ATOMIC, NONATOMIC AND CONTINUOUS FINITELY
ADDITIVE MEASURES: RESULTS AND APPLICATIONS

E. BARONE - A. GIANNONE - R. SCOZZAFAVA

Sommario - *Scopo di questo lavoro è di estendere (e di riunire in una esposizione completa) alcuni dei nostri recenti risultati sulle misure finite semplicemente additive.*

ATOMIC, NON-ATOMIC AND CONTINUOUS

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AND APPLICATIONS[(*)]

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1. Introduction and notations.

The aim of this paper is to extend (and collect in a self-contained and expository account) some of our recent results concerning a finitely additive "probability"(i.e., finite) measure μ.

Among others, new topics dealt with in this note are: the construction of countably additive sequences of sets for a non-atomic μ (and not only for a continuous one, as in [13]) and also for a particular kind of atomic μ; a deeper discussion of the case of a μ which is both non-continuous and non-atomic (studied in [1]), leading to an interesting (we hope) remark concerning the existence of measurable cardinals (Ulam's problem); moreover, we touch upon non-standard methods through finitely additive measures (as treated, e.g., in [4]) , sketching out the possibility of new trends in this field.

Let Ω be an arbitrary (infinite) set, and $\mathcal{Q} \subseteq \mathcal{P}(\Omega)$ a σ-algebra. A mass is a function

$$\mu : \mathcal{Q} \longmapsto \mathbf{R}$$

such that

_____

(a) $\mu(E) \geq 0$ for any $E \in \mathcal{Q}$,

(b) $E, F \in \mathcal{Q}$, $E \cap F = \emptyset$ $\implies \mu(E \cup F) = \mu(E) + \mu(F)$,

(c) $\mu(\Omega) < + \infty$.

In particular, μ is a (finitely additive) probability measure if

$$\mu(\Omega) = 1.$$

When axiom (b) is replaced by the (stronger) condition

(b') $F_n \in \mathcal{Q}$, $F_i \cap F_j = \emptyset$ for $i \neq j$ $\implies \mu(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} \mu(F_n)$,

we shall call μ (in short) a _measure_.

We may (and will) assume throughout the paper that, for every $x \in \Omega$,

(1) $\mu(\{x\}) = 0$,

since (if not) we might subtract from μ the (trivial) _measure_

$$m(E) = \begin{cases} 0 & \text{if} & E \subseteq \Omega - A_0 \\ \mu(E \cap A_0) & \text{if} & E \cap A_0 \neq \emptyset \end{cases},$$

where

$$A_0 = \{x \in \Omega : \mu(\{x\}) > 0\}$$

is (as it is well known) a countable set.

Proposition 1 - Let μ be a mass on a σ-algebra $\mathcal{Q}$. Then, for every
sequence $A_n \in \mathcal{Q}$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, we have

(2)
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \sum_{n=1}^{\infty} \mu(A_n) \ .$$

Proof. Apply (b) to the n+1 sets $A_1, A_2, \ldots, A_n, \bigcup_{k=n+1}^{\infty} A_k$, and then apply (a) to the latter. ∎

Definition 1 - A mass is _null_ if $\mu(E) = 0$ for each $E \epsilon \mathcal{O}$ (i.e., if $\mu(\Omega) = 0$).

Definition 2 - An _atom_ for a mass μ is a set $A \epsilon \mathcal{O}$ such that $\mu(A) > 0$ and $\mu(\mathcal{O} \cap \mathcal{P}(A)) = \{0, \mu(A)\}$.

Definition 3 - A mass μ on $\mathcal{O}$ is _atomic_ if there exists an atom for μ.

We denote by $p = \{E_1, E_2, \ldots, E_n\}$ any _finite_ partition of Ω, and by $\mathcal{D}$ the set of all _measurable_ (i.e., with every $E_n \epsilon \mathcal{O}$) p's.

Definition 4 - A mass μ on $\mathcal{O}$ is _continuous_ (or _strongly non-atomic_) if, given $\epsilon > 0$, there exists $\{E_1, E_2, \ldots, E_n\} = p \ \epsilon \ \mathcal{D}$ such that $\mu(E_k) < \epsilon$ $(k = 1, 2, \ldots, n)$.

Proposition 2 - If a mass μ on $\mathcal{O}$ is continuous, then μ is non-atomic.

