = 1$, then we denote by $M^{\mathrm{lc}} = (M_j^{\mathrm{lc}})_j$ the log-convex minorant of M which is given by

$$M_j^{\mathrm{lc}} := \sup_{t>0} \frac{t^j}{\exp(\omega_M(t))}$$

resp.

$$M_j^{\mathrm{lc}} := \inf\{M_k^{(l-j)/(l-k)} M^{(j-k)/(l-k)} : k \leq j \leq l, k \neq l\}, M_0^{\mathrm{lc}} := M_0 = 1$$

see [6, Definition 3.1] and [7] resp. [5]. Moreover we introduce

$$M^I := (M_k^I)_k, \qquad M_k^I := \left(\inf\{(M_j)^{1/j} : j \geq k\}\right)^k \text{ for } k \geq 1, \qquad M_0^I := 1,$$

see also [5]. $((M_k^I)^{1/k})_k$ is the increasing minorant of $((M_k)^{1/k})_k$, $M^I = M$ if and only if $k \mapsto (M_k)^{1/k}$ is increasing. If M is (lc), then $M = M^I$ and $(M^{\mathrm{lc}})^I = M^{\mathrm{lc}}$, so $M^{\mathrm{lc}} \leq M^I \leq M$.

Proposition 4.4. Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ with $M_0 = 1$. Then $\mathcal{E}_{[M]}$ is non-quasianalytic if and only if M^{lc} has (nq) and if and only if $\sum_{p \geq 1} \frac{1}{(M_p^I)^{1/p}} < +\infty$. In this case $\mathcal{C}^\omega \subsetneq \mathcal{E}_{[M]} = \mathcal{E}_{[M^I]} = \mathcal{E}_{[M^{\mathrm{lc}}]}$ holds.

Remark: The equivalence $\sum_{p=1}^\infty \frac{1}{(M_p^I)^{1/p}} < +\infty$ if and only if M^{lc} has (nq) can be shown directly without using the non-quasianalyticity of $\mathcal{E}_{[M]}$, see the proof of [5, Theorem 1.3.8].

PROOF. By [5, Theorem 1.3.8] and [6, Theorem 4.2] we know that $\mathcal{E}_{[M]}$ is non-quasianalytic if and only if $\sum_{p=1}^{\infty} \frac{1}{(M_p^I)^{1/p}} < +\infty$ and if and only if (nq) holds for M^{lc}. More precisely the Roumieu-case follows directly by [5, Theorem 1.3.8]. If $\mathcal{E}_{(M)}$ is non-quasianalytic, then $\mathcal{E}_{\{M\}}$ too and apply [5, Theorem 1.3.8]. If M^{lc} has (nq), then by [6, Theorem 4.2] the class $\mathcal{E}_{(M^{\mathrm{lc}})}$ is non-quasianalytic, hence $\mathcal{E}_{(M)}$ too.

Claim. If $\mathcal{E}_{[M]}$ is non-quasianalytic, then

$$\lim_{p \to \infty} (m_p)^{1/p} = +\infty \Leftrightarrow \lim_{k \to \infty} \frac{(M_p)^{1/p}}{p} = +\infty.$$

We put $a_p := \frac{1}{(M_p^I)^{1/p}}$ in the well-known Lemma 4.5 below and since $(M_p^I)^{1/p} \leq (M_p)^{1/p}$ for all $p \in \mathbb{N}_{>0}$ the claim follows.

This claim generalizes remark $(b.1)$ on page 387 in [13] since there (lc) (which is assumed in (b)) for M was necessary. Moreover it implies $\mathcal{C}^{\omega} \subsetneq \mathcal{E}_{[M]}$.

Finally by [9, Theorem 2.15] and the claim we see that $\mathcal{E}_{[M^{\mathrm{lc}}]} \subseteq \mathcal{E}_{[M^I]} \subseteq \mathcal{E}_{[M]} = \mathcal{E}_{[M^{\mathrm{lc}}]}$. $\boxed{QED}$

Lemma 4.5. Let $(a_p)_{p \geq 1}$ be a decreasing sequence of positive real numbers with $\sum_{p \geq 1} a_p < +\infty$. Then $pa_p \to 0$ as $p \to \infty$.

4.6   The general case $\mathcal{E}_{[\mathcal{M}]}$

Proposition 4.4 shows that $\mathcal{E}_{\{M\}}$ is non-quasianalytic if and only if $\mathcal{E}_{(M)}$ is. In the general case this is not true, e.g. let $\mathcal{M} = \{M^1, M^2\}$ such that $M^1 \leq M^2$, $\mathcal{E}_{[M^1]}$ is quasianalytic whereas $\mathcal{E}_{[M^2]}$ is not (take $M_p^1 := p!$ and $M_p^2 := p!^s$ for some $s > 1$).

We prove now Theorem 4.1. The Roumieu part is obvious and the Beurling part will follow from the following Proposition 4.7 which uses the idea of [14, Lemma 5.1]. We construct a non-quasianalytic sequence N which is smaller than any sequence in the matrix $\mathcal{M}$. More precisely, we will show that $N \lhd \mathcal{M}$, while in [14, Lemma 5.1] only $N(\preceq)\mathcal{M}$ was proved. Moreover the assumptions in [14] where each M^x is log-convex and $\mu_p^y \leq \mu_p^x$ for all $p \in \mathbb{N}$ and $y \leq x$ will be not needed for our proof.

