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# On The Hub Number of Some Graphs

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ABSTRACT

In this paper, we give the results for the hub numbers of the join and corona of two connected graphs, cartesian product of two complete graphs, cartesian product of a non-complete connected graph and a complete graph and the cartesian product of two paths $P_m$ and $P_n$ for $n \geq 4$ and $m = 2, 3$. Moreover, we give an upperbound for the hub number of the cartesian product of two paths $P_m$ and $P_n$ for $4 \leq m \leq n$.

*Key words*: hub set, hub number of a graph

## 1 Introduction

Let $P$ be a path with end vertices $x$ and $y$. A vertex $u$ is an internal vertex if $u \in V(P)\backslash\{x, y\}$. Suppose $S \subseteq V(G)$ and $x, y \in V(G)$, $x \neq y$. A path $P$ is an $S - path$ of $G$ if every internal vertex of $P$ is in $S$. Let $G$ be a graph; a hub set $S$ of $G$, which was introduced by Walsh, is a set of vertices with the property that for any pair of vertices outside $S$ , there is an $S - path$ between them with all internal vertices in $S$. The hub number, denoted by $h(G)$,which was introduced by Walsh, is defined to be the minimum cardinality of a hub set of $G$.

Walsh (2006) studied hub number for several classes of graphs and showed that the hub number is at least the girth minus 3 (the *girth* is the length of the shortest cycle). Grauman et al. (2008) showed the relationship of hub number, connected hub number $h_c(G)$ and connected domination number of a graph $\gamma_c(G)$ and proved that $h(G) \leq h_c(G) \leq \gamma_c(G) \leq$

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$h(G) + 1$.

The graphs being considered in this study are simple and connected. We review some several standard definitions used in this study. The *join* or *sum* of two graphs $G$ and $H$, denoted by $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \quad \text{and} \quad v \in V(H)\}$. The *corona* of two graphs $G$ and $H$, denoted by $G \circ H$, is the graph with $V(G \circ H) = V(G) \cup \bigcup_{x \in V(G)} V(H_x)$, where $H_x$ is a copy of $H$ all of whose vertices are adjacent to $x$ for $x \in V(G)$, and $E(G \circ H) = E(G) \cup \bigcup_{x \in V(G)} E(H_x) \cup \{[x, y] : x \in V(G), y \in V(H_x)\}$. The *cartesian product* of two graphs $G$ and $H$, denoted by $G \times H$, is a graph such that $V(G \times H) = V(G) \times V(H)$, and two verties $(u_1, v_1)$ and $(u_2, v_2)$ of $G \times H$ are adjacent if and only if either $u_1 = u_2$ and $[v_1, v_2] \in E(H)$ or $v_1 = v_2$ and $[u_1, u_2] \in E(G)$.

## 2 Hub Numbers of the Join and Corona of Two Connected Graphs

**Theorem 2.1.** For any connected graphs $G$ and $H$,

$$h(G + H) = \begin{cases} 0 & \text{, if } G \text{ and } H \text{ are complete ,} \\ 1 & \text{, if } G \text{ complete and } H \text{ non} - \text{complete ,} \\ min\{h(G), h(H), 2\} & \text{, if } G \text{ and } H \text{ are both non} - \text{complete .} \end{cases}$$

*Proof.* Suppose $G$ and $H$ are both complete. Then $G + H$ is also complete. By Theorem [1], $h(G + H) = 0$. Suppose $G$ is complete and $H$ is non-complete. Let $a \in V(G)$ and $S = \{a\}$. Let $x, y \in V(G + H)\backslash S$. Consider the following cases:

**Case 1**. Suppose $x, y \in V(G)\backslash\{a\}$. Since $G$ is complete, there is a path $\{x, a, y\}$ in $G$. Hence, there is an $S$-path between $x$ and $y$ in $G + H$.

**Case 2**. Suppose $x \in V(G)\backslash\{a\}$ and $y \in V(H)$. Since $G$ is complete, $x$ and $a$ are adjacent. By definition of $G + H$, $a$ is adjacent to $y$. Hence, there is a path $\{x, a, y\}$ in $G + H$. Thus, there is an $S$-path between $x$ and $y$ in $G + H$.

**Case 3**. Suppose $x, y \in V(H)$. By definition of $G + H$, $a$ is adjacent to both $x$ and $y$. Hence, there is a path $\{x, a, y\}$ in $G + H$. Thus, there is an $S$-path between $x$ and $y$ in $G + H$.

Thus, $S$ is a hub set of $G + H$. Accordingly, $h(G + H) \leq 1$. Since $H$ is non-complete, consequently $G + H$ is non-complete. By Theorem [1], $h(G + H) \neq 0$. Therefore, $h(G + H) = 1$.