Proof. Obvious. ∎

Proposition 3 - If A is an atom for μ, then

$$\mathcal{U} = \{E \epsilon \mathcal{O} : \mu(E \cap A) = \mu(A)\}$$

is an $\mathcal{O}$-ultrafilter over Ω, i.e. an ultrafilter over Ω whose elements belong to $\mathcal{O}$. Conversely, given any $\mathcal{O}$-ultrafilter $\mathcal{U}$ over Ω, the set function

$$\mu(E) = \begin{cases} 1 & \text{if} \quad E \in \mathcal{U} \\ 0 & \text{if} \quad E \notin \mathcal{U} \end{cases}$$

is an atomic mass on $\mathcal{A}$.

Proof: cfr., e.g. , [6] , p. 358.

Remark : According to our assumption 1), in this paper "ultrafilter", always means free ultrafilter, i.e. $\bigcap\limits_{E \in \mathcal{U}} E = \emptyset$ (while a fixed, or principal, ultrafilter is one whose elements are the subsets of Ω containing a given point $x \in \Omega$) .

Definition 5 - A two-valued (0 and $\mu(\Omega)$) atomic mass on $\mathcal{A}$ is called ultrafilter mass.

2. A theorem by B. de Finetti.

Given any $p \in \mathcal{P}$, choose $E^{(p)} \in p$ such that

$$\mu(E^{(p)}) \geqslant \mu(E_k) \quad ,$$

for every $E_k \in p$ $(k=1,2,\ldots,n)$, and put

(3) $$\alpha_1 = \inf_{p \in \mathcal{P}} \mu(E^{(p)}) .$$

Clearly, μ is continuous if and only if $\alpha_1 = 0$.

$$\mu(E) = \begin{cases} 1 & \text{if} \quad E \in \mathcal{U} \\ 0 & \text{if} \quad E \notin \mathcal{U} \end{cases}$$

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for every $E_k \in p$ (k=1,2,...,n), and put

   (3) $$\alpha_1 = \inf_{p \in \mathcal{D}} \mu(E^{(p)}) \; .$$

Clearly, μ is continuous if and only if $\alpha_1 = 0$.

For the sake of completeness, we recall here a decomposition theorem, essentially given by B. de Finetti in [3] ; for a different proof, see also [13]. The one given here is a direct proof avoiding the use of the "coefficient of divisibility" introduced in [3] .

Theorem 1 - Let μ be a mass on a σ-algebra $\mathcal{Q} \subseteq \mathcal{P}(\Omega)$. Then

(4)
$$\mu = \sum_{n=1}^{\infty} \beta_n + \mu_0 ,$$

where each β_n (if not null) is atomic and μ_0 is continuous (or null).

Proof - If μ is continuous, there in nothing to prove, since (4) is true with $\mu_0 = \mu$ and with each β_n null. Let now μ be non-continuous: then $\alpha_1 > 0$ and so, by (3), for every partition $p \in \mathcal{D}$ the set $E^{(p)}$ is such that $\mu(E^{(p)}) \geq \alpha_1$, and there is a partition $p_0 \in \mathcal{D}$ such that $\mu(E^{(p_0)}) < 2\alpha_1$.

Let $\mathcal{E} = \{E \epsilon p_0 : \mu(E) \geq \alpha_1\}$: there exists (again by (3), and remembering (1)) a set $E_0 \epsilon \mathcal{E}$ such that $\mu(A) \geq \alpha_1$ for at least a proper subset $A \subset E_0$. It follows then easily that

$$\mathcal{U}_1 = \{E \epsilon \mathcal{Q} : \mu(E \cap E_0) \geq \alpha_1\}$$

is an $\mathcal{Q}$-ultrafilter over Ω (the only thing which may not be completely trivial is that $A, B \epsilon \mathcal{U}_1$ implies $A \cap B \epsilon \mathcal{U}_1$; but, since only one of the four subsets into which A and B divide E_0 (i.e., $(A-B) \cap E_0, (B-A) \cap E_0, A \cap B \cap E_0, E_0 - (A \cup B)$) can have a mass $\geq \alpha_1$, it is not

difficult to see that such subset must necessarily be $A \cap B \cap E_0$). So

the mass

$$
\beta_1(E) = \begin{cases} 0 & \text{if} & E \notin \mathcal{U}_1 \\ \alpha_1 & \text{if} & E \in \mathcal{U}_1 \end{cases}
$$

is atomic. Put $\mu_1 = \mu - \beta_1$; if the mass μ_1 is non-continuous, then

$$
\alpha_2 = \inf_{p \in \mathcal{P}} \mu_1(E^{(p)}) > 0,
$$

and so it is possible to go on in the same fashion.

