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Summary. *These notes try to give a flavor of some recent research which has elucidated some unexpected connections between two apparently unrelated topics in Functional Analysis, namely, inverse function theorems and the structure theory of Fréchet spaces.*

This article is based on a series of lectures which I gave in December, 1981 at Università degli Studi di Lecce by invitation of Prof. V.B.Moscatelli for whose hospitality I am very grateful.

In these notes I would like to be rather informal, trying to give a flavor of some recent research which has elucidated some unexpected connections between two apparently unrelated topics in Functional Analysis. For the many details which will be omitted, I refer to standard texts and/or the references at the end.

INTRODUCTION.

A Fréchet space is a complete metrizable locally convex space. We will consider some details later and the reader can consult [3] for a basic discussion of these spaces, but for now I would like to mention three function spaces which are examples: $C^{\infty}(T)$, $H(\mathbb{C})$, $H(\mathbb{D})$. They are, respectively: the space of infinitely differentiable functions on the unit circle with the topology of uniform convergence of each derivative, the space of functions analytic in the complex plane with the compact - open topology and the space of functions analytic in the open unit disk in the compact - open topology.

An interesting problem, which has connections to partial differential equations and other functional equations, arises from consideration of a function $F : U \to E$ where U is a neighborhood of 0 in a Fréchet space E and $f(0) = 0$. The question is: if y is "small enough" can we always solve the equation $f(x) = y$? Putting it another way, we ask if $f(U)$ is again a neighborhood of 0 in E. As we will see,

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serious difficulties arise when we try to study this situation in the context of a general Fréchet space.

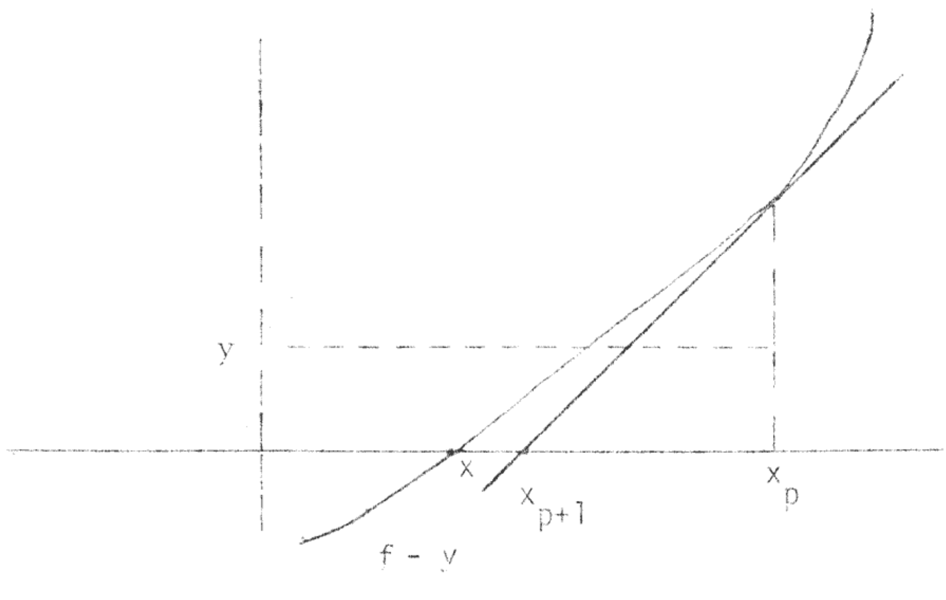
A totally different object of investigation is the structure of Fréchet spaces. There we consider a fixed space E and try to determine (up to isomorphism) all of its subspaces and quotient spaces. There are many other similar kinds of questions and this turns out to be rich area of study.

It is a little bit surprising that there are important connections between these two areas. These are being discovered in various current research activities and it is my main purpose in these lectures to describe some of them. Thus, the discussion will be divided into three parts: inverse function theorems, structure theory, and connections.

INVERSE FUNCTION THEOREMS.

We begin with $f : U \to E$ with $f(0) = 0$ and we want to solve $f(x) = y$ for small y. Of course, there are important related questions. Is the solution unique? Does it depend continuously on parameters? And so on. There are interesting things to say about such questions but, in these lectures, I will consider only the existence problem.

