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Extension properties in intersections of non quasi-analytic classes

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Abstract. Let K be a non empty compact subset of $\mathbb{R}^r$.

In [3], J. Chaumat and A.-M. Chollet introduce certain sequences of moderate growth, consider the corresponding intersections of non quasi-analytic classes of jets on K and of functions on $\mathbb{R}^r$ and give a Whitney type extension theorem in this situation. These results have already been extended by P. Beaugendre in [1]. Here we continue this research, relaxing the conditions on the sequences and obtaining analytic extensions.

Keywords: extension theorem, non quasi-analytic class, Whitney jet

MSC 2000 classification: primary 46E10, secondary 26E10

To the memory of our friend Klaus Floret

1 Introduction

For the definition of the spaces we refer to the paragraphs 2 and 7.

In [3], J. Chaumat and A.-M. Chollet consider a normalized and increasing sequence $\boldsymbol{m} = (m_p)_{p\in\mathbb{N}_0}$ such that $\lim_p M_p^{1/p} = \infty$. For every open subset Ω and compact subset K of $\mathbb{R}^r$, they introduce the spaces $\widehat{\mathfrak{M}}(\Omega)$ and $J\widehat{\mathfrak{M}}(K)$ as the projective limits of the spaces $(p!M_p^a;1)_\Omega$ and $\{p!M_p^a;1\}_K$ for $a>0$ respectively. They prove in Théorème 8 that every jet $\varphi \in J\widehat{\mathfrak{M}}(K)$ comes from an element of $\widehat{\mathfrak{M}}(\mathbb{R}^r)$ if the sequence is strongly non quasi-analytic (i.e. $\sum_{p=1}^{\infty} 1/(pm_p^a) < \infty$ for every $a>0$) and of moderate growth (i.e. there is $A>0$ such that $M_p \le A^p M_j M_{p-j}$ for every $p\in\mathbb{N}$ and $j\in\{0,\dots,p\}$).

In [1], P. Beaugendre generalizes this result as follows. He starts with a convex and increasing real function Φ on $[0,+\infty[$ such that $\lim_{t\to\infty}\Phi(t)/t = \infty$ and considers for every $a>0$ the sequence $\boldsymbol{m}_a^{(\Phi)} = (m_{a,p}^{(\Phi)})_{p\in\mathbb{N}_0}$ defined by

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$m^{(\Phi)}_{a,0} = 1$ and $m^{(\Phi)}_{a,p} = \exp(\Phi(ap) - \Phi(a(p-1)))$ for every $p \in \mathbb{N}$. He then introduces the spaces $\widehat{\Phi}(\Omega)$ and $J\widehat{\Phi}(K)$ and proves in Théorème 4.1 that every jet $\boldsymbol{\varphi} \in J\widehat{\Phi}(K)$ comes from an element of $\widehat{\Phi}(\mathbb{R}^r)$ if Φ is non quasi-analytic (i.e. $\sum_{p=1}^{\infty} 1/(pm^{(\Phi)}_{a,p}) < \infty$ for every $a > 0$).

We consider a matrix $\mathfrak{m} = (m_{j,p})_{j\in\mathbb{N},p\in\mathbb{N}_0}$ such that for every $j \in \mathbb{N}$, the sequence $\boldsymbol{m}_j = (m_{j,p})_{p\in\mathbb{N}_0}$ is normalized, increasing to $+\infty$ and such that $m_{j,p} \geq m_{j+1,p}$ for every $p \in \mathbb{N}_0$. We then introduce the spaces $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\Omega)$ and $\widehat{\mathcal{E}}_{(\mathfrak{M})}(K)$ and investigate their relations with the spaces $\widehat{\mathfrak{M}}(\Omega)$, $\widehat{\Phi}(\Omega)$ and $J\widehat{\mathfrak{M}}(K)$, $J\widehat{\Phi}(K)$. In the Borel case, this leads to the result of Chaumat and Chollet without assuming the moderate growth condition (Theorem 7). The general cases are treated as consequences of the key Theorem 13 which provides conditions under which every jet $\boldsymbol{\varphi} \in \widehat{\mathcal{E}}_{(\mathfrak{M})}(K)$ comes from an element of $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$. Indeed we obtain an extension result (Theorem 14) with the moderate condition of Chaumat and Chollet replaced by "there are $A > 1$ and $s \in \mathbb{N}$ such that $M_{p+1} \leq A^p M_p^s$ for every $p \in \mathbb{N}_0$". We also get the extension result of Beaugendre as a corollary to the Theorem 13 (cf. Theorem 21). Further we investigate the inclusion $\widehat{\mathcal{B}}_{(\mathfrak{N})}(\mathbb{R}^r) \subset \widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$ as well as the equality, obtain that the vector space $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$ never coincides with $\mathcal{B}_{\{\boldsymbol{R}\}}(\mathbb{R}^r)$ nor with $\mathcal{B}_{(\boldsymbol{R})}(\mathbb{R}^r)$ and prove that the class of the $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$ spaces is strictly larger than the one of the $\widehat{\Phi}(\mathbb{R}^r)$ spaces. Finally we obtain ultraholomorphic extensions, in particular analytic extensions on $\mathbb{R}^r \setminus K$.

2 Main notations

Whenever $\boldsymbol{m}$ is a sequence $(m_p)_{p\in\mathbb{N}_0}$ of real numbers, the notation $\boldsymbol{M}$ designates as usual the sequence $(M_p)_{p\in\mathbb{N}_0}$ where $M_p = m_0 \dots m_p$ for every $p \in \mathbb{N}_0$. Such a sequence $\boldsymbol{m}$ is

(a) *normalized* if $m_0 = 1$ and $m_p \geq 1$ for every $p \in \mathbb{N}$;
(b) *non quasi-analytic* if $\sum_{p=0}^{\infty} 1/m_p < \infty$;
(c) *verifying the condition* $(*)$ if $m_p/p \leq m_{p+1}/(p+1)$ for every $p \in \mathbb{N}$; in particular the sequence $\boldsymbol{m}$ then is strictly increasing.

Throughout the paper $\mathfrak{m}$ designates a matrix

$$\mathfrak{m} = (m_{j,p})_{\substack{j\in\mathbb{N}\\ p\in\mathbb{N}_0}}$$

of real numbers submitted to the following two conditions designated later on as *the main requirement*:

(a) for every $j \in \mathbb{N}$, the sequence $\boldsymbol{m}_j = (m_{j,p})_{p\in\mathbb{N}_0}$ is normalized and increasing to $+\infty$;

(b) $m_{j,p} \geq m_{j+1,p}$ for every $j \in \mathbb{N}$ and $p \in \mathbb{N}_0$.
Then of course, $\boldsymbol{M}_j$ designates the sequence $(M_{j,p})_{p\in\mathbb{N}_0}$ for every $j \in \mathbb{N}$ and $\mathfrak{M}$ the matrix $(M_{j,p})_{j\in\mathbb{N},p\in\mathbb{N}_0}$. Of course we say that $\mathfrak{m}$ is *non quasi-analytic* (resp. *verifying the condition* $(*)$) if, for every $j \in \mathbb{N}$, the sequence $\boldsymbol{m}_j$ is non quasi-analytic (resp. verifying the condition $(*)$).

If K is a non empty compact subset of $\mathbb{R}^r$, we consider the following notions. A *jet* $\boldsymbol{\varphi}$ *on* K is a family $(\varphi_\alpha)_{\alpha\in\mathbb{N}_0^r}$ of continuous functions on K. Given a jet $\boldsymbol{\varphi}$ on K, $m \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^r$ such that $|\alpha| \leq m$ and $x, y \in K$, we set as usual

$$(R^m\varphi_\alpha)(x,y) = \varphi_\alpha(y) - \sum_{|\beta|\leq m-|\alpha|} \varphi_{\alpha+\beta}(x)\frac{(y-x)^\beta}{\beta!}.$$

The Fréchet space $\widehat{\mathcal{E}}_{(\mathfrak{M})}(K)$ is then defined as follows. Its elements are the jets $\boldsymbol{\varphi}$ on K such that, for every $h > 0$ and $j \in \mathbb{N}$,

$$|\boldsymbol{\varphi}|_{K,h,j} := \sup_{\alpha\in\mathbb{N}_0^r}\sup_{x\in K}\frac{|\varphi_\alpha(x)|}{h^{|\alpha|}M_{j,|\alpha|}} < \infty$$

and

$$\|\boldsymbol{\varphi}\|_{K,h,j} := \sup_{m\in\mathbb{N}_0}\sup_{\substack{\alpha\in\mathbb{N}_0^r\\|\alpha|\leq m}}\sup_{\substack{x,y\in K\\x\neq y}}\frac{|(R^m\varphi_\alpha)(x,y)|\,(m+1)!}{h^{m+1}\,|\alpha|!M_{j,m+1}\,|y-x|^{m+1-|\alpha|}} < \infty.$$

Its topology is defined by the system of norms

$$\{\, |\cdot|_{K,1/j,j} + \|\cdot\|_{K,1/j,j} \mid j \in \mathbb{N}\,\}.$$

In fact it is the projective limit of the sequence of the Fréchet spaces $\mathcal{E}_{(\boldsymbol{M}_j)}(K)$.

Finally, given a non empty open subset Ω of $\mathbb{R}^r$, *the Fréchet space* $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\Omega)$ is the following space: its elements are the $\mathcal{C}^\infty$-functions f on Ω verifying

$$\|f\|_{\Omega,h,j} := \sup_{\alpha\in\mathbb{N}_0^r}\sup_{x\in\Omega}\frac{|\mathrm{D}^\alpha f(x)|}{h^{|\alpha|}M_{j,|\alpha|}} < \infty, \quad \forall h > 0, j \in \mathbb{N},$$

and it is endowed with the system of norms $\{\, \|\cdot\|_{\Omega,1/j,j} \mid j \in \mathbb{N}\,\}$. It is the projective limit of the Fréchet spaces $\mathcal{B}_{(\boldsymbol{M}_j)}(\Omega)$.

In the particular case $n = 1$ and $K = \{0\}$, i.e. in the Borel case, we replace the notation $\widehat{\mathcal{E}}_{(\mathfrak{M})}(\{0\})$ by $\widehat{\Lambda}_{(\mathfrak{M})}$ and note that the elements of this space simply are the sequences $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}_0}$ of complex numbers such that

$$|\boldsymbol{a}|_{h,j} := \sup_{p\in\mathbb{N}_0}\frac{|a_p|}{h^pM_{j,p}} < \infty, \quad \forall h > 0, j \in \mathbb{N},$$

and that its system of norms is equivalent to $\{\, |\cdot|_{1/j,j} \mid j \in \mathbb{N}\,\}$. Of course $\widehat{\mathcal{D}}_{(\mathfrak{M})}([-1,1])$ is the subspace of $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R})$ whose elements have their support contained in $[-1,1]$, a Fréchet space indeed.

If moreover we have $m_{j,p} = m_{j+1,p}$ for every $j \in \mathbb{N}$ and $p \in \mathbb{N}_0$, we set $m_p := m_{1,p}$ and of course $M_p := M_{1,p}$ for every $p \in \mathbb{N}_0$. Then $\widehat{\Lambda}_{(\mathfrak{M})}$ coincides with $\Lambda_{(\boldsymbol{M})}$, i.e. the vector space of the sequences $\boldsymbol{a} = (a_p)_{p\in\mathbb{N}_0}$ of complex numbers such that

$$|\boldsymbol{a}|_h := \sup_{p\in\mathbb{N}_0} \frac{|a_p|}{h^p M_p} < \infty, \quad \forall h > 0,$$

endowed with the system of norms $\{\, |\cdot|_j \mid j \in \mathbb{N}\,\}$.

3 Surjectivity of the restriction map $R\colon \widehat{\mathcal{D}}_{(\mathfrak{M})}([-1,1]) \to \widehat{\Lambda}_{(\mathfrak{M})}$

It is clear that the restriction map

$$R\colon \widehat{\mathcal{D}}_{(\mathfrak{M})}([-1,1]) \to \widehat{\Lambda}_{(\mathfrak{M})}; \quad f \mapsto (\mathrm{D}^p f(0))_{p\in\mathbb{N}_0}$$

is linear and continuous. Let us establish a surjectivity condition.

