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The annihilator ideal graph of a commutative ring

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Abstract. Let R be a commutative ring with nonzero identity and I be a proper ideal of R. The annihilator graph of R with respect to I, which is denoted by $AG_I(R)$, is the undirected graph with vertex-set $V(AG_I(R)) = \{x \in R \setminus I : xy \in I \text{ for some } y \notin I\}$ and two distinct vertices x and y are adjacent if and only if $A_I(xy) \neq A_I(x) \cup A_I(y)$, where $A_I(x) = \{r \in R : rx \in I\}$. In this paper, we study some basic properties of $AG_I(R)$, and we characterise when $AG_I(R)$ is planar, outerplanar or a ring graph. Also, we study the graph $AG_I(\mathbb{Z}_n)$, where $\mathbb{Z}_n$ is the ring of integers modulo n.

Keywords: Zero-divisor graph, Annihilator graph, Girth, Planar graph, Outerplanar, Ring graph

MSC 2000 classification: primary 05C10, 05C99, secondary 13A99

Introduction

Let R be a commutative ring with nonzero identity and let $Z(R)$ be the set of zero-divisors of R. Recently, there has been considerable attention in the literature to associating graphs with algebraic structures (see [1, 9, 10, 11, 12]). Probably the most attention has been to the *zero-divisor graph*, which is denoted by $\Gamma(R)$. The set of vertices of $\Gamma(R)$ is $Z(R)^* = Z(R)\setminus\{0\}$, and two distinct vertices x and y are adjacent if and only if $xy = 0$. The concept of the zero-divisor graph goes back to Beck [5], who let all elements of R be vertices and was mainly interested in colorings. The zero-divisor graph was introduced and studied by D. F.

Anderson and P. S. Livingston in [2]. For a recent survey article on zero-divisor graphs see [3]. In [4], A. Badawi introduced the *annihilator graph* $AG(R)$ for a commutative ring R. Let $a \in R$ and $\mathrm{ann}_R(a) = \{r \in R : ra = 0\}$. The annihilator graph of R is the (undirected) graph with vertices $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $\mathrm{ann}_R(xy) \neq \mathrm{ann}_R(x) \cup \mathrm{ann}_R(y)$. Clearly, each edge of $\Gamma(R)$ is an edge of $AG(R)$, and so $\Gamma(R)$ is a subgraph of $AG(R)$.

In [13], S. P. Redmond introduced the zero-divisor graph of a commutative ring R with respect to an ideal I of R. Let I be a proper ideal of R. The *zero-divisor graph of R with respect to I*, denoted by $\Gamma_I(R)$, is the graph whose vertices are the set

$$\{x \in R \setminus I : xy \in I \text{ for some } y \notin I\},$$

and two distinct vertices x and y are adjacent if and only if $xy \in I$. In [13], among other results the relationship between the graphs $\Gamma_I(R)$ and $\Gamma(R/I)$ was explored.

In this article, we introduce the *annihilator graph of R with respect to a proper ideal I of R*, which is denoted by $AG_I(R)$. Let $x \in R$ and $A_I(x) = \{r \in R : rx \in I\}$. The annihilator graph of R with respect to I is the (undirected) graph $AG_I(R)$ with vertices $V(AG_I(R)) = \{x \in R \setminus I : xy \in I \text{ for some } y \notin I\}$, and two distinct vertices x and y are adjacent if and only if $A_I(xy) \neq A_I(x) \cup A_I(y)$. In other words, two distinct vertices x and y are adjacent if and only if there exists $r \in R \setminus I$ such that $rxy \in I$ and rx, $ry \notin I$. Note that $\Gamma_I(R)$ is a subgraph of $AG_I(R)$. Also if $I = \{0\}$, then we have $AG_I(R) = AG(R)$. We call the graph $AG_I(R)$ the annihilator ideal graph.

In Section 2 of this paper, we study some basic properties of $AG_I(R)$. For instance, we show that if $AG_I(R)$ is not identical to $\Gamma_I(R)$, then the girth of the graph $AG_I(R)$ is at most 4 (see Theorem 2.7). In the third section, we study the planarity of $AG_I(R)$. In Section 4, we obtain some results on the annihilator ideal graph of $\mathbb{Z}_n$, where $\mathbb{Z}_n$ is the ring of integers modulo n for a positive integer n. We also study situations under which the graphs $AG_I(\mathbb{Z}_n)$ and $\Gamma_I(\mathbb{Z}_n)$ are isomorphic.