Proposition 4.7. Let $\mathcal{M} := \{M^x \in \mathbb{R}_{>0}^{\mathbb{N}} : x \in \Lambda = \mathbb{R}_{>0}\}$ satisfy $(\mathcal{M})$ such that $\mathcal{E}_{[M^x]}$ is non-quasianalytic for each $x > 0$. Then we get:

(i) There exists N with $N_0 = 1$ and $N^I = N$, $\mathcal{E}_{[N]}$ is non-quasianalytic and $N \lhd \mathcal{M}$, so $\mathcal{E}_{\{N\}} \subseteq \mathcal{E}_{(\mathcal{M})}$.

(ii) Let U be a non-empty open subset of $\mathbb{R}^r$. For every bounded subset B in $\mathcal{E}_{(\mathcal{M})}(U)$ there exists a sequence N as in (i) such that B is a bounded subset in $\mathcal{E}_{\{N\}}(U)$, too.

(iii) Let $\mathcal{N} := \{N^x : x \in \mathbb{R}_{>0}\}$ satisfy ($\mathcal{M}$) and ($\mathcal{M}_{\{\mathcal{C}^\omega\}}$) and such that $\mathcal{N} \lhd \mathcal{M}$. Then there exists a sequence L which satisfies (lc), (nq) and finally $\mathcal{N} \lhd L \lhd \mathcal{M}$.

PROOF. (i) Since $\Lambda = \mathbb{R}_{>0}$ and $\mathcal{M}$ satisfies ($\mathcal{M}$) we can restrict to $\Lambda = \{\frac{1}{n} : n \in \mathbb{N}_{>0}\}$, see 2.2. By [5, Theorem 1.3.8] and [6, Theorem 4.2] we get

$$\forall \, x \in \mathbb{N}_{>0} : \sum_{p \geq 1} \frac{1}{((M^{1/x})_p^I)^{1/p}} < +\infty.$$

Now, as in [14, Lemma 5.1], we introduce sequences $(a_q)_{q \geq 0}$ and $(b_q)_{q \geq 0}$ recursively as follows. Put $a_0 = b_0 = 0$, then let a_q be the first integer such that

$$a_q > b_{q-1}, \qquad \sum_{p=a_q+1}^{\infty} \frac{1}{((M^{1/(q+1)})_p^I)^{1/p}} \leq \frac{2^{-q}}{q+1}. \qquad (35)$$

b_q shall be the first integer such that $\frac{1}{q}((M^{1/q})_{a_q}^I)^{1/a_q} < \frac{1}{q+1}((M^{1/(q+1)})_{b_q}^I)^{1/b_q}$ holds. Since for each $q \in \mathbb{N}_{>0}$ separately $p \mapsto ((M^{1/q})_p^I)^{1/p}$ is increasing, tending to infinity and since $((M^{1/q})_p^I)^{1/p} \geq ((M^{1/(q+1)})_p^I)^{1/p}$ for each $p, q \geq 1$ we have $a_q < b_q$ for each q.

Now introduce $N = (N_p)_p$ as follows. We put $N_0 := 1$ and for $p \in \mathbb{N}_{>0}$ we set

$$(N_p)^{1/p} = \frac{1}{q}((M^{1/q})_p^I)^{1/p} \;\; \text{for} \;\; b_{q-1} \leq p \leq a_q,$$

$$(N_p)^{1/p} = \frac{1}{q}((M^{1/q})_{a_q}^I)^{1/a_q} \;\; \text{for} \;\; a_q + 1 \leq p \leq b_q - 1.$$

Claim. The mapping $p \mapsto (N_p)^{1/p}$ is increasing, i.e. $N^I = N$.

If $b_{q-1} \leq p < a_q$ and $a_q + 1 \leq p < b_q - 1$, then $(N_p)^{1/p} \leq (N_{p+1})^{1/(p+1)}$ holds by definition. If $p = a_q$, then

$$(N_p)^{1/p} = \frac{1}{q}((M^{1/q})_p^I)^{1/p} \leq \frac{1}{q}((M^{1/q})_{a_q}^I)^{1/a_q} = (N_{p+1})^{1/(p+1)}$$

and if $p = b_q - 1$, then

$$(N_p)^{1/p} = \frac{1}{q}((M^{1/q})_{a_q}^I)^{1/a_q} \leq \frac{1}{q+1}((M^{1/(q+1)})_{b_q}^I)^{1/b_q} = (N_{p+1})^{1/(p+1)}$$

holds by the choice of $(b_q)_q$.