1 Lemma. *Let $\mathfrak{m}$ be non quasi-analytic and verify the condition $(*)$. Then for every element $\boldsymbol{u} = (u_p)_{p\in\mathbb{N}_0}$ of $\widehat{\Lambda}_{(\mathfrak{M})}$, there is a normalized and non quasi-analytic sequence $\boldsymbol{m}'$ of real numbers verifying the condition $(*)$ and such that $\boldsymbol{u} \in \Lambda_{(\boldsymbol{M}')} \subset \widehat{\Lambda}_{(\mathfrak{M})}$.*

Moreover for every $j \in \mathbb{N}$, there is a positive constant $C(j)$ such that $m'_p \le C(j) m_{j,p}$ for every $p \in \mathbb{N}_0$.

Proof. For every $j \in \mathbb{N}$, as $\boldsymbol{u}$ belongs to $\Lambda_{(\boldsymbol{M}_j)}$, we may set

$$k_j := \sup_{p\in\mathbb{N}_0} \frac{2^{2pj}\,|u_p|}{M_{j,p}} < \infty.$$

Then an easy recursion leads to two sequences $(h_q)_{q\in\mathbb{N}_0}$ and $(n_q)_{q\in\mathbb{N}_0}$ of integers such that

$$\begin{cases} 0 = n_0 = h_0 < n_1 < h_1 < \cdots < n_q < h_q < \cdots; \\ \displaystyle\sum_{p=n_q}^{\infty} \frac{1}{m_{q+1,p}} \le 2^{-q}, \quad \forall q \in \mathbb{N}; \\ k_{q+1} 2^{-n_q} \le k_1, \quad \forall q \in \mathbb{N}; \\ \dfrac{m_{q+1,p}}{p} \le \dfrac{m_{q,n_q}}{n_q} < \dfrac{m_{q+1,h_q}}{h_q}, \quad \forall p < h_q, q \in \mathbb{N}. \end{cases}$$

[It suffices to set $n_0 = h_0 = 0$ and if $n_0, \ldots, n_{q-1}$ and $h_0, \ldots, h_{q-1}$ verify these requirements, to first choose $n_q \in \mathbb{N}$ verifying the first three requirements and then to choose as h_q the first integer such that $m_{q,n_q}/n_q < m_{q+1,h_q}/h_q$; since $m_{q+1,p} \leq m_{q,p}$ holds for every $p \in \mathbb{N}$ and $\lim_p m_{q+1,p}/p = +\infty$, it is clear that $h_q > n_q$.]

Let us now establish that the sequence $\boldsymbol{m}'$ defined by

$$m'_p := \begin{cases} m_{q,p} \text{ for } p = h_{q-1}, \ldots, n_q \text{ and } q \in \mathbb{N}, \\ \dfrac{p}{n_q} m_{q,n_q} \text{ for } p = n_q + 1, \ldots, h_q - 1 \text{ and } q \in \mathbb{N} \end{cases}$$

suits our purpose.

It clearly is normalized and verifies the condition $(*)$. As $m'_p \geq m_{j,p}$ holds for every $p \leq n_j$ and $j \in \mathbb{N}$, it also is non quasi-analytic since

$$\sum_{p=n_1}^{\infty} \frac{1}{m'_p} = \sum_{q=1}^{\infty} \sum_{p=n_q}^{n_{q+1}-1} \frac{1}{m'_p} \leq \sum_{q=1}^{\infty} \sum_{p=n_q}^{\infty} \frac{1}{m_{q+1,p}} \leq 1.$$

Moreover $\boldsymbol{u}$ belongs to $\Lambda_{(\boldsymbol{M}')}$ since for every $k \in \mathbb{N}$ and integer $p > n_k$, there is a unique integer $j > k$ such that $p \in \{n_{j-1} + 1, \ldots, n_j\}$ hence

$$\frac{2^{kp} |u_p|}{M'_p} \leq 2^{kp} k_j 2^{-2pj} \frac{M_{j,p}}{M'_p} \leq 2^{-pj} k_j \leq 2^{-n_{j-1}} k_j \leq k_1.$$

Finally the inclusion $\Lambda_{(\boldsymbol{M}')} \subset \widehat{\Lambda}_{(\mathfrak{M})}$ is a direct consequence of the fact that $m'_p \leq m_{j,p}$ holds for every $p \geq n_j$ and $j \in \mathbb{N}$ which also implies the existence of the constants $C(j)$. QED

2 Theorem. *Let* $\mathfrak{m}$ *be non quasi-analytic and verify the condition* $(*)$. *If moreover for every* $j \in \mathbb{N}$, *there is* $A(j) > 0$ *such that*

$$m_{j+1,p}^2 \leq pA(j) m_{j,p}, \quad \forall p \in \mathbb{N},$$

then every element of $\widehat{\Lambda}_{(\mathfrak{M})}$ *comes from an element of* $\widehat{\mathcal{D}}_{(\mathfrak{M})}([-1,1])$.

Proof. For every $\boldsymbol{u} \in \widehat{\Lambda}_{(\mathfrak{M})}$, the previous Lemma provides a normalized and non quasi-analytic sequence $\boldsymbol{m}'$ of real numbers verifying the condition $(*)$ and such that $u \in \Lambda_{(\boldsymbol{M}')} \subset \widehat{\Lambda}_{(\mathfrak{M})}$.

Let us now consider the sequence $\boldsymbol{m}'' = (m''_p)_{p \in \mathbb{N}_0}$ defined by $m''_0 = 1$ and $m''_p = (m'_p)^2/p$ for every $p \in \mathbb{N}$. As the sequence $\boldsymbol{m}'$ is normalized and verifies the condition $(*)$, it is an easy matter to check that $\boldsymbol{m}''$ is a normalized and increasing sequence such that $m'_p \leq m''_p$ and

$$\frac{m'_p}{p} \sum_{k=p}^{\infty} \frac{1}{m''_k} = \frac{m'_p}{p} \sum_{k=p}^{\infty} \frac{k}{m'_k} \frac{1}{m'_k} \leq \frac{m'_p}{p} \sum_{k=p}^{\infty} \frac{p}{m'_p} \frac{1}{m'_k} = \sum_{k=p}^{\infty} \frac{1}{m'_k}$$

for every $p \in \mathbb{N}$ hence $\boldsymbol{m}''$ is non quasi-analytic and even verifies

$$\sup_{p\in\mathbb{N}} \frac{m'_p}{p} \sum_{k=p}^{\infty} \frac{1}{m''_k} < \infty.$$

So we may apply the Theorem 4.2 of [6] and obtain the existence of a function $f \in \mathcal{D}_{(\boldsymbol{M}'')}([-1,1])$ such that $\mathrm{D}^p f(0) = u_p$ for every $p \in \mathbb{N}_0$.

Now to conclude, let us remark that the previous Lemma provides for every $j \in \mathbb{N}$ a constant $C(j+1) > 0$ such that $m'_p \leq C(j+1)m_{j+1,p}$ for every $p \in \mathbb{N}_0$. Therefore we get

$$m''_p = \frac{(m'_p)^2}{p} \leq \frac{1}{p}C(j+1)^2 m_{j+1,p}^2 \leq C(j+1)^2 A(j) m_{j,p}, \quad \forall j,p \in \mathbb{N},$$

hence $\mathcal{D}_{(\boldsymbol{M}'')}([-1,1]) \subset \widehat{\mathcal{D}}_{(\mathfrak{M})}([-1,1])$ and we conclude. QED

4 Surjectivity of the restriction map $R\colon \widehat{\mathfrak{M}}(\mathbb{R}) \to J\widehat{\mathfrak{M}}$

Condition (A)**.** Under condition (A), $\boldsymbol{m} = (m_p)_{p\in\mathbb{N}_0}$ is a normalized and increasing sequence of real numbers such that $\lim_p (m_0 \ldots m_p)^{1/p} = +\infty$. Therefore the sequence $\boldsymbol{M}$ is normalized, logarithmically convex and such that $M_p^{1/p} \uparrow +\infty$ and $M_p/M_{p-1} \uparrow +\infty$. Moreover $(a_j)_{j\in\mathbb{N}}$ is a sequence of positive numbers strictly decreasing to 0. Finally for every $j \in \mathbb{N}$, we set $m_{j,0} = 1$ and

$$m_{j,p} = p m_p^{a_j} = \frac{p! M_p^{a_j}}{(p-1)! M_{p-1}^{a_j}}, \quad \forall p \in \mathbb{N}.$$

For every $j \in \mathbb{N}$, it is then clear that the sequence $\boldsymbol{m}_j$ is normalized and increasing to $+\infty$. It is also clear that we have $m_{j,p} \geq m_{j+1,p}$ for every $j \in \mathbb{N}$ and $p \in \mathbb{N}_0$. Therefore the corresponding matrix $\mathfrak{m}$ satisfies the main requirement. More can be said.

3 Proposition. *If the condition (A) holds, then for every $j \in \mathbb{N}$, the sequence $\boldsymbol{m}_j$ verifies the condition $(*)$ and is such that $m_{j,p}/p \to +\infty$; in particular the matrix $\mathfrak{m}$ verifies the condition $(*)$. Moreover $M_{j,p} = p! M_p^{a_j}$ holds for every $j \in \mathbb{N}$ and $p \in \mathbb{N}_0$.*

4 Definition. Let the condition (A) hold. As in [3], we consider the following spaces.

Given a non empty open subset Ω of $\mathbb{R}^r$ and $a > 0$, *the Banach space* $(p!M_p^a; 1)_\Omega$ is the vector space whose elements are the functions $f \in \mathcal{C}^\infty(\Omega)$

verifying

$$\|f\|_{\Omega,a} := \sup_{\alpha\in\mathbb{N}_0^r} \sup_{x\in\Omega} \frac{|\mathrm{D}^\alpha f(x)|}{|\alpha|!M_{|\alpha|}^a} < \infty,$$

endowed with the norm $\|\cdot\|_{\Omega,a}$. *The Fréchet space* $\widehat{\mathfrak{M}}(\Omega)$ is then the projective limit of the Banach spaces $(p!M_p^a;1)_\Omega$ for $a>0$; of course it also is the projective limit of the Banach spaces $(p!M_p^{a_j};1)_\Omega$ for $j\in\mathbb{N}$.

Given a non empty compact subset K of $\mathbb{R}^r$ and $a>0$, *the Banach space* $\{p!M_p^a;1\}_K$ is the vector space whose elements are the Whitney jets $\boldsymbol{\varphi}$ on K such that

$$\|\boldsymbol{\varphi}\|_{K,a} := \sup_{m\in\mathbb{N}_0} \sup_{\substack{\alpha\in\mathbb{N}_0^r\\ |\alpha|\le m}} \max\left\{ \frac{\|\varphi_\alpha\|_K}{|\alpha|!M_{|\alpha|}^a}, \sup_{\substack{x,y\in K\\ x\ne y}} \frac{|(R^m\varphi_\alpha)(x,y)|}{|\alpha|!M_{m+1}^a\,|y-x|^{m+1-|\alpha|}} \right\}$$

is finite, endowed with the norm $\|\cdot\|_{K,a}$. *The Fréchet space* $J\widehat{\mathfrak{M}}(K)$ is then the projective limit of the Banach spaces $\{p!M_p^a;1\}_K$ for $a>0$; of course it also is the projective limit of the Banach spaces $\{p!M_p^{a_j};1\}_K$ for $j\in\mathbb{N}$.

In the Borel case (i.e. $r=1$ and $K=\{0\}$), we write $J\widehat{\mathfrak{M}}$ instead of $J\widehat{\mathfrak{M}}(\{0\})$ and note that the elements of $J\widehat{\mathfrak{M}}$ are the sequences $\boldsymbol{c}=(c_p)_{p\in\mathbb{N}_0}$ of complex numbers such that

$$\|\boldsymbol{c}\|_a := \sup_{p\in\mathbb{N}_0} \frac{|c_p|}{p!M_p^a} < \infty, \quad \forall a>0,$$

with $\|\cdot\|_a = \|\cdot\|_{\{0\},a}$.