Our basic approach to solving $f(x) = y$ will be Newton's method. This works equally well when E is 1-dimensional, n-dimensional or even an infinite dimensional Banach space. The following picture describes the 1-dimensional situation but leads to formulas which work in the more general context:



The idea is to set us the recursion,

(1) $$x_{p+1} = x_p - ((f'(x_p))^{-1}(y-f(x_p)) \qquad (p\geq 0) \quad x_0 = 0$$

It is clear that if $\lim_p x_p = x$ and f is continuous, then $f(x) = y$.

Before we can use such a formula it is necessary to have a theory of differentiation which works in a Banach space. This is another vast subject and eventually any investigation into non-linear phenomena will have to deal with it extensively. For these lectures we take the short-cut of using the simplest definition and appealing to various regularity conditions (which we will not state explicitly) that imply, in our context, that all definitions are equivalent. This same definition can and will be used when E is a Fréchet space.

Thus we define the derivative of f at $x \in U$ to be the continuous linear function $f'(x) : E \to E$ which satifies :

$$f'(x)v = \lim_{t\to o} \frac{f(x+tv)-f(x)}{t} \qquad (x\in U \ , \ v\in E).$$

We then have the following result (see [8] for a proof).

THEOREM 1.

If E *is a Banach space and* $f'(0)$ *is invertible, then* $f(U)$ *is a neighborhood of* 0.

This is a very nice result and has important applications in partial differential equations. Unfortunately (and this has implications for the applications) nothing so broad is true in Fréchet spaces. It is useful to try to understand what goes wrong.

A first difficulty is that in Banach spaces it suffices to assume that $f'(0)$ is invertible because this implies that there is a whole neighborhood of 0, $W \subset U$, such that $f'(x)$ is invertible for every $x\in W$ which latter property is what is really needed. It is not hard to construct examples (I think there will be one in almost every non- Banach Fréchet space) that show that no such implication holds.

Actually, this is only a minor annoyance because no important examples are lost if we go the whole way and simply assume that $f'(x)$ is invertible for all x in some neighborhood of 0. Unfortunately, as the following example shows, this is still not enough.

Let E be the Fréchet space $C(\mathbb{R})$ of continuous real- valued functions on the real line $\mathbb{R}$ with the compact-open topology and let $f : E \to E$ be defined by taking $f(x)(t) = e^{x(t)} - 1$. Then $f(0) = 0$ and f is as regular as it could be. Moreover, it is easy to compute $f'(x)$ to obtain $f'(x)v = e^{x}v$ so that the inverse of $f'(x)$ is $f'(-x)$. On the other hand, any neighborhood of 0 will contain functions which take on values less than -1, but this is not possible for a function in the range of f.

We will try to analyse more closely what is going wrong, with the goal of getting some idea how to deal with this apparently chaotic situation. Let us see what in the proof of Theorem 1 does not work when we pass to Fréchet spaces.

Once the existence of an inverse of $f'(x)$, $x \in W$, is established there are two remaining issues in the proof of Theorem 1. First, we must guarantee that $x_p \in W$ so that the formula for x_{p+1} can be used and second, once the sequence (x_p) is defined, we must show that it converges or, at least, is Cauchy. Both concerns are dealt with using the same basic calculation:

$$f(x_p)-y = f(x_p)-f(x_{p-1})-f'(x_{p-1})(x_p-x_{p-1})$$

$$= \frac{1}{2}\int_0^1 f''(x_{p-1} + t(x_p-x_{p-1}))(x_p-x_{p-1})^2 dt.$$

With appropriate (and reasonable) regularity assumptions on f, this leads to the existence of positive constants C and δ with

$$\| f(x_p) - y\| \leq C\| x_p - x_{p-1}\|^2$$

and

$$\| x_{p+1} - x_p \| \le C \| x_p - x_{p-1} \|^2 \le \delta \| x_p - x_{p-1} \| .$$

This means that if $x_p - x_{p-1}$ is sufficiently small, then x_{p+1} will stay in W and $x_{p+1} - x_p$ will be even smaller. Thus it suffices to make $y - f(x_0) = y$ sufficiently small.

In a Fréchet space, however, the topology is defined by a sequence of norms $(\| \cdot \|_k)$ so that considerations of the above type lead only to relations of the form

$$(2) \qquad \| x_{p+1} - x_p \|_k \le \delta \| x_p - x_{p-1} \|_{\sigma(k)} ,$$

where σ is a function determined by f and which is usually growing quite rapidly with k. Unfurtunately, if $\sigma(k)$ is much larger than k, the interation at each step leads to information about fewer norms and after finitely many steps, $\sigma(k) = 0$ and we have no information at all. Restrictions on the growth of σ are quite rare in the study of Fréchet spaces.