After n steps, we get

$$
\mu_n = \mu - \sum_{k=1}^{n} \beta_k
$$

and, if μ_n is continuous, eq.(4) holds with $\mu_0 = \mu_n$ and with each β_k null for $k > n$. If μ_n is non-continuous for any n, we get a sequence (β_n) such that the corresponding series $\sum_{n=1}^{\infty} \beta_n(E)$ converges for every $E \in \mathcal{Q}$ (since $\mu(E) < + \infty$). Then $\lim_{k \to \infty} \alpha_k = 0$, and it follows that

$\mu_0 = \lim_{n \to \infty} \mu_n$ is continuous. ∎

3. Non atomic masses.

In the classical case of a _measure_, non-atomicity is equivalent to

continuity. This can be seen, for example, as an easy consequence of the following

Theorem 2 (Saks) - Let μ be a measure on a σ-algebra $\mathcal{Q} \subseteq \mathcal{P}(\Omega)$. Given any $\varepsilon > 0$, there exists pe $\mathcal{D}$ such that each E_k ep is either an atom or $\mu(E_k) < \varepsilon$.

Proof : see [2], p. 308 . ∎

This equivalence cannot be carried over to the general case of a mass: in [1] it is shown (cfr. also the following Theorem 5) that, if $\mu = \nu + \lambda$, where ν is atomic and λ is continuous, with $\nu \ll \lambda$ (i.e. ν is absolu tely continuous with respect to λ), then μ is both non-atomic and non-continuous.

In other words, while atomic masses are necessarily non-continuous (see Proposition 2), non-atomic ones can be either continuous or not.

For continuous masses, the following result has been established in [13]:

Theorem 3 - Let μ be a continuous mass on a σ-algebra $\mathcal{Q} \subseteq \mathcal{P}(\Omega)$. Then there exists a sequence (F_n) of mutually disjoint measurable sets, with $\mu(F_n) > 0$, such that $\sum_{n=1}^{\infty} \mu(F_n)$ equals any preassigned α , with $0 < \alpha < \mu(\Omega)$, and

$$(5) \qquad \alpha = \sum_{n=1}^{\infty} \mu(F_n) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) .$$

So to say, the mass μ "behaves" like a measure on each collection (F_n) : then it would seem interesting a deeper investigation of the family (or of some suitable subfamily) of all such sets obtained when α ranges

in the open interval from   0 to $\mu(\Omega)$.

A simple corollary of Theorem 3 is the following: the range of a continuous μ is the whole interval $[0,\mu(\Omega)]$. The latter statement is no langer true if μ is non-continuous: a counterexample is given in [1]; moreover, there it is shown that the range of μ need not even be a closed subset of $\mathbb{R}$, contrary to the classical case of a <u>measure</u>.

As far as continuous masses are concerned, let us quote also a recent result obtained through non-standard methods: a necessary and sufficient condition for the existence of a continuous mass, which is <u>invariant</u> for a transformation of Ω into itself, is given in [16].

We want now to extend Theorem 3 to the more general case of a non-ato mic μ: the previous remarks show that we can hope, at most, in counta ble additivity on a suitable sequence of sets (and not also, as in eq. (5), in a beforehand given value of α).

<u>Theorem 4</u> - Let μ be a <u>non-atomic</u> mass on a σ-algebra $\mathcal{Q} \subseteq \mathcal{P}(\Omega)$. Then there exists a sequence (A_n) of mutually disjoint measurable sets, with $\mu(A_n) > 0$, such that

$$(6) \qquad \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) \ .$$

<u>Proof</u> - Let $B \in \mathcal{Q}$, with $0 < \mu(B) < \mu(\Omega)$: then also $B' = \Omega - B$ satisfies $0 < \mu(B') < \mu(\Omega)$. At least one of them (call it B_1) is such that $\mu(B_1) \leq \frac{1}{2} \mu(\Omega)$. Now, let $B_2 \in \mathcal{Q}$, $B_2 \subset B_1$, be such that $0 < \mu(B_2) < \mu(B_1)$ and $\mu(B_2) \leq \frac{1}{2} \mu(B_1)$. In general, we define $B_n \subset B_{n-1}$, with $0 < \mu(B_n) < \mu(B_{n-1})$ and $\mu(B_n) \leq \frac{1}{2} \mu(B_{n-1}) \leq \frac{1}{2^n} \mu(\Omega)$.