Suppose $G$ and $H$ are both non-complete. Consider the following cases:

**Case 1**. Suppose $h(G) = 1$.

Let $a \in V(G)$. Let $S = \{a\}$ be a minimum hub set of $G$. Let $x, y \in V(G + H)\backslash\{a\}$. Consider the following subcases:

**Subcase 1**. Suppose $x, y \in V(G)\backslash\{a\}$. Since $S$ is a hub set of $G$, there is an $S$-path between $x$ and $y$ in $G + H$.

**Subcase 2**. Suppose $x \in V(G)\backslash\{a\}$ and $y \in V(H)$. Since $S$ is a hub set of $G$, $x$ is incident

to $a$. By definition of $G+H$, $a$ is incident to $y$. This means that $\{x,a,y\}$ is an $S$-path in $G+H$. Hence, $S$ is a hub set of $G+H$. Thus, $h(G+H)\leq 1$.
**Subcase 3**. Suppose $x,y\in V(H)$. By definition of $G+H$, $a$ is incident to both $x$ and $y$. This means that $\{x,a,y\}$ is an $S$-path in $G+H$. Hence, $S$ is a hub set of $G+H$. Thus, $h(G+H)\leq 1$.
Combining the three subcases, $h(G+H)\leq 1$. Since $G$ and $H$ are both non-complete, $G+H$ is non-complete. So, $h(G+H)\neq 0$. Therefore, $h(G+H)=1$.
**Case 2**. Suppose $h(H)=1$.

The proof is similar to Case 1.
**Case 3**. $h(G),h(H)\geq 2$.

Let $a\in V(G)$, $b\in V(H)$ and $S=\{a,b\}$ . Consider the following subcases:
**Subcase 1**. Let $x,y\in V(G)\backslash\{a\}$. By definition of $G+H$, both $x$ and $y$ are incident to $b$. That is, there is an $S$-path $\{x,b,y\}$ in $G+H$. Hence, $S=\{a,b\}$ is a hub set of $G+H$. So, $h(G+H)\leq 2$.
**Subcase 2**. Let $x,y\in V(H)\backslash\{b\}$. The proof is similar to Subcase 1.
**Subcase 3**. Let $x\in V(G)\backslash\{a\}$ and $y\in V(H)\backslash\{b\}$. By definition of $G+H$, $x$ is incident to $b$, $b$ is incident to $a$, and $a$ is incident to $y$. That is, $\{x,b,a,y\}$ is an $S$-path in $G+H$. Hence, $S=\{a,b\}$ is a hub set of $G+H$. So, $h(G+H)\leq 2$.

Suppose $h(G+H)=1$. Assume without loss of generality, $S=\{a\}$ be a minimum hub set of $G+H$ where $a\in V(G)$. Let $x,y\in V(G)\backslash\{a\}$. Thus, $\{x,a,y\}$ is an $S$-path in $G$. This implies that $S$ is a hub set of $G$. That is, $h(G)\leq 1$. This is a contradiction to the assumption that $h(G)\geq 2$. Therefore, $h(G+H)=2$. □

**Theorem 2.2.** For any connected graphs $G$ and $H$,

$$h(G\circ H)=\begin{cases} 0 & \text{, if } |\mathrm{V(G)}|=1 \text{ and } H \text{ is complete },\\ 1 & \text{, if } |\mathrm{V(G)}|=1 \text{ and } H \text{ is non}-\text{complete },\\ |V(G)| & \text{, if } |\mathrm{V(G)}|\geq 2. \end{cases}$$

*Proof.* Suppose $|V(G)|=1$ and $H$ is complete. Obviously, $G\circ H$ is complete. By Theorem [1],
$h(G\circ H)=0$. Suppose $|V(G)|=1$ and $H$ is non-complete. Note that if $|V(G)|=1$ and $H$ is non-complete, then $G\circ H$ is just $G+H$. Thus, by Theorem 2.1, $h(G\circ H)=1$, if $H$ is non-complete.

Suppose $|V(G)|\geq 2$. Let $S=V(G)$ and $x,y\in V(G\circ H)\backslash S$.
**Case 1**. Suppose $x,y\in V(H_a)$ for some $a\in V(G)$. Clearly, $\{x,a,y\}$ is a path in $G\circ H$ and $a\in S$. Thus, there is an $S$-path between $x$ and $y$ in $G\circ H$.
**Case 2**. Suppose $x\in V(H_a)$ and $y\in V(H_b)$ for some $a,b\in V(G)$. Since $G$ is connected, there is a path $\{a,u_1,u_2,...,u_t,b\}$ in $G$ for some $u_1,u_2,...,u_t\in S$. Thus, $\{x,a,...,b,y\}$ is a path in $G\circ H$. Hence, there is an $S$-path between $x$ and $y$ in $G\circ H$. Therefore, $S$ is a hub

set of $G \circ H$. Accordingly, $h(G \circ H) \leq |V(G)|$.