If $E = C^\infty(T)$ and f is a partial differential operator, then σ is related to the order of the operator and $\| \cdot \|_k$ is calculated in terms of the first k derivates. For this reason we call the function σ the *loss of derivatives function*. One of my major points in these lectures is that many phenomena occuring in the theory of Fréchet spaces can be related to this function, both conceptually and in the actual details of calculations.

In the case of the inverse function theorem, there is a method for dealing with the loss of derivatives. It is called the Nash-Moser method and it attacks the problem directly by using an additional structure with which a Fréchet space may be equipped.

Let $(S_\Theta)_{\Theta \ge 0}$ be a family of continuous linear operators, $S_\Theta : E \to E$, on a Fréchet space E, which satisfy the following conditions for $k \le j$, $x \in E$, $\Theta > 0$ and an appropriate constant C which depends only on k, j but not on x, Θ:

(3)
$$\| S_\theta x \|_j \leq C\theta^{j-k} \| x \|_k$$
$$\| x - S_\theta x \|_k \leq C\theta^{k-j} \| x \|_j \quad .$$

Here $(\| \cdot \|_k)$ is an increasing sequence of seminorms which define the topology of E. (This will be discussed a little more in the next section). We refer to (S_θ) as a *family of smoothing operators which satisfies* (3).

The recursion relation (1) is just changed to

(4) $$x_{p+1} = x_p - S_{\theta_p}(f'(x_p))^{-1}(y-f(x_p)) \qquad (p \geq 0) \quad x_0 = 0,$$

where the sequence (θ_p) must be chosen for the convenience of further calculations. For example, it no longer follows necessarily that, if (4) is used and (x_p) converges, then the limit is a solution of $f(x) = y$. This will be guaranteed by the second inequality in (3) provided $\lim_{p\to\infty} \theta_p = \infty$. The calculations leading to (2) are then repeated by using the first relation in (3) to cancel to effect of σ. This turns out to be fairly delicate and requires that θ_p does not grow too rapidly so that a balance must be struck. A more serious restriction is that nothing works unless there is a quite severe control on the growth of σ.

Nevertheless, it is possible to push through the calculations and we do get a theorem which, although very special, does have many important applications. The original idea is due to J. Nash [7], but J. Moser [6] was the first to realize how useful it could be. Subsequent refinements have been made by many authors, especially R. Hamilton, S. Łojasiewcz and E. Zehnder. The version given here is due jointly to the last two authors [4].

THEOREM 2.

Let E *be a Fréchet space which has a family of smoothing operators which satisfy* (3) *and let* $f : U \to E$ *be a continuous function on a neighborhood* U *of* 0 *in* E *which has a derivative at each point in* U. *Let* $f(0) = 0$ *and assume that there exist* $d > 0$ *and* $\lambda \in [1,2)$ *such that for all* $k \geq 0$ *we* *have constants* $C_k > 0$ *with*

$$\| f(x) \|_k \leq C_k \| x \|_{k+d} \qquad (x \in U)$$

$$\| f'(x)v \|_k \leq C_k(\| x \|_{k+d} \| v \|_o + \| v \|_{k+d}) \qquad (x \in U,\ v \in E)$$

$$\| f(x+v) - f(x) - f'(x)v \|_k \leq C_k(\| x \|_{k+d} \| v \|_o^2 + \| v \|_{k+d} \| v \|_o) \qquad (x, x+v \in U).$$

Suppose moreover that $f'(x)$ *has a right inverse* $L(x)$ *for each* $x \in U$ *and*

$$\| L(x)y \|_k \leq C_k(\| x \|_{k+d} \| y \|_d + \| y \|_{\lambda k+d}) \qquad (x \in U,\ y \in U)$$

(here $(\| \cdot \|_k)_{k \geq 0}$ *is again an increasing sequence of seminorms which defines the topology of* E*).*

Then $f(U)$ *is a neighborhood of* 0.

It is interesting to note that the requirement $\lambda<2$ in Theorem 2 is essential. In fact, in [4] there is given an example in which all of the hypotheses of the theorem hold except that $\lambda = 2$ and the conclusion of the theorem is false!

