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Oder-bounded sets in locally solid Riesz spaces

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Abstract. Let E be Dedekind complete, Hausdorff, locally solid Riesz space and P an order bounded interval. We give a new proofs of Nakano's theorem, that if E has Fatou property, P is complete, that the restrictions on P, of all topologies on E having Lebesgue property, are identical; we also give a measure-theoretic proof of the result that if $(E, \mathcal{T})$ is a Dedekind complete, Hausdorff, locally convex-solid Riesz space with Lebesque property, then P is weakly compact and E is a regular Riesz subspace of E''.

Keywords: locally solid, band, Lebesgue property, Fatou property, order intervals, order direct sum

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1 Introduction and Notation

In this paper, for Riesz spaces, the notations are results of [1] are used. All vector spaces are over the field of real numbers. N will stand for the set of real numbers. $(E, \mathcal{T})$ will denote a Dedekind complete, Hausdorff, linear, locally solid Riesz space with Fatou property and having $\{\rho : \rho \in D\}$ a filtering upwards family of Fatou pseudo-norms generating its topology; note $(E, \mathcal{T})$ has Fatou property if it has a 0-nbd base consisting of solid and order-closed sets, and has Lebesgue property if, in E, $x_\alpha \downarrow 0$, in order, implies $x_\alpha \to 0$ in $(E, \mathcal{T})$; Lebesgue property implies Fatou property ([1], p.80). For every $\rho \in D$, A_ρ will denote the band $\rho^{-1}(0)$ in E and so $E = A_\rho \oplus A_\rho^d$ with $\varphi_\rho : E \to A_\rho^d$ the positive projection; this positive projection $\varphi_\rho : E \to A_\rho^d$ is both order and $\mathcal{T}$-continuous. For an $e \in E$, $e > 0$, P will denote the order interval $\{x \in E : |x| \leq e\}$.

In locally solid Riesz spaces, there are several deep results about P:

One is that if $(E, \mathcal{T})$ satisfies Fatou property, then P is complete; several sophisticated proofs are known ([7, 1, 3, 9]). The proof is simple when $(E, \mathcal{T})$ is metrizable and we prove that it follows easily from metrizable case (see also [11], [12] for related ideas and results).

The second result is that any two Haudroff Lebesgue topologies, when restricted to P, are identical. We obtain this result also from the metrizable case.

Still another well-known result is that if E is a Banach lattice with order-continuous norm, then P is weakly compact. In more general form, it says that if $(E, \mathcal{T})$ is a Dedekind complete, Hausdorff, locally convex-solid Riesz space with Lebesgue property, then P is weakly compact. We give a measure-theoretic proof of this.

The following lemma is simple (Cf. [1], lemma 1.25, p. 85).

1 Lemma. *Suppose E is metrizable and $\{x_n\}$ be a Cauchy sequence in P. Then there is a subsequence of $\{x_n\}$, which we denote by $\{x_{s(n)}\}$, for which $o - \lim x_{s(n)}$ exists and $\{x_{s(n)}\}$ converges to $o - \lim x_{s(n)}$ (this implies P is complete).*

Proof. Let ρ be a Fatou pseudo-norm generating its topology. $V_n = \{x \in E : \rho(x) \leq \frac{1}{2^{n+1}}$ is a 0-nbd base. Fix an $e \in E$, $e > 0$. The bounded order interval $P = \{x \in E : |x| \leq e\}$ is closed under arbitrary sup and inf. By taking subsequence of $\{x_n\}$ and denoting it by $\{x_{s(n)}\}$, we assume that, for all n, $x_{s(k)} - x_{s(l)} \in V_n$, $\forall k$ and $\forall l \geq n$. Now, $\forall p > 0$, $\forall q > 0$, $x_{s(n)} - \inf_{n+p \leq k \leq n+p+q} x_{s(k)} \leq \sum_{k=0}^{k=p+q} |x_{s(n+k)} - x_{s(n+k+1)}| \in \sum_{k=0}^{k=p+q} V_{n+k} \subset V_{n-1}$. Since ρ is a Fatou pseudo-norm, it easily follows from this that $x_{s(n)} - (o - \liminf x_{s(n)}) \in V_{n-1}$ and $o - \liminf x_{s(n)} \in P$. In a similar way, $(o - \limsup x_{s(n)}) - x_{s(n)} \in V_{n-1}$ and $o - \limsup x_{s(n)} \in P$. Thus $x_{s(n)}$ converges to $(o - \liminf x_{s(n)})$, and also to $(o - \limsup x_{s(n)})$. So that the Cauchy sequence $\{x_{s(n)}\}$ converges to $o - \lim x_{s(n)}$ in P. This complete the proof. QED

From Lemma 1, we get:

2 Corollary. *Let $\{x_\alpha\}, x_\alpha \geq 0$ be a Cauchy net in P. Then for every $\rho \in D$, there is a unique $x_\rho \in A_\rho^d \cap P$ such that $\rho(x_\alpha - x_\rho) \to 0$. Also for any two ρ and σ in D with $\rho \leq \sigma$, we have $\varphi_\rho(x_\sigma) = x_\rho$.*

Proof. Fix a $\rho \in D$ and put $P_\rho = \{y \in A_\rho^d : |y| \leq \varphi_\rho(e)\}$. Noting the facts that $\varphi_\rho(y) = y$ if $y \in A_\rho^d$ and $\varphi_\rho(e) \leq e$, we get $\varphi_\rho(P) = P_\rho \subset P$. From $\rho(x_\alpha - x_\beta) \to 0$, we get $\rho(\varphi_\rho(x_\alpha) - \varphi_\rho(x_\beta)) \to 0$. Since (A_ρ^d, ρ) is Hausdorff and metrizable, by Lemma 1, there is a unique $x_\rho \in A_\rho^d \cap P$ such that $\rho(\varphi_\rho(x_\alpha) - x_\rho) \to 0$. This implies that $\rho(x_\alpha - x_\rho) \to 0$. It is easy to see that $x_\rho \geq 0$.

Now take any $\sigma \in D$, $\sigma \geq \rho$. Since $\sigma(x_\alpha - x_\sigma) \to 0$, we get so $\rho(x_\alpha - x_\sigma) \to 0$. This means $\rho(\varphi_\rho(x_\alpha) - \varphi_\rho(x_\sigma)) \to 0$, from which it follows that $\varphi_\rho(x_\sigma) = x_\rho$. QED

Now we prove the Nakano theorem ([1], p.90, Theorem 13.1).

3 Theorem. *Every Cauchy net $\{x_\alpha\} \subset P$ is convergent in P.*

Proof. We first assume that $x_\alpha \geq 0$. By Corollary 2, for every $\sigma \in D$, we get an $x_\sigma \in P$ and $x_\sigma \uparrow$. Put $x = \sup x_\sigma$. We claim that $x_\alpha \to x$: Fix a $\rho \in D$. Now $\varphi_\rho(x_\sigma) \uparrow \varphi_\rho(x)$. By Corollary 2, for any $\sigma \geq \rho$, $\varphi_\rho(x_\sigma) = x_\rho$ and so we get

$\varphi_\rho(x) = x_\rho$. So we have $\rho(x_\alpha - x) = \rho(\varphi_\rho(x_\alpha) - \varphi_\rho(x)) = \rho(\varphi_\rho(x_\alpha) - x_\rho) \to 0$, by Corollary 2.

For the general case, one has only to note that if $\{x_\alpha\}$ is a Cauchy net then $\{x_\alpha^+\}$ and $\{x_\alpha^-\}$ are also Cauchy nets. QED

The similar method can be used to prove a well-known property for Hausdorff, Dedekind complete, linear, locally solid, Riesz space with Lebesgue property. We do it in the next theorem.

4 Theorem. *Suppose $(E, \mathcal{T})$ has Lebesgue property and $\mathcal{T}_0$ be another linear, locally solid topology on E with Lebesgue property. Then, on P, $\mathcal{T} \geq \mathcal{T}_0$ ([1], Theorem 12.9, p. 87).*

Proof. As used above, we take $\{\rho : \rho \in D\}$ to be a filtering upwards family of pseudo-norms generating the topology of $(E, \mathcal{T})$. Take a net $\{x_\alpha\} \subset P, x_\alpha \geq 0$ and assume that $x_\alpha \to 0$ in $\mathcal{T}$ but not in $\mathcal{T}_0$. Take a 0-nbd V in $\mathcal{T}_0$; we can assume that $x_\alpha \notin V$, $\forall\alpha$. Take another 0-nbd U in $\mathcal{T}_0$ such that $U + U \subset V$. Since $(\cup\{A_\rho^d : \rho \in D\})^d = \cap_{\rho\in D} A_\rho = \{0\}$ (note $\mathcal{T}$ is Hausdorff and $\cup\{A_\rho^d : \rho \in D\}$ is an ideal in E), the closure, in $\mathcal{T}_0$, of $\cup\{A_\rho^d : \rho \in D\}$, is a band and is equal to E. So take a $\rho \in D$ and an $e_0 \in A_\rho^d$ such that $0 < e_0 \leq e$ and $e - e_0 \in U$ (note $\cup A_\rho^d$ is a dense ideal in E and $\{A_\rho^d : \rho \in D\}$ is filtering upwards). Now $x_\alpha \wedge e_0 \to 0$ in $(E, \mathcal{T})$, $x_\alpha \wedge e_0 \in A_\rho^d$ and (A_ρ^d, ρ) metrizable space. By Lemma 1, there is a sequence $\{x_{\alpha_n} \wedge e_0\}$ which order converges to 0 in E. Since $(E, \mathcal{T}_0)$ is Lebesgue, we get $x_{\alpha_n} \wedge e_0$ converges to 0 in$(E, \mathcal{T}_0)$. So from some n onwards, $x_{\alpha_n} \wedge e_0 \in U$. Now $x_{\alpha_n} = x_{\alpha_n} \wedge e \leq (x_{\alpha_n} + e - e_0) \wedge (e - e_0 + e_0) \leq (e - e_0) + x_{\alpha_n} \wedge e_0 \in U + U \subset V$ which is a contradiction.