Claim. $\mathcal{E}_{[N]}$ is non-quasianalytic. First we have

$$\sum_{p=a_1+1}^{\infty} \frac{1}{(N_p)^{1/p}} = \sum_{q=1}^{\infty} \left(\sum_{p=a_q+1}^{b_q-1} \frac{1}{(N_p)^{1/p}} + \sum_{p=b_q}^{a_{q+1}} \frac{1}{(N_p)^{1/p}} \right)$$

$$= \sum_{q=1}^{\infty} \left(\sum_{p=a_q+1}^{b_q-1} \frac{q}{((M^{1/q})^I_{a_q})^{1/a_q}} + \sum_{p=b_q}^{a_{q+1}} \frac{q+1}{((M^{1/(q+1)})^I_p)^{1/p}} \right)$$

$$\underbrace{\leq}_{(\star)} \sum_{q=1}^{\infty} \underbrace{\sum_{p=a_q+1}^{\infty} \frac{q+1}{((M^{1/(q+1)})^I_p)^{1/p}}}_{\leq 2^{-q}} \leq 1.$$

$(\star)$ holds because by the choice of $(b_q)_q$ we have

$$\frac{1}{q+1}((M^{1/(q+1)})^I_p)^{1/p} \leq \frac{1}{q}((M^{1/q})^I_{a_q})^{1/a_q}$$

for $a_q+1 \leq p \leq b_q-1$. Since $N = N^I$ and by [5, Theorem 1.3.8] and [6, Theorem 4.2] we are done.

Claim. $N \lhd \mathcal{M}$, i.e. $N \lhd M^{1/x}$ for all $x \in \mathbb{N}_{>0}$.

We have $(N_p)^{1/p} \leq \frac{1}{q}((M^{1/q})^I_p)^{1/p} \leq \frac{1}{q}(M_p^{1/q})^{1/p}$ whenever $p \geq b_{q-1}$, so $\mathcal{E}_{\{N\}} \subseteq \mathcal{E}_{(\mathcal{M})}$ follows.

(ii) Let $(K_j)_{j \in \mathbb{N}_{>0}}$ be a fundamental system of compact subsets of U. For $j \in \mathbb{N}_{>0}$ put

$$k_j := \sup_{f \in B, i \in \mathbb{N}, x \in K_j} \frac{2^{2ji} j^i \|f^{(i)}(x)\|_{L^i(\mathbb{R}^r, \mathbb{R})}}{(M^{1/j})^I_i}.$$

Now introduce $(a_q)_q$ and $(b_q)_q$ as in (i) but such that a_q is the first integer satisfying (35) and additionally $k_{q+1} 2^{-a_q} \leq 1$.

Let $\|\cdot\|_{N,K,h}$ be any fundamental continuous semi-norm in $\mathcal{E}_{(N)}$, then there exists $k \in \mathbb{N}$ with $h^{-1} < 2^k$ and $K \subseteq K_k$. For all $i \in \mathbb{N}$ with $i > a_k$ there exists a unique $j > k$ with $a_{j-1} < i \leq a_j$. By definition this implies $\frac{1}{j}((M^{1/j})^I_p)^{1/p} \leq (N_p)^{1/p}$ for all $p \in \mathbb{N}$ with $p \leq i$ and so $(M^{1/j})^I_p \leq j^p N_p$ for such p. Thus we get for all i sufficiently large

$$\sup_{x \in K} \frac{\|f^{(i)}(x)\|_{L^i(\mathbb{R}^r, \mathbb{R})}}{h^i N_i} \leq \sup_{x \in K_j} \frac{2^{ki} j^i \|f^{(i)}(x)\|_{L^i(\mathbb{R}^r, \mathbb{R})}}{(M^{1/j})^I_i} \leq 2^{-ji} k_j \leq 2^{-a_{j-1}} k_j \leq 1,$$

for all $f \in B$.

We are done since by Proposition 4.4 the matrix $\mathcal{M}$ has $(\mathcal{M}_{(\mathcal{C}^\omega)})$ and so for each $M^{1/j}$ separately we get $\mathcal{E}_{[(M^{1/j})^{lc}]} = \mathcal{E}_{[(M^{1/j})^I]} = \mathcal{E}_{[M^{1/j}]}$.

(iii) By $(\mathcal{M}_{\{\mathcal{C}^\omega\}})$ and [9, Theorem 2.15 (1)] we can assume that each $N^x \in \mathcal{N}$ is log-convex since $\mathcal{E}_{\{N^x\}} = \mathcal{E}_{\{(N^x)^{lc}\}}$ for all x (we can drop all small indices for which possibly $\liminf_{k\to\infty}(n_k^x)^{1/k} = 0$ without changing the space $\mathcal{E}_{\{\mathcal{N}\}}$).

By Proposition 4.4 and (i) there exists P with $P \lhd \mathcal{M}$, $\mathcal{E}_{[P]} = \mathcal{E}_{[P^{lc}]}$ and P^{lc} has (nq). Consequently $\mathcal{E}_{[P^{lc}]}$ is non-quasianalytic and $P^{lc} \lhd \mathcal{M}$ holds, too.

On the other hand by [4, Lemma 3.5.7] there exists Q with $\mathcal{N} \lhd Q \lhd \mathcal{M}$.

Now put $Q'_k := \max\{P_k^{lc}, Q_k\}$. Since $Q' \geq P^{lc}$ we have that $\mathcal{E}_{[Q']}$ is non-quasianalytic, $\mathcal{E}_{[Q']} = \mathcal{E}_{[Q'^{lc}]}$ and Q'^{lc} satisfies (nq).

On the other hand $Q' \geq Q$ implies $\mathcal{N} \lhd Q'$. Since $\mathcal{E}_{[Q']} = \mathcal{E}_{[Q'^{lc}]}$ and each $N^x \in \mathcal{N}$ has (lc), also $\mathcal{N} \lhd Q'^{lc}$ follows.