Of course, in order to even think of applying Theorem 2 it is necessary to consider how the smoothing operators might be constructed. In the original applications of Nash and Moser, E is always $C^\infty(T)$ and the smoothing operators are obtained either by convolution or the truncation of Fourier series.

In looking over the literature on this subject, it seemed curious to me that although many authors postulated wide classes of spaces for which Theorem 2 could be used, the concrete examples of Fréchet spaces which were actually written down were almost invariably (up to isomorphism) $C^\infty(T)$ or a space closely related to it.

This is in fact the case even when it did not appear so. For example, in [4] the authors use the Fréchet space $H(\mathbb{C})$ of entire functions in one variable. But this space is what we shall call later a "coordinate subspace" of $C^\infty(T)$ and in that case, analysis just carries over. It is like the situation in which you have an inverse function theorem valid for functions in three variables. It is then trivial to obtain (by holding one variable constant) a similar theorem for functions of two variables.

In any case, I tried to see if one could use a result like Theorem 2 for Fréchet spaces very different from $C^{\infty}(T)$. As we will see, the restriction to essentially this one space was no accident and it is necessary to change things quite a bit if we want to find an implicit function theorem valid in different kinds of Fréchet space.

Before we can get very far with such a program, it is necessary to say something about these spaces.

STRUCTURE OF FRÉCHET SPACES.

Recall that a Fréchet space is a complete, metrizable, locally convex space. Equivalently, it is a vector space E which is complete under a certain translation invariant metric and on which is defined an increasing sequence of seminorms (sub-additive, positive scalar homogeneous real-valued functions) $(\|\cdot\|_k)_{k\geq 0}$ such that a sequence (x_n) in E converges to x in E iff $\lim_n \|x_n - x\|_k = 0$ $(k=0,1,2,\ldots)$.

In all of our applications we will take the seminorms $\|\cdot\|_k$ to be norms (that is, $\|x\|_k = 0$ iff $x = 0$). Somewhat more complicated is the fact that the Fréchet spaces we consider will all be nuclear. It is best to defer the definition of nuclear until we are in a more concrete situation.

The basic references for all of our discussion of the structure of nuclear Fréchet spaces will be [1],[5] and [10]. For us, the best starting point is to list some examples:

$C^\infty(T)$ - *The space of infinitely differentiable real-valued functions on the unit circle equipped with the topology of uniform convergence of each derivative.*

$H(\mathbb{C})$ - *The space of complex-valued functions of one complex variable, analytic in the complex plane, equipped with the compact-open topology.*

$H(\mathbb{D})$ - *The space of complex-valued functions of one complex variable, analytic in the open unit disk, equipped with the compact-open topology.*

For each of these spaces we give one possible choice of the norms which define the topology.

$C^\infty(T) : \|x\|_k = \sup\{|x^p(t)| : p = 0, \ldots, k, \quad t \in T\}$

$H(\mathbb{C}) : \|x\|_k = \sup\{|x(t)| : |t| \leq k\}$

$H(\mathbb{D}) : \|x\|_k = \sup\{|x(t)| : |t| \leq \frac{k}{k+1}$

Each of these examples has what is called a coordinate representation or basis. For example, in $C^\infty(T)$ we can represent functions by their Fourier series and so write,

$$C^\infty(T) = \{\xi = (\xi_n) : (\xi_n) \text{ is the sequence of Fourier coefficients of an element of } C^\infty(T)\} =$$

$$= \{\xi = (\xi_n) : \sup_n |\xi_n| e^{k \log n} < \infty, \qquad k=0,1,2,\ldots\},$$

where the second equality represents a standard fact about asymptotic behavior of Fourier coefficients of differentiable functions. More is true. If we set $\|x\|_k = \sup_n |\xi_n| e^{k \log n}$, where $x \in C^\infty(T)$ and (ξ_n) is its sequence of Fourier coefficients, then this definition of $\|\cdot\|_k$ works just as well to define the topology of $C^\infty(T)$.

The same thing can be done for $H(\mathbb{C})$ and $H(\mathbb{D})$ using power series expansions.

We obtain,

$$H(\mathbb{C}) = \{\xi = (\xi_n) : \|\xi\|_k = \sup_n |\xi_n|\, e^{kn} < \infty, \qquad k=0,1,2,\dots\}$$

$$H(\mathbb{D}) = \{\xi = (\xi_n) : \|\xi\|_k = \sup_n |\xi_n|\, e^{-\frac{n}{k}} < \infty, \qquad k=0,1,2,\dots\}$$

Clearly we can abstract all this by writing down an infinite matrix of positive numbers $a = (a_{nk})$ instead of $e^{k \log n}$, e^{kn} or $e^{-\frac{n}{k}}$.