The general case of x_α can be reduced to the positive case by taking x_α^+ and x_α^-. QED

The following corollary follows immediately from this theorem.

5 Corollary. *Let $\mathcal{T}$ and $\mathcal{T}_0$ be two Hausdorff, Dedekind complete, linear, locally solid topologies, with Lebesgue property, on a Riesz space E. Then, on P, $\mathcal{T} = \mathcal{T}_0$ ([1], Theorem 12.9, p. 87).*

Now we come to another well-known result about P. We give a measure-theoretic proof.

6 Theorem. *If $(E, \mathcal{T})$ has Lebesgue property, then P is weakly compact and E is a regular Riesz subspace of E''.*

Proof. Here $e \in E$, $e > 0$, and $P = \{y \in E : |y| \leq e\}$; take $E_0 = \{y \in E : |y| \leq ne$ for some $n \in N\}$. E_0 is a band in E and is a closed subspace of E. With the norm on E_0, $\|y\|_0 = \inf\{\lambda \geq 0 : |y| \leq \lambda e\}$, E_0 is an M-space with unit e and so, as a complete lattice, can be identified with $C(X)$ for a compact Stonian space X. Also it is a simple verification that $\|.\|_0$-topology is finer than $\mathcal{T}$-topology. Take a $\mu \in (E, \mathcal{T})'$. Now $|\mu|$ is a positive linear functional on $C(X)$ and so it extends to a positive regular Borel measure on X. Since $|\mu|$ is order continuous (note $(E, \mathcal{T})$ has Lebesgue property), for any closed set C with empty interior, $|\mu|(C) = 0$: to prove this, let $\{f_\alpha\} \subset C(X)$, $f_\alpha \downarrow \chi_C$; since C has empty interior and $C(X)$ is Dedekind complete, we get $f_\alpha \downarrow 0$ in $C(X)$, and so $|\mu|(C) = \lim |\mu|(f_\alpha) = 0$ (note $|\mu|$ is order continuous). From this it follows that $|\mu|(B) = 0$ for any meagre Borel set B. Denoting by $\beta(X)$ the set of all bounded Borel measurable functions on X, we get linear, positive, order σ-continuous mapping $\psi : \beta(X) \to C(X)$ with the property that if $\{f_\alpha\}$ is a bounded, increasing net in $C(X)$ with pointwise $\sup f_\alpha = f$, then $\psi(f) = \sup \psi(f_\alpha)$ in $C(X)$ ([5], Lemma 2, p. 379; note for $f \in C(X)$, we have $\psi(f) = f$ and, in general $\psi(f) = f$ except on a meager subset of X) and $|\mu|(g) = |\mu|(\psi(g)), \forall g \in \beta(X)$. Let $B = \{g \in \beta(X) : -1 \leq g \leq 1\}$ and $B_0 = \psi(B)$. By Hahn decomposition theorem, $X = A \cup A_1$, where A, A_1 are disjoint, positive and negative Borel subsets of X for μ ([8], p. 273). Thus $\mu = (\chi_A - \chi_{A_1})|\mu|$. Now the maximum value of μ on B is $|\mu|(1) = \int (\chi_A - \chi_{A_1}) d\mu$. Thus μ takes its maximum on B at $(\chi_A - \chi_{A_1}) \in B$, and therefore also on $\psi(\chi_A - \chi_{A_1}) \in P$. Now by Theorem 3, P is complete in $(E, \mathcal{T})$ and also every $f \in E'$ attains its maximum in P; by James theorem ([4], Theorem 6, p. 139), P is weakly compact.

Now we prove that E is a regular Riesz subspace of E''. Naturally E is a Riesz subspace of E''. Assume $0 \leq x_\alpha \uparrow e$ in E and there is a $x'' \in E''$ such that $x'' < e$ and $x_\alpha \leq x''$, $\forall \alpha$. This means $\{x_\alpha\} \subset P$ and $x'' \notin P$. Since P is weakly compact and convex, by separation theorem ([10], 9.2, p.65), there is an $\mu \in E'$ such that $< x'', \mu > > \sup\{\mu(g) : g \in P\} = |\mu|(e)$ (note P is solid). Now $< x'', \mu > \leq < x'', |\mu| > \leq < e, |\mu| >$, a contradiction. This proves the result. QED

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