Finally $Q'^{lc} \lhd \mathcal{M}$ holds because $Q'^{lc} \leq Q'$ and $P^{lc}, Q \lhd \mathcal{M}$.

The conclusion follows now by defining $L := Q'^{lc}$. $\boxed{QED}$

If $\mathcal{M} = \Omega$ is coming from $\omega \in \mathcal{W}$, then we obtain the following consequence:

Corollary 4.8. Let $\omega \in \mathcal{W}$ be given, TFAE:

(i) ω has (ω_{nq}),

(ii) $\mathcal{E}_{\{\omega\}}$ contains functions with compact support,

(iii) $\mathcal{E}_{(\omega)}$ contains functions with compact support,

(iv) some Ω^l has (nq),

(v) each Ω^l has (nq).

PROOF. By [9, 5.5] the matrix Ω is $(\mathcal{M}_{sc})$. By [9, 5.5, Corollary 5.8 (1)] we have $(i) \Leftrightarrow (iv) \Leftrightarrow (v)$. The rest follows from Theorem 4.1. $\boxed{QED}$

5 Characterization of $\mathcal{E}_{[\mathcal{M}]}$ using the Fourier transform

Using the central results from Sections 3 and 4 we are now able to characterize functions in $\mathcal{E}_{[\mathcal{M}]}$ in terms of the decay of its Fourier transform. First put

$$\mathcal{D}(\mathbb{R}^r) := \{f \in \mathcal{E}(\mathbb{R}^r) : \exists K \subset\subset \mathbb{R}^r, \, \text{supp}(f) \subseteq K\}.$$

Let $\mathcal{M} = \{M^x : x \in \Lambda\}$ satisfy $(\mathcal{M})$. If $\mathcal{E}_{(\mathcal{M})}$ respectively $\mathcal{E}_{\{\mathcal{M}\}}$ is non-quasianalytic, then

$$\mathcal{D}_{(\mathcal{M})}(U) :=$$
$$\{f \in \mathcal{E}(\mathbb{R}^r) : \exists K \subset\subset U \, \text{supp}(f) \subseteq K, \, \forall x \in \Lambda \, \forall h > 0 \, : \|f\|_{M^x, \mathbb{R}, h} < +\infty\}$$

respectively

$$\mathcal{D}_{\{\mathcal{M}\}}(K) :=$$
$$\{f \in \mathcal{E}(\mathbb{R}^r) : \exists\, K \subset\subset U \text{ supp}(f) \subseteq K, \exists\, x \in \Lambda \,\exists\, h > 0 \,:\, \|f\|_{M^x,\mathbb{R},h} < +\infty\}$$

is non-trivial.

On the other hand let $\mathcal{M} = \{M^x : x \in \Lambda\}$ be $(\mathcal{M}_{\mathrm{sc}})$ and let $K \subset\subset \mathbb{R}^r$ be compact. Then for $x \in \Lambda$ and $h > 0$ introduce the Banach space

$$\hat{\mathcal{D}}_{x,h}(K) := \{f \in \mathcal{E}(\mathbb{R}^r) : \text{ supp}(f) \subseteq K, \|f\|^{\hat{}}_{x,h} < +\infty\},$$

where $\|f\|^{\hat{}}_{x,h} := \int_{\mathbb{R}^r} |\hat{f}(t)| \exp(h\omega_{M^x}(t))dt$. So one can define

$$\hat{\mathcal{D}}_{(\omega_{\mathcal{M}})}(K) := \varprojlim_{x \in \Lambda, h>0} \hat{\mathcal{D}}_{x,h}(K) \qquad \hat{\mathcal{D}}_{\{\omega_{\mathcal{M}}\}}(K) := \varinjlim_{x \in \Lambda, h>0} \hat{\mathcal{D}}_{x,h}(K),$$

and for non-empty open $U \subseteq \mathbb{R}^r$

$$\hat{\mathcal{D}}_{(\omega_{\mathcal{M}})}(U) := \varinjlim_{K \subset\subset U} \hat{\mathcal{D}}_{(\omega_{\mathcal{M}})}(K) \qquad \hat{\mathcal{D}}_{\{\omega_{\mathcal{M}}\}}(U) := \varinjlim_{K \subset\subset U} \hat{\mathcal{D}}_{\{\omega_{\mathcal{M}}\}}(K).$$

Now we formulate our main theorem:

Theorem 5.1. Let $\mathcal{M} := \{M^x : x \in \Lambda\}$ be $(\mathcal{M}_{\mathrm{sc}})$. Moreover assume that

(i) $\mathcal{M}$ has $(\mathcal{M}_{[\mathrm{L}]})$,

(ii) $\mathcal{M}$ has $(\mathcal{M}_{[\mathrm{mg}]})$,

(iii) $\mathcal{E}_{[\mathcal{M}]}$ is non-quasianalytic.

Then we obtain the equalities

$$\mathcal{D}_{[\mathcal{M}]} = \mathcal{D}_{[\omega_{\mathcal{M}}]} = \hat{\mathcal{D}}_{[\omega_{\mathcal{M}}]}.$$

Examples. The previous theorem is valid if $\mathcal{M} = \Omega$ for some $\omega \in \mathcal{W}$ with (ω_{nq}) or also for the Gevrey-matrix $\mathcal{G}$.