Then the Fréchet space we obtain is given by

$$K(a) = \{\xi = (\xi_n) : \|\xi\|_k = \sup_n |\xi_n|\, a_{nk} < \infty, \qquad k=0,1,2,\dots\}$$

and we only need assume that $a_{nk} \leq a_{n,k+1}$. The condition that $K(a)$ is nuclear can then be expressed as follows :

$$\forall k \quad \exists j \quad \text{such that} \quad \sum_n \frac{a_{nk}}{a_{nj}} < \infty$$

If e_n is the sequence which is 0 except in the n^{th} coordinate where it is 1, then each element of $K(a)$ can be expanded as an infinte series in (e_n) and considering the coefficient sequences gives $K(a)$ back.

It is a simple but informative exercise to verify the details of this general formulation for the three examples mentioned above.

The notion of coordinate representation leads to another notion that permits us to construct other examples of nuclear Fréchet spaces. Let $\alpha = (\alpha_n)$ be a subsequence of the sequence $\mathbb{N}$ of positive integers. If E is a nuclear Fréchet space given via a coordinate representation, then $(E)_\alpha$ denotes all of those elements of E whose corresponding sequence ξ has the property that $\xi_n = 0$ unless n is one of the terms of α.

Thus $(K(a))_\alpha$ is $K(b)$ where $b_{nk} = a_{\alpha_n,k}$.

It is clear that $(E)_\alpha$ is again a nuclear Frechêt space. It is complemented in the sense that, if $\beta = N \sim \alpha$, then E is isomorphic to the product $(E)_\alpha x(E)_\beta$. We call $(E)_\alpha$ a coordinate subspace of E.

This completes the preliminaries for the structure theory and now I would like to say something about the content of this theory. Generally speaking the questions considered are of the following type. Given a nuclear Fréchet space E and another one F, find quantitative conditions which determine whether F is isomorphic to a subspace of E. Usually E is a space which can be given in a coordinate representation, say E = K(a). This may or may not be the case for F. If it is, say F is isomorphic to K(b), then the condition is given in terms of the matrices a,b. If not, then the condition is in terms of the norms.

The structure theory involves much more than I have mentioned. It is possible to replace subspace by quotient, or even complemented subspace. There are investigations which try to determine what effect this has in guaranteeing that F has a coordinate representation. Other approximation properties have been studied and there is a lot of work in determining when concrete function spaces fall under this theory. I have not tried to give a serious bibliography here. Many results are quite recent and just now beginning to find their way into print.

It will be useful for us to go a bit beyond these generalities and to give at least one definite example of how the structure theory works. We consider the possibility that a space K(b) is isomorphic to a subspace of $C^\infty(T)$. We will use the fact (mentioned above) that $C^\infty(T)$ is isomorphic to K(a) where $a_{nk} = e^{k \log n} = n^k$.

A necessary condition can be derived quite easily as follow. We have an isomorphism $A : K(b) \to K(a)$ so (after passing to a subsequence on k if necessary) we can write

$$\|x\|_k \le C_k^1 \|Ax\|_{k+1} \le C_k^2 \|x\|_{k+2} \qquad (x \in K(b),\ k=1,2,\dots).$$

Applying this with $x = e_n$ we can derive

$$\frac{1}{C_k^3} \frac{\|Ae_n\|_{k+1}}{\|Ae_n\|_k} \le \frac{\|e_n\|_{k+2}}{\|e_n\|_{k-1}} = \frac{b_{n,k+2}}{b_{n,k-1}} \le C_k^4 \frac{\|Ae_n\|_{k+3}}{\|Ae_n\|_{k-2}} \qquad (n,k=1,2,3,\dots)$$

Now if Ae_n is the sequence $(\xi_\nu^n)_\nu$ in $K(a)$ we have

$$\|Ae_n\|_k = \sup_\nu |\xi_\nu^n| \nu^k = |\xi_{g_n^k}| (g_n^k)^k ,$$

where g_n^k is the largest value of ν at which the sup occurs. (We use here the fact that $\sup_\nu |\xi_\nu^n| \nu^k < \infty$ for every k implies $\lim_\nu |\xi_\nu^n| \nu^k = 0$).