Put $A_n = B_n - B_{n+1}$: we have $\mu(A_n) = \mu(B_n) - \mu(B_{n+1}) > 0$ and

$A_i \cap A_j = \emptyset$ for $i \neq j$. Moreover

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{k} \mu(A_n) + \mu(\bigcup_{n=k+1}^{\infty} A_n) =$$

$$= \sum_{n=1}^{k} \mu(A_n) + \mu(B_{k+1}) \leq \sum_{n=1}^{k} \mu(A_n) + \frac{1}{2^{k+1}} \mu(\Omega) \ .$$

As $k \to \infty$, we get

$$\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n) \ ,$$

which, taking into account Proposition 1, gives (6). ■

The next theorem will enable us to extend (in Section 4) the previous result also to a particular class of atomic masses.

Theorem 5 - Let $\mathcal{Q} \subseteq \mathcal{P}(\Omega)$ be a σ-algebra, ν an atomic mass on $\mathcal{Q}$, and λ a continuous mass on $\mathcal{Q}$ such that $\lambda(A) > 0$ for any atom A of ν (e.g., such that $\nu \ll \lambda$). Then $\mu = \nu + \lambda$ is non atomic and non-continuous.

Proof : see [1]. ■

Remark - Put $\nu = \sum_n \beta_n$ in Theorem 1, eq.(4): ν need not be atomic

(an example is given in [10] , p. 47), and so we may have masses which are at the same time non-atomic and non-continuous, but not of the form given by Theorem 5.

Corollary 1 - Let β be an ultrafilter mass and λ a continuous mass on $\mathcal{Q} \subseteq \mathcal{P}(\Omega)$, with $\beta << \lambda$. Then $\mu = \beta + \lambda$ is non-atomic and non-continuous.

Corollary 2 - Let β be an ultrafilter mass on $\mathcal{Q} \subseteq \mathcal{P}(\Omega)$ such that $\beta << \lambda$, where λ is a continuous measure on $\mathcal{Q}$. Then β cannot be a measure on $\mathcal{Q}$.

Remark - It is interesting also to look at Theorem 5 as another counterexample to known results for measures: in [7] it is shown that, given two measures λ and ν, with $\nu << \lambda$ and λ non atomic (i.e. continuous), then ν also is non-atomic. Actually, this need not be true if ν is only a mass (and not a measure), for example if it is an ultrafilter mass β, as that of Corollary 2. The existence of such a mass (given λ) can be proved (cfr. [1]) taking an $\mathcal{Q}$-ultrafilter containing the filter

$$\mathcal{F} = \{E \in \mathcal{Q} : \lambda(E) = \lambda(\Omega)\} \ .$$

4. Atomic masses and measurable cardinals.

Since the mass μ occurring in Theorem 5 is non-atomic (and non-continuous), Theorem 4 can be suitably applied to it, giving easily a countably additive sequence of sets also for the atomic mass ν.

Theorem 6 - Let ν be an <u>atomic</u> mass on a σ-algebra $\mathcal{Q} \subseteq \mathcal{P}(\Omega)$, such that $\nu << \lambda$, where λ is a continuous <u>measure</u> on $\mathcal{Q}$. Then there exists a sequence (A_n) of mutually disjoint measurable sets, such that

$$\nu(\overset{\infty}{\underset{n=1}{\cup}} A_n) = \overset{\infty}{\underset{n=1}{\Sigma}} \nu(A_n) \ .$$

Proof - Use Theorem 4 for $\mu = \nu + \lambda$, taking into account the countable additivity of λ. ∎

Now, in order to deal with the so-called "Ulam's measure problem", we recall some known facts about ultrafilter over a set Ω; we limit owrselves to <u>free</u> ultrafilters(cfr. the remark following Proposition 3).

<u>Definition 6</u> - An ultrafilter $\mathcal{U}$ over Ω is <u>δ-complete</u> if, given any sequence of sets $A_n \in \mathcal{U}$, one has $\bigcap_{n=1}^{\infty} A_n \in \mathcal{U}$.

<u>Proposition 4</u> - Let β be an ultrafilter mass on $\mathcal{A} = \mathcal{P}(\Omega)$, and let $\mathcal{U}$ be the corresponding (free) ultrafilter. Then β is a <u>measure</u> if and only if $\mathcal{U}$ is δ-complete.

