in characteristic $p > 0$, there is different behaviour; for example, $\mathscr{U}_{2,q}$ has 28 <u>undulations</u> (points where the tangent has 4-point contact). When $g=4$, the curve $\mathscr{C}^6 = \mathscr{F}^3 \cap \mathscr{F}^2$, the intersection of a cubic and a quadric surface, has 60 <u>stalls</u> where the osculating plane meets the curve at four coincident points.

More generally, still with characteristic zero, if $\mathscr{C}$ has genus $g \geq 1$ and $P \in \mathscr{C}$, there exist integers $n_1, n_2, \ldots, n_g$ such that no function has pole divisor precisely $n_i P$. Also $\{n_1, n_2, \ldots, n_g\} = \{1, 2, \ldots, g\}$ for all but a finite number of points. We elaborate this idea and make it more precise in §§8-10.

6. FUNDAMENTAL DEFINITIONS IN ALGEBRAIC GEOMETRY

Let $\mathscr{C} \subset A^n(K)$ be an irreducible non-singular algebraic curve defined over K, let $I(\mathscr{C}) \subset K[X_1, \ldots, X_n]$ be the ideal of polynomials wich are zero at all points of $\mathscr{C}$, let $\Gamma(\mathscr{C}) = K = [X_1, \ldots, X_n]/I(\mathscr{C})$; and $K(\mathscr{C})$ be the quotient field of $\Gamma(\mathscr{C})$; then $K(\mathscr{C})$ is called the <u>function field</u> of $\mathscr{C}$. Also, for P in $\mathscr{C}$, let $O_P = \{f/g \mid f, g \in \Gamma, g(P) \neq 0\}$, the <u>local ring</u> of $\mathscr{C}$ at P. Then, by natural inclusions, $K \subset \Gamma(\mathscr{C}) \subset O_P(\mathscr{C}) \subset K(\mathscr{C})$. Also $O_P \setminus \{\text{units}\} = M_P = \langle t \rangle$, the maximal ideal, and for any z in O_P there exist a unique unit u and a unique non-negative integer m such that $z = ut^m$; write $m = \mathrm{ord}_P(z)$. Hence, if $G \in K[X_1, \ldots, X_n]$ and g is the image of G in $\Gamma(\mathscr{C})$ with $G(P) \neq 0$, define $\mathrm{ord}_P(G) = \mathrm{ord}_P(g)$. In particular, if $\mathscr{C}$ is a plane curve and V(L) the tangent at P, then $\mathrm{ord}_P(L)$ gives the multiplicity of contact of the tangent with $\mathscr{C}$.

For the extension of these definitions to the projective case, see Fulton [3], p.182. This is the situation we now consider.

A <u>divisor</u> D on $\mathscr{C}$ is $D = \sum_{P\in\mathscr{C}} n_P P$, $n_P \in \mathbf{Z}$, with $n_P=0$ for all but a finite number of points P; the <u>degree</u> of D is $\deg D = \Sigma n_P$. Then D is <u>effective</u> if $n_P \geq 0$ for all P. For z in $K(\mathscr{C})$, define

$$\begin{aligned} \operatorname{div}(z) &= \operatorname{ord}_P(z)P \\ &= (z)_0 - (z)_\infty , \end{aligned}$$

where

$$(z)_0 = \sum_{\operatorname{ord}(z)>0} \operatorname{ord}_P(z)P, \text{ the <u>divisor of zeros</u>,}$$

and

$$(z)_\infty = \sum_{\operatorname{ord}(z)<0} - \operatorname{ord}_P(z)P, \text{ <u>the divisor of poles</u>;}$$

that is, div(z) is the difference of two effective divisors and $\deg \operatorname{div}(z) = 0$.

Given $D = \Sigma n_P P$, define

$$L(D) = \{ f\in K(\mathscr{C}) \,|\, \operatorname{ord}_P(f) \geq - n_P, \ \forall P \} ;$$

that is, poles of f are no worse than n_P. In other words, $f \in L(D)$ if $f=0$ or if $\operatorname{div}(f) + D$ is effective.

The set L(D) is a vector space and its dimension is denoted $\ell(D)$.

There is an important equivalence relation on the divisors given by $D \sim D'$ if there exists g in $K(\mathscr{C})$ such that $D - D' = \operatorname{div}(g)$.