Now using the property of sup we have, for any k,j,

$$\frac{|\xi_{g_n^k}|(g_n^k)^j}{|\xi_{g_n^k}|(g_n^k)^k} \le \frac{|\xi_{g_n^j}|(g_n^j)^j}{|\xi_{g_n^k}|(g_n^k)^k} \le \frac{|\xi_{g_n^j}|(g_n^j)^j}{|\xi_{g_n^j}|(g_n^j)^k}$$

from which we conclude that

$$(g_n^k)^{j-k} \le \frac{\|Ae_n\|_j}{\|Ae_n\|_k} \le (g_n^j)^{j-k} .$$

Together with our first inequality, and writing only C for any positive constant independent of n, we obtain,

$$\frac{1}{C} \frac{b_{n,k+2}}{b_{n,k-1}} \le (g^{k+3})^5 \le \frac{\|Ae_n\|_{k+8}}{\|Ae_n\|_{k+3}} \le C \frac{b_{n,k+9}}{b_{n,k+2}}$$

which, passing to a subsequence, gives

$$\frac{b_{nk}}{b_{n,k-1}} \leq C \frac{b_{n,k+1}}{b_{nk}} .$$

It is very interesting that this simple condition which we derived is also sufficient and we have,

THEOREM 3.

A *nuclear Fréchet space* E *with a coordinate representation* $K(b)$ *is isomorphic to a subspace of* $C^{\infty}(T)$ *iff, after passing to an appropriate subsequence on* k *, we have, for every* k ,

$$\sup_n \frac{(b_{nk})^2}{b_{n,k-1}\, b_{n,k+1}} < \infty .$$

A proof of this and many similar results can be found in [1]. What is even more striking is that results of this kind with equally simple conditions can be obtained without the assumption that E has a coordinate representation (see [9], [11] and [12]).

One important consequence of this characterization is that neither $H(\mathbb{D})$ nor any of its coordinate subspaces is isomorphic to a subspace of $C^{\infty}(T)$. To see this we use the fact that any coordinate subspace of $H(\mathbb{D})$ is isomorphic to a space $K(b)$ where $b_{nk} = e^{-\frac{\alpha_n}{j(k)}}$ and $(j(k))$ is any subsequence of $\mathbb{N}$.

This gives,

$$\frac{(b_{n,j(k)})^2}{b_{n,j(k-1)}\, b_{n,j(k+1)}} = e^{\alpha_n \left(\frac{1}{j(k-1)} + \frac{1}{j(k+1)} - \frac{2}{j(k)}\right)} \geq e^{\alpha_n\left(\frac{1}{j(k-1)} - \frac{2}{j(k)}\right)} .$$

Thus, if $j(k) > 2j(k-1)$ this quantity is unbounded and passing to a subsequence on k will not help. Therefore the condition of Theorem 3 is violated.

CONNECTIONS. -

Now we turn to the main topic of these lectures which is the description of certain connections between the structure theory and inverse function theorems. Our first remark is an observation that shows how special is the relation (3) which the smoothing operators are assumed to satisfy.

If we have (3), then any $x \in E$, $\Theta > 0$ and $k < j < \ell$ we would have,

$$\| x \|_j \leq \| S_\Theta x \|_j + \| x - S_\Theta x \|_j \leq C(\Theta^{j-k} \| x \|_k + \Theta^{j-\ell} \| x \|_\ell).$$

We can use calculus to show that the right hand side achieves its minimum value when

$$\Theta = \left(\frac{\| x \|_\ell \, (\ell - j)}{\| x \|_k \, (j-k)} \right)^{\frac{1}{\ell - k}} ,$$

and substituting this value for Θ with $k = j-1$, $\ell = j+1$ yields

$$(\| x \|_j)^2 \leq C \, \| x \|_{j-1} \, \| x \|_{j+1} .$$

This immediately implies the condition of Theorem 3 so, with the above discussion, we may conclude that Theorem 2 does not hold for $H(\mathbb{D})$. Actually we have the following much stronger result of D. Vogt.

THEOREM 4. -

A nuclear Fréchet space E *has a family of smoothing operators satisfyng* (3) *if and only if* E *is isomorphic to a coordinate subspace of* $C^\infty(T)$.