For the proof we have to generalize [2, Lemma 3.3]. Let $K \subset\subset \mathbb{R}^r$ and let $H_K(t) := \sup_{s \in K} \langle t, s \rangle$ be the support function. $\lambda_r(K)$ shall denote the Lebesgue measure of K.

Lemma 5.2. Let $\mathcal{M} = \{M^x : x \in \Lambda\}$ be $(\mathcal{M}_{\mathrm{sc}})$ and $f \in \mathcal{D}(\mathbb{R}^r)$.

(i) Let $x \in \Lambda$ and $h > 0$ be arbitrary and assume that $\|f\|^{\hat{}}_{x,h} =: C < +\infty$. Then

$$\sup_{\alpha \in \mathbb{N}^r, t \in \mathbb{R}^r} |f^{(\alpha)}(t)| \exp\left(-h\varphi^*_{\omega_{M^x}}\left(\frac{|\alpha|}{h}\right)\right) \leq \frac{C}{(2\pi)^r}. \qquad (36)$$

holds.

(ii) Let $\mathcal{M}$ satisfy additionally $(\mathcal{M}_{[L]})$.

In the Roumieu case assume that there exist some $x \in \Lambda$ and $C, h > 0$ such that (36) is valid. Then there exists $D \geq 1$ depending on x, h and the dimension r and there exist $y \in \Lambda$ and $L \geq 1$ depending only on x and r such that with $K := \mathrm{supp}(f)$ we have for all $z \in \mathbb{C}^r$

$$|\hat{f}(z)| \leq \lambda_r(K) \frac{CD}{(2\pi)^r} \exp\left(H_K(\mathrm{Im}(z)) - \frac{h}{L} \omega_{M^y}(z) \right). \qquad (37)$$

In the Beurling case for arbitrary $y \in \Lambda$ and $h > 0$ there exists $D \geq 1$ depending on x, h and the dimension r and there exist $x \in \Lambda$ and $L \geq 1$ depending only on y and r such that (37) holds (with y, D, L) provided (36) is valid (with x, h, C).

For (ii) it is sufficient to assume (27) in the Roumieu and (28) in the Beurling case, see Proposition 3.12.

PROOF. (i) Since each $\omega_{M^x} \in \mathcal{W}_0$ we can replace in the proof of [2, Lemma 3.3 (1)] the weight ω by ω_{M^x}.

(ii) We consider the Roumieu case. Iterating (27) yields $\omega_{M^y}(rt) \leq \frac{L}{2}\omega_{M^x}(t) + \frac{L}{2}$ for all $t \geq 0$ and for some $y \in \Lambda$ and $L \geq 1$ both depending only on x and r. By (ω_3) for ω_{M^y} there exists some $B \geq 1$ such that $(2h/L)\omega_{M^y}(t) - \log(t) \geq (h/L)\omega_{M^y}(t) - B$ for all $t \geq 1$.

Then follow [2, Lemma 3.3 (2)]. $\boxed{QED}$

Lemma 5.2 and the Paley-Wiener theorem for $\mathcal{D}(K)$ (see [5, 7.3.1]) imply

Proposition 5.3. Let $\mathcal{M} = \{M^x : x \in \Lambda\}$ be $(\mathcal{M}_{sc})$ with $(\mathcal{M}_{[L]})$, let $K \subset\subset \mathbb{R}^r$ be a compact convex set and $f \in L^1(\mathbb{R}^r)$.

(i) The Roumieu case. The following are equivalent:

    (a) $f \in \hat{\mathcal{D}}_{\{\omega_{\mathcal{M}}\}}(K)$,

    (b) $f \in \mathcal{D}(K)$ and there exists $x \in \Lambda$ and $l > 0$ such that $\|f\|_{\omega_{M^x}, K, l} < +\infty$,

    (c) there exist $x \in \Lambda$ and $C, l > 0$ such that for all $z \in \mathbb{C}^r$ we have
$$|\hat{f}(z)| \leq C \exp(H_K(\mathrm{Im}(z)) - l\omega_{M^x}(z)).$$

(ii) The Beurling case. The following are equivalent:

    (a) $f \in \hat{\mathcal{D}}_{(\omega_{\mathcal{M}})}(K)$,

    (b) $f \in \mathcal{D}(K)$ and for all $x \in \Lambda$ and $l > 0$ we have $\|f\|_{\omega_{M^x}, K, l} < +\infty$,

    (c) for all $x \in \Lambda$ and $l > 0$ there exists $C \geq 1$ such that for all $z \in \mathbb{C}^r$ we have
$$|\hat{f}(z)| \leq C \exp(H_K(\mathrm{Im}(z)) - l\omega_{M^x}(z)).$$

Theorem 5.1 follows now by applying Theorem 3.2 and Proposition 5.3.

6   Comparison of the classes $\mathcal{E}_{[M]}$ and $\mathcal{E}_{[\omega]}$

In [1] the authors compared the classical methods which are used to introduce classes of ultradifferentiable functions, either by a weight sequence M or a weight function ω. In [9] we have introduced the technique of associating a weight matrix Ω to a given function ω. The aim of this section is to reformulate the comparison theorems in view of this new method.