Let $S'$ be another hub set of $G \circ H$. Suppose there exists $u \in V(G)$ such that $u \notin S'$. Since $|V(G)| \geq 2$, there exists $v \in V(G)$ such that $u \neq v$. Let $x \in V(H_u)$ and $y \in V(H_v)$. There exists a path $\{x, a_1, ..., a_n, y\}$ where $a_i \in S'$ for all $i = 1, 2, ..., n$ for some $n \in \mathbb{N}$. By definition of $G \circ H$, there exists $k$ such that $a_k = u$. This implies that $u \in S'$ which is a contradiction. Hence, $V(G) \subseteq S'$ for all hub set $S'$ of $G \circ H$. It follows that $|V(G)| \leq |S'|$. Thus, $|V(G)| \leq h(G \circ H)$.

Therefore, $h(G \circ H) = |V(G)|$. □

## 3 Hub Numbers of the Cartesian Product of Some Graphs

**Theorem 3.1.** Let $3 \leq m \leq n$, where $m, n \in \mathbb{Z}$. Then $h(K_m \times K_n) = m$.

*Proof.* Let $3 \leq m \leq n$, $w \in V(K_n)$ and $S = V(K_m) \times \{w\}$. Let $(u_1, v_1)$, $(u_2, v_2) \in V(K_m \times K_n)\backslash S$. Then $\{(u_1, v_1), (u_1, t_k), (u_2, t_k), (u_2, v_2)\}$ is an $S$-path in $K_m \times K_n$ for some $t_k \in V(K_n)$. Thus, $S$ is a hub set of $K_m \times K_n$. It follows that $h(K_m \times K_n) \leq |S| = m$.

Suppose $h(K_m \times K_n) < m$. Let $S'$ be a minimum hub set of $K_m \times K_n$. Then $|S'| < m$. For each $a \in V(K_m)$, let $S_a = \{(a, x) : x \in V(K_n)\}$. Since $|S'| < m \leq n$, there exists $a_1 \in V(K_m)$ such that $S' \cap S_{a_1} = \emptyset$. Suppose that $|S' \cap S_a| = 1$ for every $a \in V(K_m)\backslash\{a_1\}$. Let $(a, b) \in S' \cap S_a$. Let $b_1, b_2 \in V(K_n)$ such that $b \neq b_1$ and $b \neq b_2$. Now, $(a, b_1)$ and $(a_1, b_2)$ are joined by either paths $\{(a, b_1), (a, b), (a_1, b), (a_1, b_2)\}$ and $\{(a, b_1), (a, b_2), (a_1, b_2)\}$. But neither of these paths is an $S'$-path since $(a_1, b), (a, b_2) \notin S'$. This contradicts the assumption that $S'$ is a hub set which means that $|S'| \geq m$. Suppose $|S' \cap S_a| > 1$ for some $a \in V(K_m)$. Then there exists $a_1, a_2 \in V(K_m)$, $a_1 \neq a_2$ such that $S' \cap S_{a_1} = \emptyset$ and $S' \cap S_{a_2} = \emptyset$. Since $|S'| < m$, there exists $b_t \in V(K_n)$ such that $(a, b_t) \notin S' \cap S_a$. Now, $(a_1, b)$ and $(a_2, b_t)$ are joined by paths $\{(a_1, b), (a, b), (a, b_t), (a_2, b_t)\}$, $\{(a_1, b), (a_2, b), (a_2, b_t)\}$ and $\{(a_1, b), (a_1, b_t), (a_2, b_t)\}$. But none of these paths is an $S'$-path since $(a, b_t), (a_2, b), (a_1, b_t) \notin S'$. Thus, $|S'| \geq m$.

Therefore, $h(K_m \times K_n) = m$. □

**Theorem 3.2.** Let $G$ be a non-complete connected graph of order at least 3. Then $h(G \times K_n) = min\{|V(G)|, n \cdot h(G)\}$ for $n \geq 3$.

*Proof.* Suppose $G$ is a non-complete connected graph of order at least 3 and $n \geq 3$. Consider the following cases:

**Case 1**. $|V(G)| \leq n \cdot h(G)$.

Let $w \in V(K_n)$ and $S = V(G) \times \{w\}$. Let $(u_1, v_1), (u_2, v_2) \in V(G \times K_n)\backslash S$. Suppose $u_1 \neq u_2$. Since $G$ is connected, there exists a path $\{u_1, a_1, ..., a_r, u_2\}$ in $G$. Then

$$\{(u_1, v_1), (u_1, w), (a_1, w), ..., (a_r, w), (u_2, w), (u_2, v_2)\}$$

is an $S$-path in $G \times K_n$. It follows that $S$ is a hub set in $G \times K_n$. Consequently, $h(G \times K_n) \leq |V(G)|$.