Thus we see that Theorem 3 as presently formulated is not applicable to a very wide class of Fréchet spaces. It turns out, however, that if we look a little more closely at how the smoothing operators can be defined, the relation (3) can be derived. If we do this for $C^{\infty}(T)$ or $H(\mathbb{C})$ we get (3), but if we do it for another space, say $H(\mathbb{D})$, we get a different relation which can then be used to prove an inverse function theorem valid for $H(\mathbb{D})$.

There is a method of constructing the smoothing operators. We begin with a nuclear Fréchet space $K(a)$ given in a coordinate representation. If (e_n) is the usual sequence of sequences which are 1 at the n^{th} coordinate and 0 elsewhere, then each $\xi = (\xi_n)$ can be written $\xi = \sum_n \xi_n e_n$, where this series converges in the topology of $K(a)$. Then we define $S_\Theta : K(a) \to K(a)$, $\Theta > 0$, by

$$S_\Theta \xi = \sum_{n \leq \Theta} \xi_n e_n$$

and we may calculate for, $k \leq j$,

$$\| S_\Theta \xi \|_j = \sup_{n \leq \Theta} |\xi_n| \, a_n^j = \sup_{n \leq \Theta} |\xi_n| \, a_n^k \, \frac{a_n^j}{a_n^k} \leq \left(\sup_{n \leq \Theta} \frac{a_n^j}{a_n^k}\right) \| \xi \|_k .$$

A similar calculation for $\xi - S_\Theta \xi$ yields

$$\| \xi - S_\Theta \xi \|_k \leq \left(\sup_{n > \Theta} \frac{a_n^k}{a_n^j}\right) \| \xi \|_j .$$

If our space is $C^{\infty}(T)$ then we can take $a_n^k = n^k$, so we obtain exactly (3). If the space is $H(\mathbb{C})$ then (3) still holds since $H(\mathbb{C})$ is a coordinate subspace of $C^{\infty}(T)$. If we use the above derivation for $H(\mathbb{C})$, we get a different inequality which could still be used to prove Theorem 3. Actually, it is not so different. We get the same inequality as in (3) except that Θ is replaced by e^{Θ}. Thus if we replace S_Θ by $S_{\log\Theta}$ we get a family of smoothing operators for $H(\mathbb{C})$ satisfying (3).

For $H(\mathbb{D})$ however, this is impossible as we have seen. The inequalities we get are

$$\| S_\Theta x\|_j \leq C e^{(\frac{1}{k} - \frac{1}{j})\Theta} \| x\|_k$$

$$\| x-S_\Theta x\|_k \leq C e^{(\frac{1}{j} - \frac{1}{k})\Theta} \| x\|_j .$$

Again replacing Θ by $\log\Theta$ we have our condition as follows. For $k \leq j$, $x\epsilon E$, $\Theta > 0$ and an appropriate constant C which depends only on k,j but not on x,Θ there exists a family of smoothing operators S_Θ such that

$$\text{(3')} \qquad \begin{aligned} \| S_\Theta x\|_j &\leq C\Theta^{(\frac{1}{k} - \frac{1}{j})} \| x\|_k \\ \| x-S_\Theta x\|_k &\leq C\,\Theta^{(\frac{1}{j} - \frac{1}{k})} \| x\|_j . \end{aligned}$$

Thus we see that $H(\mathbb{D})$ does have a family of smoothing operators satisfying (3').

The ultimate goal then would be to find for each $K(a)$ a relation like (3) or (3') and then use it to prove a theorem like Theorem 2. This turns out to be not so easy. So far I have only been able to do this for (3') and, although the proof is not given so we cannot see the actual difficulties as they arise, a clue will be provided by the form of the statement of Theorem 2' and how it differs from Theorem 2.

THEOREM 2'. -

Let E *be a Fréchet space which has a family of smoothing operators which satisfy* (3') *and let* $f : U \to E$ *be a continuous function on a neighborhood of* 0, U *in* E *which has a derivative at each point in* U.