Theorem 6.1. Let $\omega \in \mathcal{W}$, TFAE:

(i) There exists $N \in \mathcal{LC}$ with $\mathcal{E}_{[N]} = \mathcal{E}_{[\omega]} = \mathcal{E}_{[\Omega]}$,

(ii) ω has (ω_6),

(iii) there exists $N \in \mathcal{LC}$ such that for each $l > 0$ we have $\mathcal{E}_{[\Omega^l]} = \mathcal{E}_{[N]}$ or equivalently $N \approx \Omega^l$.

Additionally we have:

(a) N and each Ω^l satisfy (mg).

(b) If ω has (ω_2), then $\liminf_{p \to \infty} (n_p)^{1/p} > 0$, $(\mathcal{M}_{\mathcal{H}})$ for Ω and

  * ω_N and each ω_{Ω^l} satisfy (ω_2),
  * N and each Ω^l have (β_3),
  * $\mathcal{E}_{[\omega_N]} = \mathcal{E}_{[N]} = \mathcal{E}_{[\omega]} = \mathcal{E}_{[\Omega^l]}$ for each $l > 0$.

  If ω has (ω_5), then $\lim_{p \to \infty} (n_p)^{1/p} = \infty$, $(\mathcal{M}_{(\mathcal{C}^\omega)})$ for Ω and ω_N and each ω_{Ω^l} satisfy (ω_5).

In the next theorem we start with a weight sequence N and not with a weight function ω as before.

Theorem 6.2. Let $N \in \mathcal{LC}$ with (β_3), TFAE:

(i) There exists $\omega \in \mathcal{W}$ such that $\mathcal{E}_{[\omega]} = \mathcal{E}_{[N]}$,

(ii) N satisfies (mg),

(iii) $\mathcal{E}_{[\omega_N]} = \mathcal{E}_{[N]}$ holds.

Let $\Omega := \{\Omega^l : l > 0\}$ be the matrix associated to ω arising in (i). We get for each $l > 0$:

(a) $\omega, \omega_{\Omega^l}, \omega_N \in \mathcal{W}$ satisfy (ω_6),

(b) $\omega_{\Omega^l} \sim \omega \sim \omega_N$,

(c) $\mathcal{E}_{[\omega_N]} = \mathcal{E}_{[N]} = \mathcal{E}_{[\omega]} = \mathcal{E}_{[\Omega^l]}$,

(d) $N \approx \Omega^l$,

(e) Ω^l has (mg).

(f) If N satisfies $\liminf_{p \to \infty} (n_p)^{1/p} > 0$, then

    (∗) ω, ω_N and each ω_{Ω^l} have (ω_2),

    (∗) each Ω^l has (β_3),

    (∗) $(\mathcal{M}_{\mathcal{H}})$ for Ω.

If N satisfies $\lim_{p \to \infty} (n_p)^{1/p} = +\infty$, then

    (∗) ω, ω_N and each ω_{Ω^l} have (ω_5),

    (∗) $(\mathcal{M}_{(\mathcal{C}^\omega)})$ for Ω.

Theorem 6.1 and Theorem 6.2 follow by the results below, [9, Section 5] and [1], see also [12, 6.1-6.4].

Theorem 6.3. Let $\omega \in \mathcal{W}$, $U \subseteq \mathbb{R}^r$ non-empty open. Then we get:

(1) ω has (ω_6) if and only if $\mathcal{E}_{[\Omega^l]}(U) = \mathcal{E}_{[\omega]}(U)$ holds for each $l > 0$. Moreover for each $l > 0$

    (a) $\omega \sim \omega_{\Omega^l}$,

    (b) $\Omega^l \in \mathcal{LC}$,

    (c) $\omega_{\Omega^l} \in \mathcal{W}$ with (ω_6),

    (d) $\Omega^x \approx \Omega^y$ holds for all $x, y > 0$,

    (e) Ω^l satisfies (mg).

(2) Let ω be as in (1) with (ω_2), then

    (a) Ω has $(\mathcal{M}_{\mathcal{H}})$,

    (b) each Ω^l satisfies (β_3),

    (c) each ω_{Ω^l} has (ω_2).

If ω is as in (1) with (ω_5), then

    (d) Ω has $(\mathcal{M}_{(\mathcal{C}^\omega)})$,

    (e) each ω_{Ω^l} has (ω_5).

PROOF. (1) This was already shown in [9, Section 5].

(2) To prove (β_3) for each Ω^l we proceed similarly as in [1, Lemma 12 $(1) \Rightarrow (2)$] (each sequence Ω^l satisfies the required assumptions).

If ω has (ω_2) or (ω_5), then by [9, Lemma 5.7] each ω_{Ω^l} too and by [9, Proposition 4.6 (1), Corollary 5.15] we get $(\mathcal{M}_{\mathcal{H}})$ or $(\mathcal{M}_{(\mathcal{C}^\omega)})$ for Ω. $\boxed{QED}$

In the next result we start with a weight sequence M and not with ω.

Theorem 6.4. Let $M \in \mathcal{LC}$ with (β_3) and (mg). Let $r \in \mathbb{N}_{>0}$ and $U \subseteq \mathbb{R}^r$ be non-empty open. Then

(1) $\omega_M \in \mathcal{W}$ has (ω_6).