5 Proposition. *If the condition* (A) *holds, then the Fréchet spaces* $\widehat{\mathfrak{M}}(\Omega)$ *and* $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\Omega)$ *coincide for every non empty open subset* Ω *of* $\mathbb{R}^r$*.*

PROOF. We first prove that $\widehat{\mathfrak{M}}(\Omega)$ is a vector subspace of $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\Omega)$, endowed with a finer topology. Indeed given any continuous norm $\|\cdot\|_{\Omega,h,j}$ on $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\Omega)$ and $f\in\widehat{\mathfrak{M}}(\Omega)$, we first note that we successively have

$$\|\mathrm{D}^\alpha f\|_\Omega \le \|f\|_{\Omega,a_{j+1}}\,|\alpha|!M_{|\alpha|}^{a_j}M_{|\alpha|}^{a_{j+1}-a_j} = \|f\|_{\Omega,a_{j+1}}\,M_{j,|\alpha|}M_{|\alpha|}^{a_{j+1}-a_j}.$$

Then from

$$\lim_{|\alpha|\to\infty} M_{|\alpha|}^{(a_{j+1}-a_j)/|\alpha|} = 0,$$

we deduce the existence of an integer $p_0\in\mathbb{N}$ such that $M_{|\alpha|}^{a_{j+1}-a_j} \le h^{|\alpha|}$ if $|\alpha|\ge p_0$. This clearly implies that f belongs to $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\Omega)$ as well as the existence of a positive constant $C(j,h)$ (independent from f) such that $\|f\|_{\Omega,h,j} \le C(j,h)\,\|f\|_{\Omega,a_{j+1}}$.

Conversely we first note that $\{\,\|\cdot\|_{\Omega,a_j} \mid j \in \mathbb{N}\,\}$ clearly is a fundamental system of continuous norms on $\widehat{\mathfrak{M}}(\Omega)$. To conclude it suffices then to check that $\|\cdot\|_{\Omega,a_j} \le \|\cdot\|_{\Omega,1/j,j}$ holds on $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\Omega)$ which directly comes from the inequalities

$$\|\mathrm{D}^\alpha g\|_\Omega \le \|g\|_{\Omega,1/j,j}\, j^{-|\alpha|} M_{j,|\alpha|} \le \|g\|_{\Omega,1/j,j}\, |\alpha|! M_{|\alpha|}^{a_j}$$

valid for every $g \in \widehat{\mathcal{B}}_{(\mathfrak{M})}(\Omega)$ and $\alpha \in \mathbb{N}_0^r$. QED

6 Proposition. *If the condition (A) holds, then the Fréchet spaces $J\widehat{\mathfrak{M}}(K)$ and $\widehat{\mathcal{E}}_{(\mathfrak{M})}(K)$ coincide for every non empty compact subset of $\mathbb{R}^r$.*

In particular the Fréchet spaces $J\widehat{\mathfrak{M}}$ and $\widehat{\Lambda}_{(\mathfrak{M})}$ coincide.

Proof. We first prove that $J\widehat{\mathfrak{M}}(K)$ is a vector subspace of $\widehat{\mathcal{E}}_{(\mathfrak{M})}(K)$, endowed with a finer topology. On one hand, for every $h > 0$ and $j \in \mathbb{N}$, acting as in the previous proof, we obtain immediately the existence of a positive constant $A(j,h)$ such that

$$|\boldsymbol{\varphi}|_{K,h,j} \le A(j,h)\, \|\boldsymbol{\varphi}\|_{K,a_{j+1}}, \quad \forall \boldsymbol{\varphi} \in J\widehat{\mathfrak{M}}(K).$$

On the other hand, for every $h > 0$ and $j \in \mathbb{N}$, acting in the same way, we obtain

$$\sup_{\substack{x,y\in K\\ x\neq y}} \frac{|(R^m\varphi_\alpha)(x,y)|}{|y-x|^{m+1-|\alpha|}} \le \|\boldsymbol{\varphi}\|_{K,a_{j+1}}\, |\alpha|! M_{m+1}^{a_j} M_{m+1}^{a_{j+1}-a_j}$$

for every $\alpha \in \mathbb{N}_0^r$ such that $|\alpha| \le m$. As there is a positive integer p_0 such that $M_{m+1}^{a_{j+1}-a_j} \le h^{m+1}$ for every $m \ge p_0$, we obtain immediately the existence of a positive constant $B(j,h)$ such that

$$\|\boldsymbol{\varphi}\|_{K,h,j} \le B(j,h)\, \|\boldsymbol{\varphi}\|_{K,a_{j+1}}, \quad \forall \boldsymbol{\varphi} \in J\widehat{\mathfrak{M}}(K),$$

hence the conclusion.

Conversely we first note that $\{\,\|\cdot\|_{K,a_j} \mid j \in \mathbb{N}\,\}$ clearly is a fundamental system of continuous norms on $J\widehat{\mathfrak{M}}(K)$. To conclude it suffices then to check that we have

$$\|\boldsymbol{\psi}\|_{K,a_j} \le |\boldsymbol{\psi}|_{K,1/j,j} + \|\boldsymbol{\psi}\|_{K,1/j,j}, \quad \forall \boldsymbol{\psi} \in \widehat{\mathcal{E}}_{(\mathfrak{M})}(K),$$

for every $j \in \mathbb{N}$: one has just to proceed as previously starting with

$$\|\psi_\alpha\|_K \text{ and } \sup_{\substack{x,y\in K\\ x\neq y}} \frac{|(R^m\psi_\alpha)(x,y)|}{|y-x|^{m+1-|\alpha|}}$$

separately. QED

7 Theorem. *If the condition* (A) *holds and if* $\mathfrak{m}$ *is non quasi-analytic (i.e.* $\sum_{p=1}^{\infty}(pm_p^a)^{-1} < \infty$ *for every* $a > 0$*), then every element of* $J\widehat{\mathfrak{M}}$ *comes from an element of* $\widehat{\mathfrak{M}}(\mathbb{R})$ *having a compact support.*

PROOF. The use of the sequence $(a_j := 2^{-j})_{j\in\mathbb{N}}$ to define the topologies of the spaces $J\widehat{\mathfrak{M}}$ and $\widehat{\mathfrak{M}}(\mathbb{R})$ leads to

$$m_{j+1,p}^2 = \left(pm_p^{2^{-(j+1)}}\right)^2 = p^2 m_p^{2^{-j}} = pm_{j,p}, \quad \forall p, j \in \mathbb{N}.$$

As $J\widehat{\mathfrak{M}}$ coincides with $\widehat{\Lambda}_{(\mathfrak{M})}$ and $\widehat{\mathfrak{M}}(\mathbb{R})$ with $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R})$ and as $\mathfrak{m}$ is non quasi-analytic and verifies the condition $(*)$, the conclusion follows at once from the Theorem 2. QED

5 Surjectivity of the restriction map $R\colon \widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r) \to \widehat{\mathcal{E}}_{(\mathfrak{M})}(K)$

For the sake of completeness, let us recall the following two results.

8 Result. [2, Lemme 14] *If* $(a_p)_{p\in\mathbb{N}}$ *is a sequence of non negative numbers and* $(b_p)_{p\in\mathbb{N}}$ *a sequence of positive numbers, the following three conditions are equivalent:*
(a) for every $h > 0$*, there is a positive constant* $C(h)$ *such that* $a_p \leq C(h)h^p b_p$ *for every* $p \in \mathbb{N}$*;*
(b) $\lim_{p\to\infty}(a_p/b_p)^{1/p} = 0$*;*
(c) there is a sequence $(\varepsilon_p)_{p\in\mathbb{N}}$ *of positive numbers decreasing to* 0 *and such that* $a_p \leq \varepsilon_1 \ldots \varepsilon_p b_p$ *for every* $p \in \mathbb{N}$*.*

9 Result. [2, Lemme 16], *Let* $(\alpha_p)_{p\in\mathbb{N}}$ *be a sequence of non negative numbers such that* $\sum_{p=1}^{\infty}\alpha_p < \infty$ *and let* $(\beta_p)_{p\in\mathbb{N}}$ *and* $(\gamma_p)_{p\in\mathbb{N}}$ *be sequences of positive numbers such that* $\beta_p \to 0$ *and* $\gamma_p \downarrow 0$*. Then there is a sequence* $(\lambda_p)_{p\in\mathbb{N}}$ *of real numbers such that* $1 < \lambda_1$*,* $\lambda_p \uparrow \infty$*,* $\lambda_p\gamma_p \downarrow$*,* $\lambda_p\beta_p \to 0$ *and*

$$\sum_{k=p}^{\infty}\lambda_k\alpha_k \leq 8\lambda_p\sum_{k=p}^{\infty}\alpha_k, \quad \forall p \in \mathbb{N}.$$

10 Proposition. *Let the sequence* $(b_p)_{p\in\mathbb{N}_0}$ *be normalized, non quasi-analytic and verify the condition* $(*)$*. Let moreover* $(u_p)_{p\in\mathbb{N}_0}$ *be a sequence of positive numbers such that, for every* $h > 0$*, there is a positive constant* $C(h)$ *such that* $u_p \leq C(h)h^p b_0 \ldots b_p$ *for every* $p \in \mathbb{N}_0$*.*

Then there are a normalized and non quasi-analytic sequence $(c_p)_{p\in\mathbb{N}_0}$ *verifying the condition* $(*)$*, a positive number* C *such that* $u_p \leq Cc_0 \ldots c_p$ *for every* $p \in \mathbb{N}_0$ *and a sequence* $(\theta_p)_{p\in\mathbb{N}}$ *of positive numbers, decreasing to* 0 *and such that* $\prod_{k=0}^{p} c_k \leq \theta_1 \ldots \theta_p \prod_{k=0}^{p} b_k$ *for every* $p \in \mathbb{N}$*.*

Proof. A direct application of the Result 8 to the sequences $(u_p)_{p\in\mathbb{N}}$ and $(\prod_{k=0}^p b_k)_{p\in\mathbb{N}}$ instead of $(a_p)_{p\in\mathbb{N}}$ and $(b_p)_{p\in\mathbb{N}}$ respectively, leads to the existence of a sequence $(\varepsilon_p)_{p\in\mathbb{N}}$ of positive numbers decreasing to 0 and such that $u_p \leq \varepsilon_1 \dots \varepsilon_p \prod_{k=0}^p b_k$ for every $p \in \mathbb{N}$.

Then we set $\alpha_p = 1/b_p$, $\beta_p = \varepsilon_p$ and $\gamma_p = p/b_p$ for every $p \in \mathbb{N}$ and apply the Result 9: we get the existence of a sequence $(\lambda_p)_{p\in\mathbb{N}}$ of real numbers such that $0 < \lambda_1 \leq b_1$, $\lambda_p \uparrow \infty$, $\lambda_p\gamma_p \downarrow$, $\lambda_p\beta_p \to 0$ and

$$\sum_{k=p}^{\infty} \lambda_k\alpha_k \leq 8\lambda_p \sum_{k=p}^{\infty} \alpha_k, \quad \forall p \in \mathbb{N}.$$

Now we set $c_0 = 1$, $\theta_p = 1/\lambda_p$ and $c_p = \theta_p b_p$ for every $p \in \mathbb{N}$. From $p/c_p = \lambda_p\gamma_p$ with $c_1 \geq 1$, we deduce that the sequence $(c_p)_{p\in\mathbb{N}_0}$ is normalized and verifies the condition $(*)$. It also is non quasi-analytic since

$$\sum_{p=0}^{\infty} \frac{1}{c_p} = \frac{1}{c_0} + \sum_{p=1}^{\infty} \lambda_p\alpha_p \leq \frac{1}{c_0} + 8\lambda_1 \sum_{p=1}^{\infty} \frac{1}{b_p} < \infty.$$

From

$$u_p \leq \varepsilon_1 \dots \varepsilon_p \prod_{k=0}^{p} b_k = (\varepsilon_1\lambda_1)\dots(\varepsilon_p\lambda_p) \prod_{k=0}^{p} c_k, \quad \forall p \in \mathbb{N},$$

and $\lambda_p\varepsilon_p = \lambda_p\beta_p \to 0$, we get the existence of a positive constant C such that $u_p \leq Cc_0 \dots c_p$ for every $p \in \mathbb{N}_0$. To conclude, we note that $(\theta_p)_{p\in\mathbb{N}}$ is a sequence of positive numbers decreasing to 0 and such that $\prod_{k=0}^p c_k = \theta_1 \dots \theta_p \prod_{k=0}^p b_k$ for every $p \in \mathbb{N}$. QED

11 Proposition. *Let K be a non empty compact subset of $\mathbb{R}^r$ and let $\mathfrak{m}$ be non quasi-analytic and verify the condition $(*)$.*

If, for every $j \in \mathbb{N}$, there is a positive constant $A(j)$ such that

$$M_{j+1,p}M_{j+1,p+1} \leq A(j)^{p+1}M_{j,p}, \quad \forall p \in \mathbb{N}_0,$$

then, for every $f \in \widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$, the jet $(\varphi_\alpha := D^\alpha f|_K)_{\alpha\in\mathbb{N}_0^r}$ belongs to $\widehat{\mathcal{E}}_{(\mathfrak{M})}(K)$.