Now, we give a brief necessary background of graph theory. Let X be an undirected graph. We use the notation $V(X)$ for the set of vertices of X. Also, for a vertex $x \in V(X)$, $N(x)$ denotes the set of vertices adjacent to x, and $|N(X)|$ is called the *degree* of x. We say that the graph X is connected if there is a path between each pair of distinct vertices of X. For two distinct vertices x and y, we define $\mathrm{d}(x, y)$ to be the length of a shortest path between x and y ($d(x, y) = \infty$ if there is no such path). The *diameter* of X is $\mathrm{diam}(X) = \sup\{\mathrm{d}(x, y) : x \text{ and } y \text{ are distinct vertices of } X\}$. The *girth* of X is the length of a shortest cycle in X, denoted by $\mathrm{gr}(X)$ ($\mathrm{gr}(X) = \infty$ if X has no cycles). A *clique* of a graph is any complete subgraph of the graph and the number of vertices in a largest clique of X, denoted by $\omega(X)$, is called the clique number of X. A *planar graph* is a graph that can be embedded in the plane, that is, it can be drawn in the plane in such a way that its edges intersect only at their endpoints. Kuratowski provided a characterization of planar graphs, which is now known as Kuratowski's Theorem: A finite graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ [6, Theorem 9.10]. Suppose that X is a graph with n vertices and q edges. Also, assume that C is a cycle of X. A chord in X is any edge of X joining two non-adjacent vertices in C. A primitive cycle is a cycle without chords. Moreover, we say that a graph X has the primitive cycle property (PCP) if any two primitive cycles intersect in at most one edge. The free rank of X, denoted by $\mathrm{frank}(X)$, is the number of primitive cycles of X. Also, the number $\mathrm{rank}(X) = q - n + r$, where r is the number of connected components of X, is called the *cycle rank* of X. The cycle rank of X can be expressed as the dimension of the cycle space of X. These two numbers satisfy the inequality $\mathrm{rank}(X) \leq \mathrm{frank}(X)$, as is seen in [7, Proposition 2.2]. The precise definition of a *ring graph* can be found in Section 2 of [7]; the authors showed that, for the graph X, the following conditions are equivalent:

(i) X is a ring graph,

(ii) $\mathrm{rank}(X) = \mathrm{frank}(X)$,

(iii) X satisfies PCP and X does not contain a subdivision of K_4 as a subgraph.

Clearly ring graphs are planar. An undirected graph is an *outerplanar* graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization of outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of the complete graph K_4 or the complete bipartite graph $K_{2,3}$ [8, Theorem 11.10]. Clearly, every outerplanar graph is planar. Let G and H be graphs. We use the notations $G = H$ and $G \cong H$ to denote identical and isomorphic graphs, respectively.

As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the rings of integers and integers modulo n, respectively.

1 Basic properties of $AG_I(R)$

In this section, we investigate the basic properties of $AG_I(R)$. Also, we study the relations between the graphs $AG_I(R)$ and $AG(R/I)$. The following proposition immediately follows from the definition of the graph $AG_I(R)$.

Proposition 1.1. Let I be a proper ideal of R. Then $AG_I(R) = \emptyset$ if and only if I is a prime ideal.

Proposition 1.2. Let $x + I$ and $y + I$ be distinct elements in R/I. Then $x+I$ is adjacent to $y+I$ in $AG(R/I)$ if and only if x is adjacent to y in $AG_I(R)$.

Proof. Since $A_I(xy) \neq A_I(x) \cup A_I(y)$ if and only if $ann_{R/I}(xy+I) \neq ann_{R/I}(x+I) \cup ann_{R/I}(y + I)$, we have $x + I$ is adjacent to $y + I$ in $AG(R/I)$ if and only if x is adjacent to y in $AG_I(R)$. QED

Theorem 1.3. If $x + i$ is adjacent to $y + i$ in $AG_I(R)$, for some $i \in I$, then all elements of $x + I$ and $y + I$are adjacent in $AG_I(R)$.