<u>Proof</u> - Countable additivity of β implies that, given any sequence of sets $A_n \in \mathcal{U}$, for $A_n' = \Omega - A_n \notin \mathcal{U}$ we must have $\bigcup_{n=1}^{\infty} A_n' \notin \mathcal{U}$.
Therefore $\Omega - \bigcup_{n=1}^{\infty} A_n' = \bigcap_{n=1}^{\infty} A_n \in \mathcal{U}$, i.e. $\mathcal{U}$ is δ-complete. The

converse is also easily seen, since β is two-valued.

<u>Definition 7</u> - Let Ω be a set: card Ω is said <u>measurable</u> when there exists a δ-complete free ultrafilter over Ω.

<u>Corollary 3</u> - An ultrafilter <u>measure</u> exists on $\mathcal{A} = \mathcal{P}(\Omega)$ if and only if card Ω is measurable.

(Notice that the latter measure is finite,defined for all subsets of Ω, and zero on singletons).

The question concerning the existence of measurable cardinals (known also under the name of <u>Ulam's measure problem</u>) cannot be settled in ZFC (Zermelo-Fraenkel set theory with the Axiom of Choice).

It was shown that a measurable cardinal (assuming its existence) must be very large and, in fact, must be an inaccessible cardinal: really, if k is a measurable cardinal, then there are k inaccessible cardinals preceding it (cfr., e.g., [11], p. 26 and [14], p. 26).

Moreover, the existence of a measurable cardinal settles many mathematical problems: see [8].

On the other hand, if we assume that "all" sets are constructible (the so-called "axiom of constructibility" V = L), no measurable cardinal exists: in fact, if there is a measurable cardinal, then V = L " is as false as it possibly can be" (cfr. [14], p. 31).

Notice that, by Corollary 3, the existence of an ultrafilter measure on $\mathscr{P}(\Omega)$ is equivalent to the statement that card Ω is measurable , while an ultrafilter mass always exists, by a classical result due to Tarski [15].

Proposition 5 - Let card $\Omega = c$ and assume the continuum hypothesis (CH). Then no ultrafilter measure exists on $\mathscr{P}(\Omega)$ (i.e., under CH, c is not a measurable cardinal).

Proof - See [17] or [11]. ∎

We point out that Corollary 2 (cfr. Section 3) gives non-existence of a particular class of ultrafilter measures, without any assumption on the cardinality of Ω.

We end this Section with a necessary condition for a cardinal to be measurable, which gives an interesting remark to Ulam's measure problem; we state first the following obvious

Lemma - Let Ω be a set such that card Ω is measurable, and let β be the corresponding ultrafilter measure. Then $\beta(E)>0$ implies card $E > \aleph_0$.

Theorem 7 - Let β be an ultrafilter measure on $\mathcal{P}(\Omega)$. Then, given any continuous measure λ on $\mathcal{Q} \subseteq \mathcal{P}(\Omega)$, necessarily $\beta \perp \lambda$ (i.e., β is singular with respect to λ) and there are sets $E \subset \Omega$, with card $E > \aleph_0$, such that $\lambda(E) = 0$.

Proof - It is essentially a reformulation of Corollary 2, taking into account the preceding Lemma. ∎

Remark - Theorem 7 can be looked at to give some grounds for the acceptance or not of the axiom concerning the existence of measurable cardinals: for example, if we assume that, given a set Ω, there exists at least a continuous measure on a σ-algebra $\mathcal{Q} \subseteq \mathcal{P}(\Omega)$, vanishing only on countable (*) sets, then card Ω is not measurable.

This result is also a partial converse to a theorem given by Ulam (cfr. Satz 2, p. 147) in [17]: he proved that, if card Ω is not measurable and there exists a measure on Ω, then this measure is necessarily continuous.

(*) Here it would be possible to replace "countable sets" by "sets of cardinality less than card Ω", just using a suitable definition of measure, in which countable additivity is replaced by the "natural" stronger requirement (cfr.[14], p. 20).

5. <u>Non-standard methods</u>.

It is well-known that ultrafilter masses (or measures: this stronger terminology may be used when the cardinality of the index set is measu<u>rable)</u> are a tool in non-standard analysis for the construction of the relevant superstructure.

In [4] it is shown how some fundamental ideas in this field may be introduced through a finitely additive probability measure μ (i.e., a mass) on the index set, extending the concept of ultrapower to that of "μ-power".