Proof. Let us consider any $h > 0$ and $j \in \mathbb{N}$.

On one hand it is clear that $|\boldsymbol{\varphi}|_{K,h,j}$ is finite since it certainly is $\leq \|f\|_{\mathbb{R}^r,h,j}$.

On the other hand, if we fix $m \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^r$ such that $|\alpha| \leq m$ as well as two distinct points x and y of K, the limited Taylor formula applied to $g = \Re f$ or $g = \Im f$ provides the existence of some $\theta \in]0, 1[$ such that

$$\mathrm{D}^\alpha g(y) = \sum_{|\beta|\leq m-|\alpha|} \mathrm{D}^{\alpha+\beta} g(x)\frac{(y-x)^\beta}{\beta!} + \sum_{|\beta|=m+1-|\alpha|} \mathrm{D}^{\alpha+\beta} g(x+\theta(y-x))\frac{(y-x)^\beta}{\beta!}$$

hence

$$|(R^m\varphi_\alpha)(x,y)| \le 2\,\|f\|_{\mathbb{R}^r,h/t,j+1}\,\big(\frac{h}{t}\big)^{m+1} M_{j+1,m+1}\frac{r^{m+1-|\alpha|}}{(m+1-|\alpha|)!}\,|y-x|^{m+1-|\alpha|}$$

for every $t>0$ since we have $\sum_{|\beta|=k} 1/\beta! = r^k/k!$ for every $k\in\mathbb{N}$. So if we use the inequality mentioned in the statement and if we note that $\lim_m (m+1)!/M_{j+1,m+2}=0$ implies the existence of a constant $B(j)>0$ such that $(m+1)!/M_{j+1,m+2}\le B(j)$ for every $m\in\mathbb{N}$, we successively get

$$\begin{aligned}\frac{|(R^m\varphi_\alpha)(x,y)|\,(m+1)!}{h^{m+1}\,|\alpha|!M_{j,m+1}\,|y-x|^{m+1-|\alpha|}} &\le \frac{|(R^m\varphi_\alpha)(x,y)|\,A(j)B(j)}{(h/A(j))^{m+1}\,|\alpha|!M_{j+1,m+1}\,|y-x|^{m+1-|\alpha|}}\\ &\le 2A(j)B(j)\,\|f\|_{\mathbb{R}^r,h/(rA(j)),j+1}\,.\end{aligned}$$

Hence the conclusion. *QED*

In the proof of the following key theorem (a kind of converse to the Proposition 11), we will use the following Result which is a direct consequence of the Théorème 3.1 of [1].

12 Result ([1], Théorème 3.1). *Let K be a non empty compact subset of $\mathbb{R}^r$ and let also $\boldsymbol{m}$ be a normalized and increasing sequence such that $\sum_{p=1}^{\infty}(pm_p)^{-1} < \infty$. Then every $\boldsymbol{\varphi}\in\{p!M_p;1\}_K$ comes from a function $f\in\mathcal{C}^\infty(\mathbb{R}^r)$ with compact support for which there are positive constants A and B such that*

$$\|\mathrm{D}^\alpha f\|_{\mathbb{R}^r} \le AB^{|\alpha|}\,|\alpha|!M_{|\alpha|}M_{|\alpha|+1},\quad \forall\alpha\in\mathbb{N}_0^r.$$

13 Theorem. *Let K be a non empty compact subset of $\mathbb{R}^r$ and let $\mathfrak{m}$ be non quasi-analytic and verify the condition $(*)$.*

If, for every $j\in\mathbb{N}$, there is a positive constant $A(j)$ such that

$$M_{j+1,p}M_{j+1,p+1} \le A(j)^{p+1}(p+1)!M_{j,p},\quad \forall p\in\mathbb{N}_0,$$

then every jet $\boldsymbol{\varphi}\in\widehat{\mathcal{E}}_{(\mathfrak{M})}(K)$ comes from an element of $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$ with compact support.

Proof. Let us first obtain an extension f of $\boldsymbol{\varphi}$.

Let us first remark that, for every $j\in\mathbb{N}$, there is a constant $B(j)>0$ such that

$$M_{j+1,p+1} \le B(j)A(j)^{p+1}M_{j,p},\quad \forall p\in\mathbb{N}_0.$$

Indeed as the sequence $\boldsymbol{m}_{j+1}$ is normalized, increasing and non quasi-analytic, the sequence $((p+1)/m_{j+1,p})_{p\in\mathbb{N}_0}$ converges to 0 and this implies the existence of a constant $B(j)>0$ such that $(p+1)!/M_{j+1,p}\le B(j)$ for every $p\in\mathbb{N}_0$, which leads to

$$M_{j+1,p}M_{j+1,p+1} \le A(j)^{p+1}(p+1)!M_{j,p} \le B(j)A(j)^{p+1}M_{j,p}M_{j+1,p}$$

for every $p \in \mathbb{N}_0$.

For every $p \in \mathbb{N}_0$, let us now set

$$v_p := \sup_{\substack{\alpha\in\mathbb{N}_0^r \\ |\alpha|=p}} \|\varphi_\alpha\|_K \text{ and } w_p := \sup_{\substack{\alpha\in\mathbb{N}_0^r \\ |\alpha|\leq p}} \sup_{\substack{x,y\in K \\ x\neq y}} \frac{|(R^p\varphi_\alpha)(x,y)|\,(p+1)!}{|\alpha|!\,|y-x|^{p+1-|\alpha|}}.$$

For every $h > 0$ and $j \in \mathbb{N}$, we then have

$$v_p \leq |\boldsymbol{\varphi}|_{K,h,j}\, h^p M_{j,p}$$

and

$$w_p \leq \|\boldsymbol{\varphi}\|_{K,h,j+1}\, h^{p+1} M_{j+1,p+1} \leq \|\boldsymbol{\varphi}\|_{K,h,j+1}\, h^{p+1} B(j) A(j)^{p+1} M_{j,p}.$$

Therefore the sequence $\boldsymbol{u} := (u_p := v_p + w_p)_{p\in\mathbb{N}_0}$ belongs to $\widehat{\Lambda}_{(\mathfrak{M})}$ and the Lemma 1 provides a normalized and non quasi-analytic sequence $\boldsymbol{m}'$ verifying the condition $(*)$ and such that $\boldsymbol{u} \in \Lambda_{(\boldsymbol{M}')} \subset \widehat{\Lambda}_{(\mathfrak{M})}$.

As $\boldsymbol{u}$ belongs to $\Lambda_{(\boldsymbol{M}')}$, an application of the Proposition 10 with the sequence $(b_p)_{p\in\mathbb{N}_0}$ replaced by $\boldsymbol{m}'$ provides a normalized and non quasi-analytic sequence $(c_p)_{p\in\mathbb{N}_0}$ verifying the condition $(*)$, a positive constant C such that $u_p \leq Cc_0 \ldots c_p$ for every $p \in \mathbb{N}_0$ and a sequence $(\theta_p)_{p\in\mathbb{N}_0}$, decreasing to 0 and such that $\prod_{k=0}^p c_k \leq \theta_1 \ldots \theta_p \prod_{k=0}^p m'_k$ for every $p \in \mathbb{N}$.

Let us set $C_p := c_0 \ldots c_p$ and $P_p := C_p/p!$ for every $p \in \mathbb{N}_0$. We then have

$$\|\varphi_\alpha\|_K \leq v_{|\alpha|} \leq u_{|\alpha|} \leq C\,|\alpha|!P_{|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^r,$$

as well as, for every $p \in \mathbb{N}_0$,

$$\sup_{\substack{\alpha\in\mathbb{N}_0^r \\ |\alpha|\leq p}} \sup_{\substack{x,y\in K \\ x\neq y}} \frac{|(R^p\varphi_\alpha)(x,y)|\,(p+1)!}{|\alpha|!\,|y-x|^{p+1-|\alpha|}} = w_p \leq u_p \leq Cp!P_p \leq C(p+1)!P_{p+1}$$

hence $\boldsymbol{\varphi}$ belongs to $\{p!P_p; 1\}_K$.

The Result 12 then provides a function $f \in \mathcal{C}^\infty(\mathbb{R}^r)$ with compact support and constants $A > 1$ and $B > 0$ such that $\boldsymbol{\varphi}$ comes from f and

$$\|\mathrm{D}^\alpha f\|_{\mathbb{R}^r} \leq AB^{|\alpha|}\,|\alpha|!P_{|\alpha|}P_{|\alpha|+1}, \quad \forall \alpha \in \mathbb{N}_0^r.$$

To conclude, let us prove that f belongs to $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$.

Let us fix any $j \in \mathbb{N}$.

By the Lemma 1, there also is a constant $C(j+1) > 0$ such that

$$M'_p M'_{p+1} \leq C(j+1)^{2p+1} M_{j+1,p} M_{j+1,p+1} \leq C(j+1)^{2p+1} A(j)^{p+1} (p+1)! M_{j,p}.$$

Therefore for every $\alpha \in \mathbb{N}_0^r$, we successively get

$$\begin{aligned}\|\mathrm{D}^\alpha f\|_{\mathbb{R}^r} &\leq AB^{|\alpha|}\frac{1}{(|\alpha|+1)!}\theta_1^2\ldots\theta_{|\alpha|}^2\theta_{|\alpha|+1}M'_{|\alpha|}M'_{|\alpha|+1}\\ &\leq AB^{|\alpha|}C(j+1)^{2|\alpha|+1}A(j)^{|\alpha|+1}\theta_1^2\ldots\theta_{|\alpha|}^2\theta_{|\alpha|+1}M_{j,|\alpha|}\end{aligned}$$

hence we get the existence of a constant $D(j) > 0$ such that

$$a_p := \sup_{\substack{\alpha\in\mathbb{N}_0^r\\ |\alpha|=p}} \|\mathrm{D}^\alpha f\|_{\mathbb{R}^r} \leq AD(j)^p\theta_1^2\ldots\theta_p^2 M_{j,p}$$

for every $p \in \mathbb{N}_0$.

So everything is in order to apply the Result 8 to the sequences $(a_p)_{p\in\mathbb{N}_0}$ and $(AD(j)^p M_{j,p})_{p\in\mathbb{N}_0}$: for every $h > 0$, there is a positive constant $E(h)$ such that

$$a_p \leq E(h)\Big(\frac{h}{AD(j)}\Big)^p AD(j)^p M_{j,p} \leq E(h)h^p M_{j,p}, \quad \forall p \in \mathbb{N}.$$

Hence the conclusion. *QED*

6 Surjectivity of the restriction map $R\colon \widehat{\mathfrak{M}}(\mathbb{R}^r) \to J\widehat{\mathfrak{M}}(K)$

14 Theorem. *Let K be a non empty compact subset of $\mathbb{R}^r$.*

If the condition (A) holds, if $\mathfrak{m}$ is non quasi-analytic and if there are $A > 1$ and $s \in \mathbb{N}$ such that $M_{p+1} \leq A^p M_p^s$ for every $p \in \mathbb{N}_0$, then every jet $\boldsymbol{\varphi} \in J\widehat{\mathfrak{M}}(K)$ comes from an element of $\widehat{\mathfrak{M}}(\mathbb{R}^r)$ with compact support.