Proof. Assume that $x+i$ is adjacent to $y+i$, for some $i \in I$. Hence there exists $r \in R \setminus I$ such that $r(x + i)(y + i) \in I$, $r(x + i) \notin I$, and $r(y + i) \notin I$. Then, for every $j, k \in I$, we have $r(x + j)(y + k) \in I$, $r(x + j) \notin I$, and $r(y + k) \notin I$. So all vertices of $x + I$ and $y + I$ are adjacent in $AG_I(R)$. QED

In the following example, we show that if $AG(R/I) \cong AG(S/J)$, where I and J are ideals of the rings R and S, respectively, then the graphs $AG_I(R)$ and $AG_J(S)$ are not necessarily isomorphic.

Example 1.4. Let $R = \mathbb{Z}_6 \times \mathbb{Z}_3$ and $I = 0 \times \mathbb{Z}_3$. Then $AG(R/I) \cong AG(\mathbb{Z}_6)$ with vertex-set $\{(\overline{2}, 0), (\overline{3}, 0), (\overline{4}, 0)\}$, where $(\overline{3}, 0)$ is adjacent to both vertices $(\overline{2}, 0)$ and $(\overline{4}, 0)$. Let $S = \mathbb{Z}_{24}$ and $J =< 8 >$. Then $AG(S/J) \cong AG(\mathbb{Z}_8)$ with vertex-set $\{\overline{2}, \overline{4}, \overline{6}\}$, where the vertex $\overline{4}$ is adjacent to the vertices $\overline{2}$ and $\overline{6}$. Now the graph $AG_I(R)$ is pictured in Figure 1 while $AG_J(S)$ is a complete graph. Hence $AG(R/I) \cong AG(S/J)$, but the graphs $AG_I(R)$ and $AG_J(S)$ are not isomorphic.

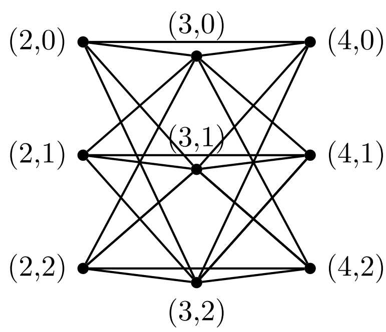

Figure 1

Let $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq R$ be a set of coset representatives of the vertices of $AG(R/I)$. In [13], S. P. Redmond defined the concept of a column and a connected column. Here, we call the subset $a_\lambda + I = \{a_\lambda + i : i \in I\}$ a column of $AG_I(R)$. Moreover, if there exists $r \in R \setminus I$ such that $ra_\lambda \notin I$ and $ra_\lambda^2 \in I$, then we say that $a_\lambda + I$ is a connected column of $AG_I(R)$.

Now the following two theorems follow immediately from the previous paragraph.

Theorem 1.5. If $a + I$ is a connected column of $AG_I(R)$, then $a + I$ is a complete subgraph of $AG_I(R)$ and $\omega(AG_I(R)) \geq |I|$.

Theorem 1.6. If $AG_I(R)$ has a connected column and $|V(AG(R/I))| \geq 2$, then $\omega(AG_I(R)) \geq |I| + 1$.

In the next theorem, we study the girth of the graph $AG_I(R)$.

Theorem 1.7. Let $AG_I(R) \neq \Gamma_I(R)$. Then $\mathrm{gr}(AG_I(R)) \leq 4$.

Proof. Since $AG_I(R) \neq \Gamma_I(R)$, there exist adjacent vertices x and y such that $xy \notin I$. Thus there exists $1 \neq r \in R \setminus I$ such that $rxy \in I$, $rx \notin I$ and $ry \notin I$. Now, if $r \notin \{x, y\}$, then it is easy to see that r is a vertex of $AG_I(R)$ which is adjacent to both vertices x and y, and so we have the cycle $x - r - y - x$. Without loss of generality, we may assume that $r = x$. Then clearly $x \neq x^2$, $y \neq xy$ and we have the cycle $x - y - x^2 - xy - x$. So the result holds. QED

2 Planar, outerplanar and ring graph annihilator ideal graphs

We begin this section with the following theorem.

Theorem 2.1. Let $|V(AG(R/I))| = 1$. Then $AG_I(R)$ is planar if and only if $|I| \leqslant 4$.

Proof. Suppose that $V(AG(R/I)) = \{x + I\}$. Clearly $x^2 \in I$ and $x + I$ is a connected column of $AG_I(R)$. Hence, by Theorem 1.5, $AG_I(R)$ is isomorphic to the complete graph $K_{|I|}$. Therefore $AG_I(R)$ is planar if and only if $|I| \leq 4$. QED

Lemma 2.2. Let $|V(AG(R/I))| \geqslant 2$ and $|I| \geq 3$. Then $AG_I(R)$ is not planar.