(2) $\mathcal{E}_{[\omega_M]}(U) = \mathcal{E}_{[N^l]}(U) = \mathcal{E}_{[M]}(U)$ for each $l > 0$, where $N_p^l := \exp(\frac{1}{l}\varphi_{\omega_M}^*(lp))$. Moreover $N^1 = M$ and for each $l > 0$

   (a) $N^l \in \mathcal{LC}$ and has (mg),

   (b) $\omega_{N^l} \sim \omega_M$, $\omega_{N^l} \in \mathcal{W}$ with (ω_6),

   (c) $M \approx N^l$.

(3) If M satisfies $\liminf_{p\to\infty}(m_p)^{1/p} > 0$, then

   (a) (ω_2) for ω_M and each ω_{N^l},

   (b) each N^l has (β_3) and $\liminf_{p\to\infty}(n_p^l)^{1/p} > 0$.

   If M satisfies $\lim_{p\to\infty}(m_p)^{1/p} = \infty$, then

   (c) (ω_5) for ω_M and each ω_{N^l},

   (d) each N^l has $\lim_{p\to\infty}(n_p^l)^{1/p} = +\infty$.

PROOF. (1) By 2.8 we get $\omega_M \in \mathcal{W}_0$, by [1, Lemma 12 (2) $\Rightarrow$ (4)] we get (ω_1) and by [6, Proposition 3.6] we get (ω_6) for ω_M.

(2) In Theorem 6.3 consider $\omega = \omega_M$ and then $\mathcal{E}_{[\omega_M]}(U) = \mathcal{E}_{[N^l]}(U)$ for each $l > 0$. By [9, 5.5] we have $N^l \in \mathcal{LC}$ and so

$$M_p = \sup_{t\geq 0} \frac{t^p}{\exp(\omega_M(t))} = \exp\left(\sup_{t\geq 0}(p\log(t) - \omega_M(t))\right) = \exp\left(\varphi_{\omega_M}^*(p)\right) =: N_p^1,$$

for all $p \in \mathbb{N}$. Thus $\mathcal{E}_{[M]} = \mathcal{E}_{[N^1]} = \mathcal{E}_{[\omega_M]} = \mathcal{E}_{[N^l]}$ which implies $M \approx N^l$ and (mg) follows for each N^l.

By 2.8 we have $\omega_{N^l} \in \mathcal{W}_0$, hence [9, Lemma 5.7] applied to ω_M implies $\omega_{N^l} \sim \omega_M$ for each $l > 0$ and so (ω_1) and (ω_6) for each ω_{N^l} follow.

(3) By 2.8 the assumption $\liminf_{p\to\infty}(m_p)^{1/p} > 0$ implies (ω_2) for ω_M and each ω_{N^l}. Again by [9, Proposition 4.6 (1), Corollary 5.15] we get

$$\liminf_{p\to\infty}(n_p^l)^{1/p} > 0 \quad \text{for each} \quad l > 0$$

and similarly for $\lim_{p\to\infty}(m_p)^{1/p} = +\infty$.

To show (β_3) for each N^l we follow again [1, Lemma 12 (1) $\Rightarrow$ (2)]. $\boxed{QED}$

7   Appendix: nuclearity of the connecting mappings for $\mathcal{E}_{[\mathcal{M}]}$

First we recall [6, Lemma 2.3]:

Lemma 7.1. The identity mapping

$$\mathcal{C}^{r+1}(K,\mathbb{R}) \longrightarrow \mathcal{C}(K,\mathbb{R})$$

is nuclear for each compact set $K \subset\subset \mathbb{R}^r$ with smooth boundary.

Let $\mathcal{M} := \{M^x : x \in \Lambda\}$ be $(\mathcal{M})$. For $x \le y$, $h \le k$ and a compact set $K \subset\subset \mathbb{R}^r$ with smooth boundary consider the inclusion

$$\mathcal{E}_{M^x,h}(K,\mathbb{R}) \longrightarrow \mathcal{E}_{M^y,k}(K,\mathbb{R}), \tag{38}$$

and we are going to prove the matrix generalization of [6, Proposition 2.4]:

Proposition 7.2. Let $\mathcal{M}$ satisfy $(\mathcal{M})$.

(a) If $(\mathcal{M}_{\{dc\}})$, then $\forall\, x \in \Lambda\; \forall\, h > 0\; \exists\, y \in \Lambda\; \exists\, k > 0 :$ (38) is nuclear.

(b) If $(\mathcal{M}_{(dc)})$, then $\forall\, y \in \Lambda\; \forall\, k > 0\; \exists\, x \in \Lambda\; \exists\, h > 0 :$ (38) is nuclear.

PROOF. As already pointed out in [6, Proposition 2.4], since each inclusion mapping is a product of two inclusion mappings of the same type, it is enough to show quasi-nuclearity, see [8, Theorem 3.3.2]. For convenience put $X := \mathcal{E}_{M^x,h}(K,\mathbb{R})$ and $Y := \mathcal{E}_{M^y,k}(K,\mathbb{R})$. So we have to show that there exists $(u_j)_j$, $u_j \in X'$, such that $\sum_{j=1}^{\infty} \|u_j\|_{X'} < +\infty$ and

$$\|f\|_Y \le \sum_{j=1}^{\infty} |\langle f, u_j\rangle_X| \qquad \forall\, f \in X.$$