Proof. The use of the sequence $(a_j := 2^{-js})_{j\in\mathbb{N}}$ to define the topologies of the spaces $J\widehat{\mathfrak{M}}(K)$ and $\widehat{\mathfrak{M}}(\mathbb{R}^r)$ leads to

$$M_p^{a_{j+1}} M_{p+1}^{a_{j+1}} \leq A^{pa_{j+1}} M_p^{(s+1)a_{j+1}} \leq A^{pa_{j+1}} M_p^{2^{-js}} \leq A^p M_p^{a_j}$$

hence to

$$M_{j+1,p}M_{j+1,p+1} = p!M_p^{a_{j+1}}(p+1)!M_{p+1}^{a_{j+1}} \leq A^p p!(p+1)!M_p^{a_j} \leq A^{p+1}(p+1)!M_{j,p}$$

for every $p \in \mathbb{N}$. As $\widehat{\mathfrak{M}}(\mathbb{R}^r)$ coincides with $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$ and $J\widehat{\mathfrak{M}}(K)$ with $\widehat{\mathcal{E}}_{(\mathfrak{M})}(K)$, the conclusion follows at once from the Theorem 13. *QED*

7 Surjectivity of the restriction map $R\colon \widehat{\Phi}(\mathbb{R}^r) \to J\widehat{\Phi}(K)$

15 Proposition. *If* $\Phi\colon [0,\infty[\to \mathbb{R}$ *is a convex and increasing function such that* $\Phi(0) = 0$ *and* $\lim_{t\to\infty} \Phi(t)/t = +\infty$*, then, for every* $a > 0$*, the sequence* $\boldsymbol{m}_a = (m_{a,p})_{p\in\mathbb{N}_0}$ *defined by* $m_{a,0} := 1$ *and*

$$m_{a,p} := p\exp(\Phi(ap) - \Phi(a(p-1))), \quad \forall p \in \mathbb{N},$$

is normalized, verifies the condition $(*)$ *and is such that* $\lim_p m_{a,p}/p = +\infty$*. Moreover the inequality* $m_{b,p} \le m_{a,p}$ *holds for every* $0 < b < a$ *and* $p \in \mathbb{N}_0$*.*

Proof. It is clear that the sequence $\boldsymbol{m}_a$ is normalized. Moreover for every $p \in \mathbb{N}$, the convexity of Φ leads to

$$\Phi(ap) - \Phi(a(p-1)) \le \Phi(a(p+1)) - \Phi(ap)$$

which implies of course the condition $(*)$. It also provides

$$\frac{\Phi(ap) - \Phi(a(p-1))}{a} \ge \frac{\Phi(ap)}{ap}$$

hence $\lim_p m_{a,p}/p = +\infty$. If $0 < b < a$, it finally gives

$$\frac{\Phi(bp) - \Phi(b(p-1))}{b} \le \frac{\Phi(ap) - \Phi(a(p-1))}{a}$$

hence $\Phi(bp) - \Phi(b(p-1)) \le \Phi(ap) - \Phi(a(p-1))$ and therefore $m_{b,p} \le m_{a,p}$. QED

16 Definition. Let $\Phi\colon [0,+\infty[\to \mathbb{R}$ be a convex and increasing function such that $\lim_{t\to\infty} \Phi(t)/t = +\infty$. For every $a > 0$ and $p \in \mathbb{N}_0$, set also $M_{ap}^{(\Phi)} := \exp(\Phi(ap))$.

Then, as in [1], let us consider the following notions.

Let Ω be a non empty open subset of $\mathbb{R}^r$. For every $a > 0$, *the Banach space* $\{p!M_{ap}^{(\Phi)}, 1, \Omega\}$ is the following space: its elements are the $\mathcal{C}^\infty$-functions f on Ω such that

$$\|f\|_{\Omega,a} := \sup_{\alpha\in\mathbb{N}_0^r} \sup_{x\in\Omega} \frac{|\mathrm{D}^\alpha f(x)|}{|\alpha|!M_{a|\alpha|}^{(\Phi)}} < \infty$$

and it is endowed with the norm $\|\cdot\|_{\Omega,a}$. *The Fréchet space* $\widehat{\Phi}(\Omega)$ is then the projective limit of the Banach spaces $\{p!M_{ap}^{(\Phi)}, 1, \Omega\}$ for $a > 0$.

Let now K be a non empty compact subset of R^r. For every $a > 0$, *the Banach space* $\{p!M_{ap}^{(\Phi)}, 1, K\}$ is the vector space of the jets $\boldsymbol{\varphi}$ on K such that

$$|\boldsymbol{\varphi}|_{K,a} := \sup_{\alpha \in \mathbb{N}_0^r} \sup_{x \in K} \frac{|\varphi_\alpha(x)|}{|\alpha|! M_{a|\alpha|}^{(\Phi)}} < \infty$$

and

$$\|\boldsymbol{\varphi}\|_{K,a} := \sup_{m \in \mathbb{N}_0^r} \sup_{\substack{\alpha \in \mathbb{N}_0^r \\ |\alpha| \leq m}} \sup_{\substack{x,y \in K \\ x \neq y}} \frac{|(R^m \varphi_\alpha)(x,y)|}{|\alpha|! M_{a(m+1)}^{(\Phi)} |y-x|^{m+1-|\alpha|}} < \infty$$

endowed with the norm $|\cdot|_{K,a} + \|\cdot\|_{K,a}$. *The Fréchet space* $J\widehat{\Phi}(K)$ is then the projective limit of the Banach spaces $\{p!M_{ap}^{(\Phi)}, 1, K\}$ for $a > 0$.

If we set $\Psi(t) = \Phi(t) - \Phi(0)$ for every $t \geq 0$, Ψ also is a convex and increasing function on $[0, +\infty[$ such that $\lim_{t \to +\infty} \Psi(t)/t = +\infty$ and it is a direct matter to check that the Fréchet spaces $\widehat{\Phi}(\Omega)$ and $\widehat{\Psi}(\Omega)$ as well as $J\widehat{\Phi}(K)$ and $J\widehat{\Psi}(K)$ coincide.

Condition (B)**.** *Condition* (B) means that Φ is a real, convex and increasing function on $[0, +\infty[$ such that $\Phi(0) = 0$. It also means that $(a_j)_{j \in \mathbb{N}}$ is a sequence of positive numbers strictly decreasing to 0 and that, for every $j \in \mathbb{N}$, we set $m_{j,0} := 1$ and

$$m_{j,p} := \frac{p! M_{a_j p}^{(\Phi)}}{(p-1)! M_{a_j(p-1)}^{(\Phi)}} = p \exp(\Phi(a_j p) - \Phi(a_j(p-1))), \quad \forall p \in \mathbb{N},$$

as well as of course $M_{j,p} = m_{j,0} \ldots m_{j,p} = p! M_{a_j p}^{(\Phi)}$ for every $p \in \mathbb{N}_0$.

If the condition (B) holds, the Proposition 15 tells us that, for every $j \in \mathbb{N}$, the sequence $\boldsymbol{m}_j$ is normalized, verifies the condition $(*)$ (hence is increasing) and verifies $\lim_p m_{j,p}/p = \infty$ (hence $\lim_p m_{j,p} = \infty$). It also tells us that the inequality $m_{j+1,p} \leq m_{j,p}$ holds for every $j \in \mathbb{N}$ and $p \in \mathbb{N}_0$. Therefore the matrix $\mathfrak{m} = (m_{j,p})_{j \in \mathbb{N}, p \in \mathbb{N}_0}$ verifies the main requirement and the condition $(*)$.

If the condition (B) holds, it is also clear that the Fréchet space $\widehat{\Phi}(\Omega)$ is the projective limit of the Banach spaces $\{p!M_{a_j p}^{(\Phi)}, 1, \Omega\}$ and the Fréchet space $J\widehat{\Phi}(K)$ the one of the Banach spaces $\{p!M_{a_j p}^{(\Phi)}, 1, K\}$.

17 Lemma. *If the condition* (B) *holds, then, for every* $h > 0$ *and* $j \in \mathbb{N}$, *there is* $p_0 \in \mathbb{N}$ *such that*

$$\frac{M_{a_{j+1}|\alpha|}^{(\Phi)}}{M_{a_j|\alpha|}^{(\Phi)}} \leq h^{|\alpha|} \text{ if } |\alpha| \geq p_0.$$

Proof. It suffices to note that we even have

$$\lim_{|\alpha|\to\infty}\left(M^{(\Phi)}_{a_{j+1}|\alpha|}/M^{(\Phi)}_{a_j|\alpha|}\right)^{1/|\alpha|}=0$$

since the convexity of Φ provides

$$\frac{\Phi(a_j\,|\alpha|)-\Phi(a_{j+1}\,|\alpha|)}{(a_j-a_{j+1})\,|\alpha|}\geq\frac{\Phi(a_j\,|\alpha|)}{a_j\,|\alpha|}$$

for every $j\in\mathbb{N}$ and $\alpha\in\mathbb{N}_0^r$. QED

18 Proposition. *If the condition (B) holds, then the Fréchet spaces $\widehat{\Phi}(\Omega)$ and $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\Omega)$ coincide for every non empty open subset Ω of $\mathbb{R}^r$.*

Proof. We first prove that $\widehat{\Phi}(\Omega)$ is a vector subspace of $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\Omega)$, endowed with a finer topology. Indeed given any continuous norm $\|\cdot\|_{\Omega,h,j}$ on $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\Omega)$ and $f\in\widehat{\Phi}(\Omega)$, we certainly have

$$\|\mathrm{D}^\alpha f\|_\Omega\leq\|f\|_{\Omega,a_{j+1}}\,|\alpha|!M^{(\Phi)}_{a_{j+1}|\alpha|}=\|f\|_{\Omega,a_{j+1}}\,|\alpha|!M^{(\Phi)}_{a_j|\alpha|}\frac{M^{(\Phi)}_{a_{j+1}|\alpha|}}{M^{(\Phi)}_{a_j|\alpha|}}$$

for every $\alpha\in\mathbb{N}_0^r$. To conclude, it suffices then to note that the Lemma 17 provides a constant $C(j,h)>0$ (independent from f) such that $\|f\|_{\Omega,h,j}\leq C(j,h)\,\|f\|_{\Omega,a_{j+1}}$.

Conversely since $\{\,\|\cdot\|_{\Omega,h,j}\mid j\in\mathbb{N}\,\}$ is a fundamental system of continuous norms on $\widehat{\Phi}(\Omega)$, it suffices to check that the inequality $\|\cdot\|_{\Omega,a_j}\leq\|\cdot\|_{\Omega,1/j,j}$ holds on $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\Omega)$, which is immediate. QED

19 Proposition. *If the condition (B) holds, then the Fréchet spaces $J\widehat{\Phi}(K)$ and $\widehat{\mathcal{E}}_{(\mathfrak{M})}(K)$ coincide for every non empty compact subset K of $\mathbb{R}^r$.*

Proof. We first prove that $J\widehat{\Phi}(K)$ is a vector subspace of $\widehat{\mathcal{E}}_{(\mathfrak{M})}(K)$, endowed with a finer topology. Given any continuous norm $|\cdot|_{K,h,j}+\|\cdot\|_{K,h,j}$ on $\widehat{\mathcal{E}}_{(\mathfrak{M})}(K)$ and $\boldsymbol{\varphi}\in J\widehat{\Phi}(K)$, we have

$$\|\varphi_\alpha\|_K\leq|\boldsymbol{\varphi}|_{K,a_{j+1}}\,|\alpha|!M^{(\Phi)}_{a_{j+1}|\alpha|}=|\boldsymbol{\varphi}|_{K,a_{j+1}}\,|\alpha|!M^{(\Phi)}_{a_j|\alpha|}\frac{M^{(\Phi)}_{a_{j+1}|\alpha|}}{M^{(\Phi)}_{a_j|\alpha|}}$$

for every $\alpha\in\mathbb{N}_0^r$ and the Lemma 17 provides a constant $A(j,h)>0$ (independent from f) such that $|\boldsymbol{\varphi}|_{K,h,j}\leq A(j,h)\,|\boldsymbol{\varphi}|_{K,a_{j+1}}$. For every $m\in\mathbb{N}$ and $\alpha\in\mathbb{N}_0^r$ such that $|\alpha|\leq m$, we also have

$$\sup_{\substack{x,y\in K\\x\neq y}}\frac{|(R^m\varphi_\alpha)(x,y)|}{|y-x|^{m+1-|\alpha|}}\leq\|\boldsymbol{\varphi}\|_{K,a_{j+1}}\,|\alpha|!M^{(\Phi)}_{a_j(m+1)}\frac{M^{(\Phi)}_{a_{j+1}(m+1)}}{M^{(\Phi)}_{a_j(m+1)}}$$

and again the Lemma 17 provides a constant $B(j,h) > 0$ (independent from f) such that $\|\boldsymbol{\varphi}\|_{K,h,j} \leq B(j,h)\,\|\boldsymbol{\varphi}\|_{K,a_{j+1}}$.

Conversely since $\{\,|\cdot|_{K,a_j} + \|\cdot\|_{K,a_j} \mid j \in \mathbb{N}\}$ is a fundamental system of continuous norms on $J\widehat{\Phi}(K)$, it suffices to check that we have

$$|\cdot|_{K,a_j} \leq |\cdot|_{K,1/j,j} \text{ and } \|\cdot\|_{K,a_j} \leq \|\cdot\|_{K,1/j,j}$$

on $\widehat{\mathcal{E}}_{(\mathfrak{M})}(K)$, which is a direct matter. QED

20 Definition. If the condition (B) holds, let us say as in [1] that Φ is *non quasi-analytic* if the matrix $\mathfrak{m}$ is non quasi-analytic.