Proof. Suppose that $a+I$ is a vertex of $AG(R/I)$. Note that $AG(R)$ is connected since $\Gamma(R)$ is connected [2, Theorem 2.3], and $\Gamma(R)$ is a subgraph of $AG(R)$ with $V(\Gamma(R)) = V(AG(R))$. Now since $|V(AG(R/I))| \geqslant 2$ and $AG(R/I)$ is connected, there exists $b \in R \setminus I$ such that the vertices a and b are adjacent in $AG_I(R)$. Hence $AG_I(R)$ has a subgraph isomorphic to $K_{3,3}$ with vertex-set $\{a, a+x_1, a+x_2\} \cup \{b, b+x_1, b+x_2\}$, where x_1 and x_2 are distinct elements in $I \setminus \{0\}$. So $AG_I(R)$ contains a subgraph which is isomorphic to $K_{3,3}$. Thus $AG_I(R)$ is not planar. $\boxed{QED}$

Theorem 2.3. Let $|V(AG(R/I))| = 3$ and $|I| = 2$. Then $AG_I(R)$ is not planar if and only if $AG(R/I)$ is complete such that at least one of its vertices is a connected column in $AG_I(R)$.

Proof. First assume that $AG(R/I)$ is a complete graph with vertices $a+I, b+I$ and $c+I$. Without loss of generality, we may assume that $a+I$ is a connected column. Then $AG_I(R)$ has a subgraph isomorphism to $K_{3,3}$ with vertex-set $\{b, b+x, a\} \cup \{c, c+x, a+x\}$, where $x \in I \setminus \{0\}$. Hence $AG_I(R)$ contains a copy of $K_{3,3}$, which means that it is not planar.

For the converse statement, assume to the contrary that if $AG(R/I)$ is complete, then none of its vertices is a connected column, or $AG(R/I)$ is not complete. If $AG(R/I)$ is complete and no vertex is a connected column, then $AG_I(R)$ is pictured in Figure 2, which is planar, and this is a contradiction.

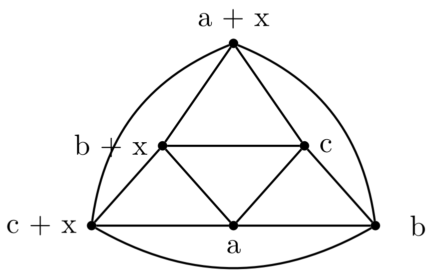

Figure 2

Now assume that $AG(R/I)$ is not a triangle, say that $a+I$ and $c+I$ are not adjacent. Then $AG_I(R)$ is pictured in Figure 3, which is planar, and this is a contradiction.

Hence the result holds. $\boxed{QED}$

Theorem 2.4. Let $|V(AG(R/I))| \geqslant 4$ and $|I| = 2$. Then $AG_I(R)$ is not planar if and only if there exists a subdivision of the complete graph K_3 in $AG(R/I)$.

Proof. First assume that there exists a subdivision of the complete graph K_3 in $AG(R/I)$. Suppose that $a+I$, $b+I$, $c+I$ and $d+I$ are distinct vertices in $AG(R/I)$. Now if these vertices form a square in $AG(R/I)$, then $AG_I(R)$ has a

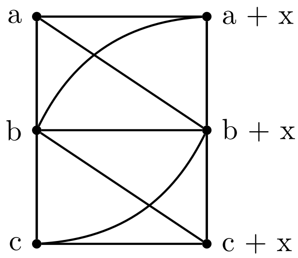

Figure 3

subgraph isomorphic to $K_{3,3}$ with vertex-set $\{a, a+x, c\} \cup \{b, b+x, d\}$, where $x \in I \setminus \{0\}$. Hence $AG_I(R)$ contains a subgraph isomorphic to $K_{3,3}$. So $AG_I(R)$ is not planar. If a, b and c form a triangle in $AG_I(R/I)$, then, in view of Figure 4, the graph $AG_I(R)$ contains a subdivision of $K_{3,3}$. So $AG_I(R)$ is not planar.