Now we point out that

$$\|f\|_Y := \sup_{\alpha\in\mathbb{N}^r, x\in K} \frac{|f^{(\alpha)}(x)|}{k^{|\alpha|}M_{|\alpha|}^y} = \sup_{\alpha\in\mathbb{N}^r} \frac{\|f^{(\alpha)}\|_{\mathcal{C}(K,\mathbb{R})}}{k^{|\alpha|}M_{|\alpha|}^y} \le \sum_{\alpha\in\mathbb{N}^r} \frac{\|f^{(\alpha)}\|_{\mathcal{C}(K,\mathbb{R})}}{k^{|\alpha|}M_{|\alpha|}^y}. \tag{39}$$

By Lemma 7.1 there exists $(v_j)_j$, $v_j \in (\mathcal{C}^{r+1}(K,\mathbb{R}))'$ such that

$$\sum_{j=1}^{\infty} \|v_j\|_{(\mathcal{C}^{r+1}(K,\mathbb{R}))'} < +\infty, \qquad \|f^{(\alpha)}\|_{\mathcal{C}(K,\mathbb{R})} \le \sum_{j=1}^{\infty} \left| \langle f^{(\alpha)}, v_j \rangle_{\mathcal{C}^{r+1}(K,\mathbb{R})} \right|. \quad (40)$$

Now let $u_{\alpha,j}$ be the linear functional on X defined by

$$\langle f, u_{\alpha,j} \rangle := \frac{\langle f^{(\alpha)}, v_j \rangle_{\mathcal{C}^{r+1}(K,\mathbb{R})}}{k^{|\alpha|} M^y_{|\alpha|}}. \quad (41)$$

By (39) and (40) we get:

$$\|f\|_Y \le \sum_{\alpha \in \mathbb{N}^r, j \in \mathbb{N}} |\langle f, u_{\alpha,j} \rangle|.$$

Moreover, by (41) we have

$$|\langle f, u_{\alpha,j} \rangle| = \frac{|\langle f^{(\alpha)}, v_j \rangle_{\mathcal{C}^{r+1}(K,\mathbb{R})}|}{k^{|\alpha|} M^y_{|\alpha|}} \le \frac{\|f^{(\alpha)}\|_{\mathcal{C}^{r+1}(K,\mathbb{R})} \|v_j\|_{\mathcal{C}^{r+1}(K,\mathbb{R})}}{k^{|\alpha|} M^y_{|\alpha|}}$$

$$\le \sup_{0 \le |q| \le r+1} \frac{\|f^{(\alpha+q)}\|_{\mathcal{C}(K,\mathbb{R})} \|v_j\|_{\mathcal{C}^{r+1}(K,\mathbb{R})}}{k^{|\alpha|} M^y_{|\alpha|}}$$

$$\le \sup_{0 \le |q| \le r+1} \frac{\|f\|_X h^{|\alpha+q|} M^x_{|\alpha+q|} \|v_j\|_{\mathcal{C}^{r+1}(K,\mathbb{R})}}{k^{|\alpha|} M^y_{|\alpha|}}$$

$$\le \frac{h^{|\alpha|}}{k^{|\alpha|}} \sup_{0 \le |q| \le r+1} \frac{h^{|\alpha+q|} M^x_{|\alpha+q|}}{h^{|\alpha|} M^y_{|\alpha|}} \|f\|_X \|v_j\|_{\mathcal{C}^{r+1}(K,\mathbb{R})}.$$

(a) *Roumieu case.* By $(\mathcal{M}_{\{dc\}})$ for given $x \in \Lambda$ we can find $x_1 \in \Lambda$ and $H \ge 1$ such that $M^x_{|\alpha+q|} = M^x_{|\alpha|+|q|} \le H^{|\alpha|} M^{x_1}_{|\alpha|}$ for all $\alpha \in \mathbb{N}^r$ and $q \in \mathbb{N}^r$ with $0 \le |q| \le r+1$. $M^y \ge M^{x_1}$ holds for $y \ge x_1$ and so

$$\sup_{0 \le |q| \le r+1} \frac{h^{|\alpha+q|} M^x_{|\alpha+q|}}{h^{|\alpha|} M^y_{|\alpha|}} \le A H^{|\alpha|} (1 + h^{r+1})$$

for some constant $A > 0$. Hence if we choose k such that $k > Hh \Leftrightarrow \frac{Hh}{k} < 1$, then by (40) we get

$$\sum_{\alpha \in \mathbb{N}^r, j \in \mathbb{N}} \|u_{\alpha,j}\|_{X'} \le A \sum_{\alpha \in \mathbb{N}^r, j \in \mathbb{N}} \left(\frac{Hh}{k} \right)^{|\alpha|} \|v_j\|_{(\mathcal{C}^{r+1}(K,\mathbb{R}))')} (1 + h^{r+1}) < +\infty.$$

(b) *Beurling case.* By $(\mathcal{M}_{(dc)})$ for given $y \in \Lambda$ we can find $y_1 \in \Lambda$ and $H \ge 1$ such that $M^{y_1}_{|\alpha+q|} \le H^{|\alpha|} M^y_{|\alpha|}$ for all $\alpha \in \mathbb{N}^r$ and $q \in \mathbb{R}^r$ with $0 \le |q| \le r+1$.

So for given $y \in \Lambda$ and $k > 0$ (both small) we can take $x \le y_1$, $h < \frac{k}{H}$ and estimate as for the Roumieu case. $\boxed{QED}$

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