To facilitate the exposition, the attention was confined to a stru<u>c</u>ture $\mathcal{E} = <E, \mathcal{R}>$, consisting of a non-empty set E and a set $\mathcal{R}$ of relations on E. Given an index set J, let μ be a <u>mass</u> on $\mathcal{P}(J)$, with $\mu(J) = 1$: the superstructure $^*\mathcal{E} = <^*E, ^*\mathcal{R}>$ consists of the set $^*E$ of all functions $f : J \to E$ (modulo μ-null sets) and any relation $R \in {}^*\mathcal{R}$ "is true" (loosely speaking) in the "model" $^*\mathcal{E}$ if and only if it is true in $\mathcal{E}$ for <u>almost all</u> $j \in J$ (here each value of one of the "equivalent" functions, with domain J, defining an element of $^*E$, is a point of E; $^*E$ is a proper extension of E, by virtue of condi<u>-</u>tion (1) : cfr. [4]) .

Let us consider, to be definite, the structure given by the ordered field $\mathbf{R}$. It is clear that, using μ-powers instead of ultrapowers (i.e., arbitrary masses on J instead of ultrafilter ones), $^*R$ is <u>not</u> necessarily a field (and, moreover, it is only partially ordered): take, for example, $A \subset J$ with $0 < \mu(A) < 1$. Its characteristic function χ_A is not equivalent to the null function 0, and so gives rise, in $^*\mathcal{E}$, to an element (i.e., an equivalence class modulo μ-null sets) $[\chi_A]$ which

is <u>not</u> $^*0 = [0]$; the same is true for X_{J-A} . But $X_A \cdot X_{J-A} \equiv 0$, and

so in $^*\mathcal{E}$ the product $[X_A] \cdot [X_{J-A}]$ of this two non-null elements is

null (i.e., $^*R$ has zero divisors).

This fact, from the usual point of view adopted in the costruction of non-standard models of the reals, should be considered a "defect", since it is possible (as it is well known) to build up an enlargement $^*R$ which is an ordered field (though, of course, a non-archimedean one). But it is possible to look at the question from different viewpoints, similar (apart from the dropping out of the condition of σ-additivity for the measure on the index set) to those sketched by D . Scott in $[12]$.

A problem of interest in probability theory is the following: if we take μ to be a <u>measure</u>, it does <u>not</u> exist a denumerable "uniform" (and measurable) partition of the index set $J = \overset{\infty}{\underset{n=1}{U}} E_n$, with $\mu(E_n)=0$ for each n, otherwise

$$0 = \overset{\infty}{\underset{n=1}{\Sigma}}\mu(E_n) = \mu(\overset{\infty}{\underset{n=1}{U}}E_n) = 1 \quad \text{(impossible)}.$$

On the other hand, if μ is only a <u>mass</u> and such a partition exists, the latter inequality is consistent with Prop. 1; moreover we show the pos̲ sibility of looking at it as a sort of "non-standard" countable additivity.

To begin with, we may give a meaning to $\overset{\infty}{\underset{n=1}{\Sigma}} {}^*a_n$ (with ${}^*a_n \in {}^*R$) by

choosing <u>suitable</u> representatives a_n of ${}^*a_n$ (n = 1,2,...) such that

$\overset{\infty}{\underset{n=1}{\Sigma}}a_n(i)$ converges for almost all i $\in$ J, and by putting

(7) $$\sum_{n=1}^{\infty} {}^{*}a_n = \left[\left(\sum_{n=1}^{\infty} a_n(i) \right)_{i \in J} \right] \quad .$$

Now, we remark that, since $x_{E_n} = 0$ for almost all $j \in J$, we have $\left[x_{E_n} \right] = {}^{*}0$; on the other hand, $\sum_{n=1}^{\infty} x_{E_n} = x_J \equiv 1$, and so $[x_J] = {}^{*}1$.

If we apply (7) to $\sum_{n=1}^{\infty} {}^{*}0 = \sum_{n=1}^{\infty} [x_{E_n}]$, choosing x_{E_n} as representative of $[x_{E_n}]$, we get

$$\sum_{n=1}^{\infty} {}^{*}0 = \sum_{n=1}^{\infty} [x_{E_n}] = [\sum_{n=1}^{\infty} x_{E_n}] = [x_J] = {}^{*}1 \quad .$$

So an uniform probability distribution on a countable set does not conflict, in ${}^{*}\mathbb{R}$, with "countable additivity" of μ .

We point out that this approach differs from the well-know one (see, e.g., [5] , [9]) through *-finite sets: is there some hope that such "models"would open new trends in this field?

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