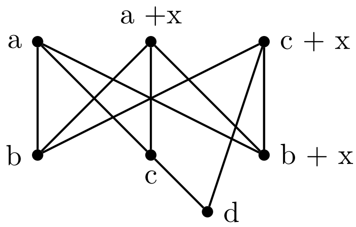

Figure 4

The converse statement is clear. QED

Theorem 2.5. Let $|V(AG(R/I))| = 1$. Then $|I| \leq 3$ if and only if $AG_I(R)$ is a ring graph and an outerplanar graph.

Proof. Since $|V(A(R/I))| = 1$, then, in view of the proof of Theorem 2.1, $AG_I(R)$ is isomorphic to $K_{|I|}$. If $|I| \geqslant 4$, then $AG_I(R)$ contains a subgraph which is isomorphic to K_4. So $AG_I(R)$ is neither a ring graph nor an outerplanar graph.

The converse statement is clear. QED

Since the graph $AG_I(R)$ is isomorphic to $AG(R)$ whenever $|I| = 1$, in the following theorem, we verify the case that $|I| \geqslant 2$ and $|V(A(R/I))| \geq 2$.

Theorem 2.6. Assume that $|I|, |V(AG(R/I))| \geq 2$, and at least one of the sets $V(AG(R/I))$ or I has three elements. Then $AG_I(R)$ is neither outerplanar nor a ring graph.

Proof. First, suppose that $|I| \geq 3$. Then the set of vertices $\{a, a+x_1, a+x_2\} \cup \{b, b+x_1\}$, where x_1 and x_2 are distinct nonzero elements of I and a, b are distinct elements of $R \setminus I$, forms the graph $K_{2,3}$. So $AG_I(R)$ contains a subgraph which is isomorphic to $K_{2,3}$. Assume that $|V(AG(R/I))| \geq 3$ and that $a+I, b+I, c+I$ are distinct vertices of $AG(R/I)$. Since $AG(R/I)$ is connected, we may assume that $a+I$ and $c+I$ are adjacent to $b+I$. Thus the vertices a and c are adjacent to b in $AG_I(R)$. Therefore the set of vertices $\{a, a+x_1, c\} \cup \{b, b+x_1\}$ forms the graph $K_{2,3}$. Hence $AG_I(R)$ contains a subgraph which is isomorphic to $K_{2,3}$. So $AG_I(R)$ is neither outerplanar nor a ring graph. QED

In the following theorem, we study the outerplanar and ring graph annihilator ideal graphs in the remaining case $|I| = 2$ and $|V(A(R/I))| = 2$.

Theorem 2.7. Let $|I| = |V(AG(R/I))| = 2$. Then $AG_I(R)$ is neither outerplanar nor a ring graph if and only if there exist two connected columns in $AG_I(R)$.

Proof. Note that $AG_I(R)$ is connected since $\Gamma_I(R)$ is connected [13, Theorem 2.4], and $\Gamma_I(R)$ is a subgraph of $AG_I(R)$ with $V(\Gamma_I(R)) = V(AG_I(R))$. We have $|I| = |V(AG(R/I))| = 2$, and $|V(AG_I(R))| = 4$. So it is easy to see that there exist two connected columns in $AG_I(R)$ if and only if $AG_I(R)$ is isomorphic to K_4. Therefore the result holds. QED

3 Annihilator ideal graph of $\mathbb{Z}_n$

In this section, we assume that n is a positive integer and p, q are distinct prime numbers that divide n. Also, for $x \in \mathbb{Z}_n$, $< x >$ denotes the ideal generated by x in $\mathbb{Z}_n$.

Theorem 3.1. Let $R = \mathbb{Z}_n$ and $I =< pq >$. Then $AG_I(\mathbb{Z}_n) = \Gamma_I(\mathbb{Z}_n)$.

Proof. Let x and y be adjacent vertices of $AG_I(\mathbb{Z}_n)$. Then there exists $r \in R \setminus I$ such that $rxy \in I$, $rx \in R \setminus I$, and $ry \in R \setminus I$. It is enough to show that $xy \in I$. Since $rxy \in I$ and $p \mid pq$, we have $p \mid rxy$, which implies that $p \mid r$, $p \mid x$ or $p \mid y$, and $q \mid r$, $q \mid x$ or $q \mid y$. Let $p \mid r$. Since $pq \nmid rx$ and $pq \nmid ry$, we have $q \mid r$. Thus $pq \mid r^2$, and this implies that $pq \mid r$. So $r \in I$, which is a contradiction. Let $p \mid x$. Then $q \mid y$. So $xy \in I$. The case that $p \mid y$ is obtained in a similar way. QED

Theorem 3.2. Let $R = \mathbb{Z}_n$ and $I =< p^2 >$. Then $AG_I(\mathbb{Z}_n) = \Gamma_I(\mathbb{Z}_n)$.