21 Theorem. *If the condition (B) holds with Φ non quasi-analytic and if K is a non empty compact subset of $\mathbb{R}^r$, then every jet $\boldsymbol{\varphi} \in J\widehat{\Phi}(K)$ comes from an element of $\widehat{\Phi}(\mathbb{R}^r)$ with compact support.*

Proof. The use of the sequence $(a_j := 3^{-j})_{j\in\mathbb{N}}$ to define the topologies of the spaces $J\widehat{\Phi}(K)$ and $\widehat{\Phi}(\mathbb{R}^r)$ leads to

$$\begin{aligned} M_{j+1,p}M_{j+1,p+1} &= p!M^{(\Phi)}_{a_{j+1}p}(p+1)!M^{(\Phi)}_{a_{j+1}(p+1)} \\ &= p!(p+1)!e^{\Phi(a_{j+1}p)+\Phi(a_{j+1}(p+1))} \end{aligned}$$

with $\Phi(a_{j+1}p) + \Phi(a_{j+1}(p+1)) \leq \Phi(a_{j+1}(2p+1))$ for every $p \in \mathbb{N}_0$ by use of the properties of Φ. Therefore we have

$$M_{j+1,0}M_{j+1,1} \leq e^{\Phi(a_{j+1})} \leq e^{\Phi(1)}(0+1)!M_{j,0}$$

as well as for every $p \in \mathbb{N}$

$$\begin{aligned} M_{j+1,p}M_{j+1,p+1} &\leq p!(p+1)!e^{\Phi(3a_{j+1}p)} = p!(p+1)!e^{\Phi(a_jp)} = (p+1)!M_{j,p} \\ &\leq e^{\Phi(1)}(p+1)!M_{j,p}. \end{aligned}$$

As $\widehat{\Phi}(\mathbb{R}^r)$ coincides with $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$ and $J\widehat{\Phi}(K)$ with $\widehat{\mathcal{E}}_{(\mathfrak{M})}(K)$, the conclusion now follows at once from the Theorem 13. QED

8 Mixed problem

In this paragraph, we consider a second matrix $\mathfrak{r} = (r_{j,p})_{j\in\mathbb{N},p\in\mathbb{N}_0}$ verifying the main requirement, i.e. for every $j \in \mathbb{N}$, the sequence $\boldsymbol{r}_j = (r_{j,p})_{p\in\mathbb{N}_0}$ is normalized and increasing to $+\infty$ and the inequality $r_{j,p} \geq r_{j+1,p}$ holds for every $p \in \mathbb{N}_0$. Moreover we of course set $R_{j,p} = r_{j,0}\dots r_{j,p}$ for every $j \in \mathbb{N}$ and $p \in \mathbb{N}_0$.

22 Proposition. *If the matrix $\mathfrak{r}$ verifies*

$$\liminf_p \frac{R_{j,p}}{2^p R_{j+1,p}} > 1, \quad \forall j \in \mathbb{N},$$

then the inclusion $\widehat{\Lambda}_{(\mathfrak{R})} \subset \widehat{\Lambda}_{(\mathfrak{M})}$ implies for every $j \in \mathbb{N}$ the existence of an integer $c(j) > j$ such that $R_{c(j),p} \leq M_{j,p}$ for every $p \geq c(j)$.

Proof. If it is not the case, there is an integer $j_0 \in \mathbb{N}$ such that for every integer $q > j_0$, there is an integer $s_q \geq q$ such that $R_{q,s_q} > M_{j_0,s_q}$.

Since replacing $\mathfrak{m}$ and $\mathfrak{r}$ by use of the same subsequence of their lines does not modify the hypothesis, we may very well suppose the existence of a strictly increasing sequence $(s_q)_{q\in\mathbb{N}}$ of positive integers such that $R_{q,s_q} > M_{1,s_q}$ for every $q \in \mathbb{N}$.

Let us now prove that the sequence $\boldsymbol{u} = (u_p)_{p\in\mathbb{N}_0}$ defined by $u_{s_q} := M_{1,s_q}$ for every $q \in \mathbb{N}$ and $u_p := 0$ otherwise belongs to $\Lambda_{(\boldsymbol{R}_k)}$ for every $k \in \mathbb{N}$ hence to $\widehat{\Lambda}_{(\mathfrak{R})}$. Indeed let $k \geq 2$ be an integer and fix any $h > 0$. Then we let l be the first integer such that $l > k$ and $2^{k-l} < h$. On one hand, from

$$u_{s_q} = M_{1,s_q} < R_{q,s_q} \leq R_{l,s_q}, \quad \forall q \geq l,$$

we get that $u_p < R_{l,p}$ holds for every $p \geq s_l$. On the other hand, by hypothesis, there is a positive integer p_0 such that

$$\frac{R_{j,p}}{2^p R_{j+1,p}} > 1, \quad \forall j \in \{k, \ldots, l-1\}, p \geq p_0,$$

hence such that

$$\frac{h^p R_{k,p}}{R_{l,p}} > 2^{p(k-l)} \frac{R_{k,p}}{R_{l,p}} = \frac{R_{k,p}}{2^p R_{k+1,p}} \cdots \frac{R_{l-1,p}}{2^p R_{l,p}} > 1, \quad \forall p \geq p_0.$$

These two informations put together provide clearly the existence of a positive constant $A(h)$ such that $u_p \leq A(h) h^p R_{k,p}$ for every $p \in \mathbb{N}_0$.

Now as $\boldsymbol{u}$ belongs to $\widehat{\Lambda}_{(\mathfrak{R})}$, it also belong to $\widehat{\Lambda}_{(\mathfrak{M})}$ hence to $\Lambda_{(\boldsymbol{M}_1)}$ and there is a constant $A > 0$ such that $u_p \leq A 2^{-p} M_{1,p}$ for every $p \in \mathbb{N}_0$. But there also is $q \in \mathbb{N}$ such that $A 2^{-s_q} < 1$. This leads to the contradiction $u_{s_q} = M_{1,s_q} < M_{1,s_q}$. QED

23 Proposition. *Let $\mathfrak{m}$ and $\mathfrak{r}$ be non quasi-analytic and verify the condition $(*)$. Suppose moreover that, for every $j \in \mathbb{N}$, there is a constant $A(j) > 0$ such that*

$$\begin{cases} M_{j+1,p} M_{j+1,p+1} \leq A(j)^{p+1} (p+1)! M_{j,p} \\ R_{j+1,p} R_{j+1,p+1} \leq A(j)^{p+1} (p+1)! R_{j,p} \end{cases}$$

for every $p \in \mathbb{N}_0$.

Then the equality of the vector spaces $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$ *and* $\widehat{\mathcal{B}}_{(\mathfrak{R})}(\mathbb{R}^r)$ *for some* $r \in \mathbb{N}$ *implies that*
(a) the vector spaces $\widehat{\Lambda}_{(\mathfrak{M})}$ *and* $\widehat{\Lambda}_{(\mathfrak{R})}$ *coincide;*
(b) for every $j \in \mathbb{N}$*, there are positive integers* $a(j)$*,* $b(j)$ *and* $c(j)$ *such that*

$$\begin{cases} j < a(j) < a(j) + 1 < b(j), \\ R_{j,p} \geq M_{a(j),p} \geq M_{a(j)+1,p} \geq R_{b(j),p}, \quad \forall p \geq c(j). \end{cases}$$

PROOF. (a) By the Theorem 13, every element $\boldsymbol{u}$ of $\widehat{\Lambda}_{(\mathfrak{M})}$ comes from an element f of $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R})$. Setting $x = (x_1, \ldots, x_r)$ for the generic points of $\mathbb{R}^r$, it is immediate that the function $g(x) = f(x_1)$ belongs to $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$ hence to $\widehat{\mathcal{B}}_{(\mathfrak{R})}(\mathbb{R}^r)$. For every $\alpha \in \mathbb{N}_0^r$ such that $\alpha_2 = \ldots = \alpha_r = 0$, we obviously have $\mathrm{D}^\alpha g(0) = u_{\alpha_1}$ hence $\boldsymbol{u}$ belongs to $\widehat{\Lambda}_{(\mathfrak{R})}$. Hence the conclusion since the conditions imposed on $\mathfrak{m}$ and $\mathfrak{r}$ are identical.

(b) On one hand, the hypothesis implies the inequalities

$$\frac{M_{j,p}}{2^p M_{j+1,p}} \geq \frac{M_{j+1,p+1}}{2^{p+1} A(j)^{p+1} (p+1)!}, \quad \forall j \in \mathbb{N}, p \in \mathbb{N}_0.$$

On the other hand, for every $j \in \mathbb{N}$, the sequence $\boldsymbol{m}_j$ is normalized, increasing and non quasi-analytic; it is well known that this implies $p/m_{j+1,p} \to 0$ hence

$$\lim_p \left(\frac{(p+1)!}{M_{j+1,p+1}} \right)^{1/(p+1)} = 0, \quad \forall j \in \mathbb{N}.$$

Putting these two informations together leads to

$$\liminf_p \frac{M_{j,p}}{2^p M_{j+1,p}} > 1, \quad \forall j \in \mathbb{N}.$$

Since $\mathfrak{m}$ and $\mathfrak{r}$ satisfy identical conditions, analogous inequalities hold for $\mathfrak{r}$.

As part (a) provides the equality $\widehat{\Lambda}_{(\mathfrak{M})} = \widehat{\Lambda}_{(\mathfrak{R})}$, we conclude at once by use of the Proposition 22. *QED*

9 Autonomy of the spaces $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$

24 Proposition. *Let* $\mathfrak{m}$ *be non quasi-analytic and verify the condition* $(*)$ *and let* $\boldsymbol{r} = (r_p)_{p \in \mathbb{N}_0}$ *be a normalized sequence increasing to* $+\infty$*. Suppose moreover that, for every* $j \in \mathbb{N}$*, there is* $A(j) > 0$ *such that*

$$M_{j+1,p} M_{j+1,p+1} \leq A(j)^{p+1} (p+1)! M_{j,p}, \quad \forall p \in \mathbb{N}_0,$$

then, for every $r \in \mathbb{N}$*, the vector space* $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$ *does not coincide with the vector space* $\mathcal{B}_{\{\boldsymbol{R}\}}(\mathbb{R}^r)$ *nor with the vector space* $\mathcal{B}_{(\boldsymbol{R})}(\mathbb{R}^r)$*.*

Proof. (a) If the vector spaces $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$ and $\mathcal{B}_{\{\boldsymbol{R}\}}(\mathbb{R}^r)$ coincide, it is clear that the vector spaces $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R})$ and $\mathcal{B}_{\{\boldsymbol{R}\}}(\mathbb{R})$ coincide too. It also is an easy matter to check that the Fréchet space $\widehat{\Lambda}_{(\mathfrak{M})}$ is not isomorphic to a Banach space. By the Theorem 13, we know that every element of $\widehat{\Lambda}_{(\mathfrak{M})}$ comes from an element of $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R})$. As a direct consequence of the closed graph theorem, we obtain that $\widehat{\Lambda}_{(\mathfrak{M})}$ is a Hausdorff quotient of the (LB)-space $\mathcal{B}_{\{\boldsymbol{R}\}}(\mathbb{R})$ hence is a (LB)-space, which is contradictory.

(b) If the vector spaces $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$ and $\mathcal{B}_{(\boldsymbol{R})}(\mathbb{R}^r)$ coincide, it is clear that the vector spaces $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R})$ and $\mathcal{B}_{(\boldsymbol{R})}(\mathbb{R})$ coincide too. By the Theorem 13, we know that every element of $\widehat{\Lambda}_{(\mathfrak{M})}$ comes from an element of $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R})$ hence from an element of $\mathcal{B}_{(\boldsymbol{R})}(\mathbb{R})$. If we define the matrix $\mathfrak{r} = (r_{j,p})_{j\in\mathbb{N},p\in\mathbb{N}_0}$ by $r_{j,p} = r_p$ for every $j \in \mathbb{N}$ and $p \in \mathbb{N}_0$, $\mathfrak{r}$ verifies the main requirement and what precedes leads to $\widehat{\Lambda}_{(\mathfrak{M})} \subset \Lambda_{(\boldsymbol{R})} \subset \widehat{\Lambda}_{(\mathfrak{R})}$. Now proceeding as at the beginning of the proof of the part (b) of the Proposition 23, we get

$$\liminf_p \frac{M_{j,p}}{2^p M_{j+1,p}} > 1, \quad \forall j \in \mathbb{N}.$$

Therefore we may apply the Proposition 22 and get the existence of an integer $c > 1$ such that $M_{c,p} \leq R_{1,p} = R_p$ for every integer $p \geq c$.