Let $f(0) = 0$ *and assume that there exist* $d_0, d_1 > 0$ *and strictly increasing unbounded functions* $\alpha, \lambda_0, \lambda_1 : [0,\infty) \to [0,\infty)$ *such that, for all* $k \geq 0$, *we have constants* $C_k > 0$ *with*

$$\| f(x)\|_k \leq C_k \| x\|_{\alpha(k)} \qquad (x\epsilon U)$$

$$\| f'(x)v \|_{d_1} \le C_0 \| v \|_{d_0} \qquad (x \in U, v \in E)$$

$$\| f(x+v) - f(x) - f'(x)v \|_{d_1} \le C_0 \| v \|^2_{d_0} \qquad (x, x+v \in U)$$

Suppose further that $f'(x)$ *has a right inverse* $L(x)$ *for each* $x \in U$ *and*

$$\| L(x)y \|_k \le C_k (\| x \|_{\lambda_0(k)} \| y \|_{d_1} + \| y \|_{\lambda_1(k)}) \qquad (x \in U, y \in E).$$

Finally we suppose the following relations hold,

$$\lambda_0(0) \le d_0, \ \lambda_1(0) \le d_1, \ d_0 < \frac{1}{2}, \quad \alpha(d_1) \le d_0 .$$

Then $f(U)$ *is a neighborhood of* 0.

There are important differences between this result and Theorem 2 which go beyond the difference between inequalities (3) and (3'). First observe that the restrictions on f' and its approximation by a difference (essentially a condition on f'') need only be made for a single norm here while in Theorem 2 it was on every norm and also the loss could only be from k to k+d. This relaxation is also present for the conditions on f and L. In Theorem 2 it could only be a constant linear loss of d while in Theorem 2' the loss (measured by $\alpha, \lambda_0, \lambda_1$) can have any gowth for large k but is mildly restricted for small k. Also in Theorem 2' there is the strange requirement that $d_0 < \frac{1}{2}$.

These variations are direct consequences of the proofs. About half of the argument is the same for both theorems and probabily will work for a wide class of space K(a). The other half is quite different and seems to reflect fundamental differences between spaces like $C^\infty(T)$ and $H(\mathbb{C})$ on the one hand and $H(\mathbb{D})$ on the other. This is very similar to other phenomena in the structure theory. It appears that these basic differences will render it unlikely that a unified proof valid for all spaces K(a) can be constructed.

There remains a major consideration in comparing Theorems 2 and 2'. Although, on the face of it, the hypotheses in Theorem 2' are, at least in some respects, weaker than those of Theorem 2, it is necessary to pin this down with examples. Thus we would want to have a function on $H(\mathbb{D})$ that satisfies the hypotheses of Theorem 2' but not of Theorem 2. Such examples have not yet been discovered and I would consider it to be a major question in this research.

On the other, unsuccessful attempts to find such examples have brought into focus other phenomena which turn out to be important in this and other contexts. I would like to close these notes with a brief explanation.

Perhaps the simplest example of a non-linear function is what we might call a binomial, which is defined as follows. Let $B : E\times E \to E$ be bilinear, symmetric and continuous. Then we define $f : E \to E$ by

$$f(x) = x + B(x,x).$$

We can calculate,

$$f'(x)v = \lim_{t\to\infty} \frac{x+t\ v+B(x+tv,x+tv)-x-B(x,x)}{t} = v+2B(x,v).$$

It is necessary to assume that $f'(x)$ is invertible. That is, for each x in a suitable neighborhood of 0, the operator $v \to v+2B(x,v)$ is invertible. Then we would try to find a B such that the hypotheses of Theorem 2' hold but those of Theorem 2 fail. Without going into details, I can say that one kind of calculation leads to the conclusion that B should satisfy the following condition, which we might call *separately bounded:*

There is a k_0 *such that for every* k *there is* $\sigma(k)$ *and* $C_k > 0$ *such that*

$$\| B(x,y)\|_k \le C_k \| x\|_{k_0} \| y\|_k \qquad (x,y\in E)$$

(and then, by symmetry, the same result would hold with x,y interchanged), but that B should *not* satisfy the following condition which we might call *jointly bounded:*

There is a k_0 *such that for every* k *there is* $C_k > 0$ *such that*

$$\| B(x,y)\|_k \le C_k \|x\|_{k_o} \|y\|_{k_o} \qquad (x,y\in E).$$

However, we have the following somewhat surprising result from the structure theory:

THEOREM 5.

In $H(\mathbb{D})$ *and its coordinate subspaces, every continuous, symmetric, bilinear, separately bounded function is jointly bounded. In* $C^\infty(T)$ *and each of its coordinate subspaces, this statement is false.*

In the light of this result, I feel quite uncertain as to the exact reason for the difference between Theorem 2 and Theorem 2'. Is the latter in some sense stronger or are the hypotheses actually equivalent? Perhaps future research will explain the matter.

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