Proof. The proof is similar to the proof of Theorem 3.1. QED

Theorem 3.3. Let $R = \mathbb{Z}_{p^n}$ and $I =< p^{n-i} >$, for some $1 \leq i \leq n-1$. Then $AG_I(R)$ is a complete graph.

Proof. Let x and y be distinct vertices of $V(AG_I(\mathbb{Z}_{p^n}))$. Then $xx' \in I$ and $yy' \in I$, for some vertices $x', y' \in V(A_I(\mathbb{Z}_{p^n}))$. Hence $p^{n-i} \mid xx'$ and $p^{n-i} \mid yy'$. Set $\alpha = \mathrm{Max}\{\alpha' \in \mathbb{N} : p^{\alpha'} \mid x\}$ and $\beta = \mathrm{Max}\{\beta' \in \mathbb{N} : p^{\beta'} \mid y\}$. Then we have $p^{\alpha+\beta} \mid xy$. Now, if $\alpha + \beta \geqslant n - i$, then $xy \in I$, which implies that x is adjacent to y. Let $\alpha + \beta < n - i$ and put $r = p^{n-i-(\alpha+\beta)}$. This implies that $rxy \in I$, and we have that $rx \in R \setminus I$. Assume that $rx \in I$. Then $p^{n-i} \mid p^{n-i-(\alpha+\beta)}p^{\alpha}a$, where there exists $a \in \mathbb{Z}$ such that $p^{\alpha}a = x$. Thus $p^{\beta} \mid a$, and hence $p^{\alpha+\beta} \mid x$. Therefore we have $\alpha + \beta \leq \alpha$ which implies that $\beta = 0$ and $p^{n-i} \mid y'$. This is a contradiction. QED

Theorem 3.4. Suppose that $R = \mathbb{Z}_n$ and $I = \langle k \rangle$, where $k \in \mathbb{Z}_n$. If there exist vertices $a, b, k \in \mathbb{Z}_n$ such that $\gcd(a, k) = \gcd(b, k)$, then $\deg(a) = \deg(b)$ and $N(a) \setminus \{b\} = N(b) \setminus \{a\}$.

Proof. Let $s \neq b$ be an adjacent vertex to a. Then there exists $r \in R \setminus I$ such that $rsa \in I, rs \notin I$, and $ra \notin I$. Thus $k \mid rsa$, $\;k \nmid rs$, and $k \nmid ra$. In order to show that $s \in N(b)$, it is enough to prove that $k \mid rsb$, and $k \nmid rb$. Set $d = \gcd(a, k) = \gcd(b, k)$. Then we have $\frac{k}{d} \mid rs\frac{a}{d}$, and so $\frac{k}{d} \mid rs$, $k \mid rsd$ and $k \mid rsb$. Now, assume on the contrary that $k \mid rb$. Then $\frac{k}{d} \mid r\frac{b}{d}$, and so $\frac{k}{d} \mid r$, $\frac{k}{d} \mid r\frac{a}{d}$, and $k \mid ra$, which is contradiction. Therefore $s \in N(b) \setminus \{a\}$, and so the results hold. QED

Theorem 3.5. Suppose that $R = \mathbb{Z}_n$ and $I = \langle k \rangle$, where $k \in \mathbb{Z}_n$. If there exists an integer $d > 1$ such that $d^2 \mid k$, then $\mathrm{gr}(AG_I(\mathbb{Z}_n)) = 3$.

Proof. Consider two distinct integers s_1, s_2 such that $\gcd(s_1, k) = \gcd(s_2, k) = 1$. Then $\gcd(s_1 d, k) = \gcd(s_2 d, k) = d$ and, by Theorem 3.4, $N(s_1 d) \setminus \{s_2 d\} = N(s_2 d) \setminus \{s_1 d\}$. Now it is easy to see that $s_1 d$ is adjacent to $s_2 d$ and there exist at least three vertices in $AG_I(\mathbb{Z}_n)$, which completes the proof. QED

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