Using the hypothesis provides

$$M_{c+2,p} M_{c+2,p+1} \leq A(c+1)^{p+1}(p+1)! M_{c+1,p}, \quad \forall p \in \mathbb{N}_0,$$

hence

$$\lim_p \Big(\frac{M_{c+1,p}}{M_{c+2,p}}\Big)^{1/p} \geq \lim_p A(c+1)^{-(p+1)/p} \Big(\frac{M_{c+2,p+1}}{(p+1)!}\Big)^{1/p} = \infty.$$

Hence by the result ([4], 6.7.III), there is a function g belonging to $\mathcal{B}_{\{M_{c+1}\}}(\mathbb{R})$ and not to $\mathcal{B}_{\{M_{c+2}\}}(\mathbb{R})$, hence not belonging to $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R})$.

However the hypothesis also provides

$$\lim_p \Big(\frac{M_{c,p}}{M_{c+1,p}}\Big)^{1/p} \geq \lim_p A(c)^{-(p+1)/p} \Big(\frac{M_{c+1,p+1}}{(p+1)!}\Big)^{1/p} = \infty.$$

Therefore, for every $h > 0$, there is $p_0 \in \mathbb{N}$ such that

$$M_{c+1,p} \leq h^p M_{c,p}, \quad \forall p \geq p_0.$$

This implies that g belongs to $\mathcal{B}_{(\boldsymbol{M}_c)}(\mathbb{R})$ hence to $\mathcal{B}_{(\boldsymbol{R})}(\mathbb{R}) = \widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R})$.

Hence a contradiction. QED

10 About the equality $\widehat{\Phi}(\mathbb{R}^r) = \widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$

25 Proposition. *Let $\mathfrak{m}$ be non-quasi-analytic and verify the condition $(*)$. Suppose moreover that, for every $j \in \mathbb{N}$, there is $A(j) > 0$ such that*

$$M_{j+1,p} M_{j+1,p+1} \leq A(j)^{p+1}(p+1)! M_{j,p}, \quad \forall p \in \mathbb{N}_0.$$

Let finally the condition (B) hold with Φ non quasi-analytic and set

$$r_{j,p} = \frac{p! M^{(\Phi)}_{a_j p}}{(p-1)! M^{(\Phi)}_{a_j(p-1)}} = p e^{\Phi(a_j p) - \Phi(a_j(p-1))}, \quad \forall j \in \mathbb{N}, p \in \mathbb{N}_0,$$

to avoid any confusion in the notations.

If the vector spaces $\widehat{\Phi}(\mathbb{R}^r)$ and $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$ coincide for some $r \in \mathbb{N}$, then, for every $j \in \mathbb{N}$, there are positive integers $a(j) > j$, $c(j)$ and $r(j)$ such that

$$\frac{M_{a(j),p}}{p!} \leq \frac{M_{a(j)+1, r(j)p}}{(r(j)p)!}, \quad \forall p \geq c(j).$$

Proof. Let us use the sequence $(a_j := 3^{-j})_{j \in \mathbb{N}}$ to define the topologies of the spaces $\widehat{\Phi}(\mathbb{R}^r)$ and $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$. The beginning of the proof of the Theorem 21 gives then the existence of a constant $A > 0$ such that

$$R_{j+1,p} R_{j+1,p+1} \leq A(p+1)! R_{j,p}, \quad \forall j \in \mathbb{N}, p \in \mathbb{N}_0.$$

As we know by the Proposition 18 that the Fréchet spaces $\widehat{\Phi}(\mathbb{R}^r)$ and $\widehat{\mathcal{B}}_{(\mathfrak{R})}(\mathbb{R}^r)$ coincide, we are in position to apply the part (b) of the Proposition 23: for every $j \in \mathbb{N}$, there are positive integers $a(j)$, $b(j)$ and $c(j)$ such that

$$\begin{cases} j < a(j) < a(j)+1 < b(j), \\ R_{j,p} \geq M_{a(j),p} \geq M_{a(j)+1,p} \geq R_{b(j),p}, \quad \forall p \geq c(j). \end{cases}$$

Therefore if we choose a positive integer $r(j)$ such that $r(j) a_{b(j)} > a(j)$, we successively have

$$\begin{aligned} M_{a(j),p} \leq R_{j,p} = p! e^{\Phi(a_j p)} &\leq \frac{p!}{(r(j)p)!}(r(j)p)! e^{\Phi(r(j) a_{b(j)} p)} \\ &\leq \frac{p!}{(r(j)p)!} R_{b(j), r(j)p} \leq p! \frac{M_{a(j)+1, r(j)p}}{(r(j)p)!}, \quad \forall p \geq c(j). \end{aligned}$$

Hence the conclusion. QED

Construction Let us obtain the sequences $\boldsymbol{a}_j = (a_{j,p})_{p\in\mathbb{N}_0}$ by the following recursion on $j \in \mathbb{N}$.

As starting point we take the sequence $\boldsymbol{a}_1 = (a_{1,p} := 2^p)_{p\in\mathbb{N}_0}$ and observe that, replacing $a_{1,p}$ by $b_{j,p}$ for every $p \in \mathbb{N}_0$, the sequence $\boldsymbol{a}_1$ verifies the following five conditions for every $p \in \mathbb{N}_0$:
(a) $b_{j,p} < b_{j,p+1}$,
(b) $b_{j,p}$ is of the type $q(j,p)2^{r(j,p)}$ with $q(j,p)$, $r(j,p) \in \mathbb{N}_0$ such that $q(j,p)$ is odd and $\lim_p r(j,p) = \infty$;
(c) there is $s(j,p) \in \mathbb{N}_0$ such that $b_{j,p+1} - b_{j,p} = 2^{s(j,p)}$;
(d) $b_{j,p+2} - b_{j,p+1} \geq b_{j,p+1} - b_{j,p}$;
(e) $\lim_p(b_{j,p+1} - b_{j,p}) = \infty$.

Now if the sequences $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_j$ are obtained, we construct the sequence $\boldsymbol{a}_{j+1}$ by use of the following recursion which leads to the fact that the sequences $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{j+1}$ verify the conditions (a) to (e) here above when setting $a_{k,p} = b_{k,p}$ for each $k \in \{1, \ldots, j+1\}$ and all $p \in \mathbb{N}_0$.

As starting point we define the sequence $\boldsymbol{a}_j^{(0)}$ by setting $a_{j,p}^{(0)} = a_{j,p}$ for every $p \in \mathbb{N}_0$. Then if the sequences $\boldsymbol{a}_j^{(0)}, \ldots, \boldsymbol{a}_j^{(k)}$ are obtained, we construct the sequence $\boldsymbol{a}_j^{(k+1)}$ by use of the following construction which also leads to the fact that the sequences $\boldsymbol{a}_j^{(0)}, \ldots, \boldsymbol{a}_j^{(k+1)}$ verify the conditions (a) to (e) when setting $a_{j,p}^{(l)} = b_{j,p}$ for some $l \in \{0, \ldots, k+1\}$ and all $p \in \mathbb{N}_0$. We let n_k be the first integer fulfilling the following two conditions:
(1) considering the $r(j,p)$'s coming from $a_{j,p}^{(k)} = b_{j,p}$ for every $p \in \mathbb{N}_0$, we have $r(j,p) > k+1$ for every $p > n_k$;
(2) $a_{j,n_k+2}^{(k)} - a_{j,n_k+1}^{(k)} > a_{j,n_k+1}^{(k)} - a_{j,n_k}^{(k)}$.
Then ordering the set

$$\{a_{j,p}^{(k)} : p \in \mathbb{N}_0\} \cup \left\{ \frac{a_{j,p+1}^{(k)} - a_{j,p}^{(k)}}{2} : p > n_k \right\}$$

provides the sequence $\boldsymbol{a}_j^{(k+1)}$. Finally ordering the set $\cup_{k=1}^{\infty}\{a_{j,p}^{(k)} : p \in \mathbb{N}_0\}$ provides the sequence $\boldsymbol{a}_{j+1}$.

Here is the aim of the choice of the starting sequence $\boldsymbol{a}_1$ and of this construction: for every j, $r \in \mathbb{N}$, it is a direct matter to check that there is a positive integer $b(j,r)$ such that for every integer $n > b(j,r)$, there are at least 2^r elements of the sequence $\boldsymbol{a}_{j+1}$ belonging to $]a_{j,n}, a_{j,n+1}[$.

Condition (C)**.** Condition (C) means that the sequences $\boldsymbol{a}_j$ of the previous construction are at our disposal. It also means that $\boldsymbol{m}$ is a normalized, strictly increasing and non quasi-analytic sequence and that we set $M_0^{(j)} := 1$ and $M_p^{(j)} := M_{a_{j,p}}$ for every $j \in \mathbb{N}$ and $p \in \mathbb{N}_0$.

Let us immediately note that, if condition (C) holds, then we certainly have

$$\begin{cases} \displaystyle\sum_{p=1}^{\infty} \frac{1}{m_p^{(j)}} = \sum_{p=1}^{\infty} \frac{M_{p-1}^{(j)}}{M_p^{(j)}} < \infty, & \forall j \in \mathbb{N}, \\ \big(M_p^{(j)}\big)^2 \leq M_{p-1}^{(j)} M_{p+1}^{(j)}, & \forall j \in \mathbb{N}, p \in \mathbb{N}_0, \end{cases}$$

(for the last inequalities just recall that we have $a_{j,p+2} - a_{j,p+1} \geq a_{j,p+1} - a_{j,p}$ for every $j \in \mathbb{N}$ and $p \in \mathbb{N}_0$).

26 Proposition. *If the condition (C) holds, then, for every $j \in \mathbb{N}$, there is $A(j) > 0$ such that*

$$M_p^{(j+1)} M_{p+1}^{(j+1)} \leq A(j) M_p^{(j)}, \quad \forall p \in \mathbb{N}_0.$$

PROOF. By the construction, there is $p_0 \in \mathbb{N}$ such that, for every $n \geq p_0$, the number of the elements of $\boldsymbol{a}_{j+1}$ belonging to $]a_{j,n}, a_{j,n+1}[$ is $\geq 2^2$. For every integer $p > 2p_0$, $M_p^{(j)}$ is then the element $p+1$ of the sequence $\boldsymbol{M}^{(j)}$ and also the element $M_q^{(j+1)}$ of the sequence $\boldsymbol{M}^{(j+1)}$ for some integer $q \geq 4(p - p_0)$. We therefore get

$$M_p^{(j+1)} M_{p+1}^{(j+1)} \leq M_{2p+1}^{(j+1)} \leq M_q^{(j+1)} = M_p^{(j)}, \quad \forall p > 2p_0,$$

since $p > 2p_0$ implies $4(p - p_0) > 2p + 1$. QED

27 Proposition. *If the condition (C) holds, then, for every j, r, $p_0 \in \mathbb{N}$, there is an integer $p > p_0$ such that $M_p^{(j)} > M_{rp}^{(j+1)}$.*

PROOF. By the construction, there is an integer $p_1 > p_0$ such that, for every integer $n \geq p_1$, the number of the elements of $\boldsymbol{a}_{j+1}$ belonging to $]a_{j,n}, a_{j,n+1}[$ is $\geq 2^r$. Now let p_2 be an integer such that $p_2 > p_1$ and $2^r(p - p_1) > rp$ for every integer $p \geq p_2$. Then, for every integer $p \geq p_2$, $M_p^{(j)}$ is the element $p+1$ of the sequence $\boldsymbol{M}^{(j)}$ and also the element $M_q^{(j+1)}$ of the sequence $\boldsymbol{M}^{(j+1)}$ for some integer $q \geq 2^r(p - p_1) > rp$ hence $M_p^{(j)} = M_q^{(j+1)} > M_{rp}^{(j+1)}$. QED

28 Theorem. *If the condition (C) holds, then there is no non quasi-analytic function Φ for which the condition (B) holds and such that $\widehat{\Phi}(\mathbb{R}) = \widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R})$.*

PROOF. For every $j \in \mathbb{N}$, let us set $m_{j,0} := 1$ and

$$m_{j,p} := \frac{p! M_p^{(j)}}{(p-1)! M_{p-1}^{(j)}}, \quad \forall p \in \mathbb{N}.$$

It is clear that, for every $j \in \mathbb{N}$, the sequence $\boldsymbol{m}_j = (m_{j,p})_{p\in\mathbb{N}_0}$ is normalized and non quasi-analytic and verifies the condition $(*)$. Moreover by the Proposition 26, we also have

$$\begin{aligned} M_{j+1,p}M_{j+1,p+1} &= p!M_p^{(j+1)}(p+1)!M_{p+1}^{(j+1)} \\ &\leq p!(p+1)!A(j)M_p^{(j)} = A(j)(p+1)!M_{j,p} \end{aligned}$$

for every $p \in \mathbb{N}_0$ and some $A(j) > 0$. Therefore by the Theorem 13, every element of $J\widehat{\mathfrak{M}}$ comes from an element of $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R})$.

For every j, r, $p_0 \in \mathbb{N}$, the Proposition 27 provides an integer $p > p_0$ such that $M_p^{(j)} > M_{rp}^{(j+1)}$ which implies

$$\frac{M_{j,p}}{p!} = M_p^{(j)} > M_{rp}^{(j+1)} = \frac{M_{j+1,rp}}{(rp)!}.$$

Hence the conclusion by the Proposition 25. QED

11 Ultraholomorphic extensions

Let $\boldsymbol{m}$ be a normalized, increasing and non quasi-analytic sequence. In [5], given a proper open subset Ω of $\mathbb{R}^r$, *the Fréchet space* $\mathcal{C}(\boldsymbol{M}, \Omega)$ is defined as the vector space of the $\mathcal{C}^\infty$-functions f on Ω such that

$$\|f\|_p := \sup_{\alpha\in\mathbb{N}_0^r} \frac{2^{(p+1)|\alpha|}\,\|\mathrm{D}^\alpha f\|_\Omega}{M_{|\alpha|}} < \infty, \quad \forall p \in \mathbb{N},$$

endowed with the system of norms $\{\, \|\cdot\|_p \mid p \in \mathbb{N}\,\}$. In fact, the Fréchet spaces $\mathcal{C}(\boldsymbol{M}, \Omega)$ and $\mathcal{B}_{(\boldsymbol{M})}(\Omega)$ coincide. Given a proper open subset U of $\mathbb{C}^r$, *the Fréchet space* $\mathcal{H}_\infty(\boldsymbol{M}, U)$ is the vector space of the holomorphic functions g on U such that

$$\|g\|_p := \sup_{\alpha\in\mathbb{N}_0^r} \frac{2^{(p+1)|\alpha|}\,\|\mathrm{D}^\alpha g\|_U}{M_{|\alpha|}} < \infty, \quad \forall p \in \mathbb{N},$$

endowed with the system of norms $\{\, \|\cdot\|_p \mid p \in \mathbb{N}\,\}$. This leads to *the Fréchet space* $\widehat{\mathcal{H}}_\infty(\mathfrak{M}, U)$, the projective limit of the spaces $\mathcal{H}_\infty(\boldsymbol{M}_j, U)$. Moreover an open subset D_Ω of $\mathbb{C}^r$ is constructed such that, in particular, $D_\Omega \cap \mathbb{R}^r = \Omega$ and u belongs to Ω if $u + iv \in D_\Omega$. This leads to the following key result.

29 Result. [5, Theorem 4.3] *Let $\boldsymbol{m}$ be a normalized, increasing and non quasi-analytic sequence. For every proper open subset Ω of $\mathbb{R}^r$, there is a continuous linear mapT_Ω from $\mathcal{C}(\boldsymbol{M}, \Omega)$ into $\mathcal{H}_\infty\{(\boldsymbol{M}, D_\Omega)\}$ such that for every$f \in \mathcal{C}(\boldsymbol{M}, \Omega)$,$\varepsilon > 0$ and$s \in \mathbb{N}$, there is a compact subset K of Ω such that*

$$|D^\alpha(T_\Omega f)(u + iv) - D^\alpha f(u)| \leq \varepsilon$$

for every $u + iv \in D_\Omega$ and $\alpha \in \mathbb{N}_0^n$verifying $u \in \Omega \setminus K$ and $|\alpha| \leq s$.

With these elements at our disposal, we can obtain the following enhancement of the Theorem 2.

30 Theorem. *Let $\mathfrak{m}$ be non quasi-analytic and verify the condition $(*)$. If moreover for every $j \in \mathbb{N}$, there is $A(j) > 0$ such that*

$$m_{j+1,p}^2 \leq pA(j)m_{j,p}, \quad \forall p \in \mathbb{N},$$

then every $\boldsymbol{u} \in \widehat{\Lambda}_{(\mathfrak{M})}$ comes from an element of $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}) \cap \widehat{\mathcal{H}}_\infty(\mathfrak{M}, D_{\mathbb{R}\setminus\{0\}})$; in particular, it comes from an element of $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R})$ which is analytic on $\mathbb{R} \setminus \{0\}$.

Proof. In the proof of the Theorem 2, we obtain that every $\boldsymbol{u} \in \widehat{\Lambda}_{(\mathfrak{M})}$ comes from an element f of $\widehat{\mathcal{D}}_{(\mathfrak{M}'')}([-1, 1])$, $\boldsymbol{m}''$ being a normalized, increasing and non quasi-analytic sequence. As f certainly belongs to $\mathcal{C}^\infty(\boldsymbol{M}'', \mathbb{R} \setminus \{0\})$, we may apply the Result 29 and get that the function g defined on $\mathbb{R} \cup D_{\mathbb{R}\setminus\{0\}}$ by $g(0) = u_0$ and $g(z) = T_{\mathbb{R}\setminus\{0\}}(f|_{\mathbb{R}\setminus\{0\}})(z)$ for every $z \in D_{\mathbb{R}\setminus\{0\}}$ is an extension of $\boldsymbol{u}$ belonging to $\mathcal{B}_{(\boldsymbol{M}'')}(\mathbb{R}) \cap \mathcal{H}_\infty(\boldsymbol{M}'', D_{\mathbb{R}\setminus\{0\}})$. The conclusion then follows as in the proof of the Theorem 2. QED

A look at the proof of the Theorem 7 provides then immediately the following enhancement of it.

31 Theorem. *If the condition (A) holds and if $\mathfrak{m}$ is non quasi-analytic (i.e. $\sum_{p=1}^\infty (pm_p^a)^{-1} < \infty$ for every $a > 0$), then every element of $J\widehat{\mathfrak{M}}$ comes from an element of $\widehat{\mathfrak{M}}(\mathbb{R}) \cap \widehat{\mathcal{H}}_\infty(\mathfrak{M}, D_{\mathbb{R}\setminus\{0\}})$.*

In order to enhance the Theorem 13, we need the following information coming from [5] as well.

Let $\boldsymbol{m}$ be a normalized, increasing and non quasi-analytic sequence. For every proper open subset Ω of $\mathbb{R}^r$ and $p \in \mathbb{N}$, *the Banach space* $\mathcal{C}_p\{\boldsymbol{M}, \Omega\}$ is the vector space of the $\mathcal{C}^\infty$-functions f on Ω such that

$$|f|_p := \sup_{\alpha \in \mathbb{N}_0^r} \frac{\|\mathrm{D}^\alpha f\|_\Omega}{p^{|\alpha|} M_{|\alpha|}} < \infty,$$

endowed with the norm $|\cdot|_p$ and *the Hausdorff* (LB)-*space* $\mathcal{C}\{\boldsymbol{M}, \Omega\}$ is the inductive limit of these spaces. It is a direct matter, for every proper open subset U of $\mathbb{C}^r$, to introduce in a similar way *the Hausdorff* (LB)-*space* $\mathcal{H}_\infty\{\boldsymbol{M}, U\}$. This leads to the following key result.

32 Theorem. *Let $\boldsymbol{m}$ be a normalized, increasing and non quasi-analytic sequence. For every proper open subset Ω of $\mathbb{R}^r$, there is a continuous linear mapT_Ω from $\mathcal{C}\{\boldsymbol{M}, \Omega\}$ into $\mathcal{H}_\infty\{\boldsymbol{M}, D_\Omega\}$ such that for every$f \in \mathcal{C}\{\boldsymbol{M}, \Omega\}$,$\varepsilon > 0$ and$s \in \mathbb{N}$, there is a compact subset K of Ωsuch that*

$$|D^\alpha(T_\Omega f)(u + iv) - D^\alpha f(u)| \leq \varepsilon$$

for every $u+iv \in D$ and $\alpha \in \mathbb{N}_0^n$verifying $u \in \Omega \setminus K$ and $|\alpha| \leq s$.In fact, for every $p \in \mathbb{N}$, T_Ω is a continuous linear map from$\mathcal{C}_p\{\boldsymbol{M}, \Omega\}$ into $\mathcal{H}_{4p}\{\boldsymbol{M}, D_\Omega\}$.

33 Theorem. *Let K be a non empty compact subset of $\mathbb{R}^r$ and let $\mathfrak{m}$ be non quasi-analytic and verify the condition $(*)$.*

If, for every $j \in \mathbb{N}$, there is a positive constant $A(j)$ such that

$$M_{j+1,p}M_{j+1,p+1} \leq A(j)^{p+1}(p+1)!M_{j,p}, \quad \forall p \in \mathbb{N}_0,$$

then every $\boldsymbol{\varphi} \in \widehat{\mathcal{E}}_{(\mathfrak{M})}(K)$ comes from an element of $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r) \cap \widehat{\mathcal{H}}_\infty(\mathfrak{M}, D_{\mathbb{R}\setminus K})$; in particular, $\boldsymbol{\varphi}$ comes from an element of $\widehat{\mathcal{B}}_{(\mathfrak{M})}(\mathbb{R}^r)$ which is analytic on $\mathbb{R}^r \setminus K$.

Proof. In the proof of the Theorem 13, we obtain an extension $f \in \mathcal{C}^\infty(\mathbb{R}^r)$ of $\boldsymbol{\varphi}$, with compact support and for which there are constants $A > 1$ and $B > 0$ such that

$$\|\mathrm{D}^\alpha f\|_{\mathbb{R}^r} \leq AB^{|\alpha|}\,|\alpha|!P_{|\alpha|}P_{|\alpha|+1}, \quad \forall \alpha \in \mathbb{N}_0^r.$$

As $\boldsymbol{c}$ is a normalized and non quasi-analytic sequence verifying the condition $(*)$, it is a direct matter to check that the sequence $\boldsymbol{q} = (q_k)_{k\in\mathbb{N}_0}$ defined by $q_0 := 1$ and $q_k := kp_kp_{k+1}$ for every $k \in \mathbb{N}$ is normalized, increasing and non quasi-analytic hence f belongs to the space $\mathcal{C}_B\{\boldsymbol{Q}, \mathbb{R}^r\}$. So the function g defined on $\mathbb{R}^r \cup D_{\mathbb{R}^r\setminus K}$ by $g(x) := f(x)$ for every $x \in K$ and $g(z) := T_{\mathbb{R}^r\setminus K}(f|_{\mathbb{R}^r\setminus K})$ for every $z \in D_{\mathbb{R}^r\setminus K}$ belongs to $\mathcal{C}_{4B}\{\boldsymbol{Q}, \mathbb{R}^r\} \cap \mathcal{H}_{4B}\{\boldsymbol{Q}, D_{\mathbb{R}^r\setminus K}\}$. The conclusion then follows as in the proof of the Theorem 13. QED

This leads immediately to the following enhancements of the Theorems 14 and 21.

34 Theorem. *Let K be a non empty compact subset of $\mathbb{R}^r$.*

If the condition (A) holds, if $\mathfrak{m}$ is non quasi-analytic and if there are $A > 1$ and $s \in \mathbb{N}$ such that $M_{p+1} \leq A^pM_p^s$ for every $p \in \mathbb{N}_0$, then every jet $\boldsymbol{\varphi} \in J\widehat{\mathfrak{M}}(K)$ comes from an element of $\widehat{\mathfrak{M}}(\mathbb{R}^r) \cap \widehat{\mathcal{H}}_\infty(\mathfrak{M}, D_{\mathbb{R}\setminus K})$; in particular, the extension is analytic on $\mathbb{R}^r \setminus K$.

35 Theorem. *If the condition (B) holds with Φ non quasi-analytic and if K is a non empty compact subset of $\mathbb{R}^r$, then every jet $\boldsymbol{\varphi} \in J\widehat{\Phi}(K)$ comes from an element of $\widehat{\Phi}(\mathbb{R}^r) \cap \widehat{\mathcal{H}}_\infty(\mathfrak{M}, D_{\mathbb{R}\setminus K})$; in particular, the extension is analytic on $\mathbb{R}^r \setminus K$.*

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