Suppose $h(G \times K_n) < |V(G)|$. Let $S'$ be a minimum hub set of $G \times K_n$ and for each $a \in V(G)$, let $S_a = \{(x,a) : x \in V(G)\}$. Then there exist $a_1 \in V(G)$ such that $S' \cap S_{a_1} = \emptyset$. Suppose $|S' \cap S_a| = 1$ for all $a \in V(G)$ and $a \neq a_1$. Let $(a,b) \in S' \cap S_a$. Let $b_1, b_2 \in V(K_n)$ such that $b \neq b_1$ and $b \neq b_2$. Then $(a,b_1)$ and $(a_1,b_2)$ are joined by the paths $\{(a,b_1),(a_1,b_1),(a_1,b_2)\}$ and $\{(a,b_1),(a,b_2),(a_1,b_2)\}$. But none of these paths is an $S'$-path since $(a_1,b_1),(a,b_2) \notin S'$. This contradicts the assumption that $S'$ is a hub set. Suppose $|S' \cap S_a| > 1$ for some $a \in V(K_m)$. Then there exists $a_1, a_2 \in V(K_m)$, $a_1 \neq a_2$ such that $S' \cap S_{a_1} = \emptyset$ and $S' \cap S_{a_2} = \emptyset$. Since $|S'| < m$, there exists $b_t \in V(K_n)$ such that $(a,b_t) \notin S' \cap S_a$. Now, $(a_1,b)$ and $(a_2,b_t)$ are joined by paths $\{(a_1,b),(a,b),(a,b_t),(a_2,b_t)\}$, $\{(a_1,b),(a_2,b),(a_2,b_t)\}$ and $\{(a_1,b),(a_1,b_t),(a_2,b_t)\}$. But none of these paths is an $S'$-path since $(a,b_t),(a_2,b),(a_1,b_t) \notin S'$. Thus, $|S'| \geq |V(G)|$.
Therefore, $h(G \times K_n) = |V(G)|$.

**Case 2**. $n \cdot h(G) \leq |V(G)|$.

Let $A$ be a minimum hub set in $G$ and $S = \{(a,b) : a \in A, b \in V(K_n)\}$. This implies that

$|S| = n(h(G))$. Let $(u_1,v_1),(u_2,v_2) \in V(G \times K_n) \backslash S$. If $u_1 = u_2$, then we can use the degenerate case. Suppose $u_1 \neq u_2$. Then $\{u_1, t_1, t_2, ..., t_r, u_r\}$ is an $A$-path in $G$ and

$$\{(u_1,v_1),(t_1,v_1),...,(t_r,v_1),(t_r,v_2),(u_2,v_2)\}$$

is an $S$-path in $G \times K_n$. It follows that $S$ is a hub set of $G \times K_n$. Consequently, $h(G \times K_n) \leq n \cdot h(G)$.

Suppose $h(G \times K_n) < n \cdot h(G)$. Let $S'$ be a minimum hub set of $G \times K_n$ and for each $v \in V(K_n)$, let $T_v = \{(x,v) : x \in V(G)\}$. Let $v \in V(K_n)$, $A \subseteq V(G)$ such that $A \times \{v\} = S' \cap T_v$, and

$a_1, a_2 \in V(G) \backslash A$. Then $(a_1,v),(a_2,v) \in V(G \times K_n) \backslash S'$. Since $S'$ is a hub set in $G \times K_n$, there exists a path $\{(a_1,v),(u_1,v_1),...,(u_t,v_t),(a_2,v)\}$ in $G \times K_n$ such that $(u_i,v_i) \in S'$, with $1 \leq i \leq t$. Then there exists a path $\{a_1, u_1^{'}, ..., u_s^{'}, a_2\}$ in $G$. This implies that $A$ is a hub set in $G$. Thus,

$$\begin{aligned}
|S'| &= \left| \bigcup_{v \in V(K_n)} (S' \cap T_v) \right| \\
&= \sum_{v \in V(K_n)} |S' \cap T_v| = \sum_{v \in V(K_n)} |A| \\
&\geq \sum_{v \in V(K_n)} h(G) \\
&= n \cdot h(G).
\end{aligned}$$

Hence, $h(G \times K_n) = n \cdot h(G)$.

Therefore, $h(G \times K_n) = min\{|V(G)|, n \cdot h(G)\}$. □

**Theorem 3.3.** Let $n \geq 4$. If $m = 2$ or $m = 3$ , then $h(P_m \times P_n) = n$.

*Proof.* Let $S = \{(i, 2) : 1 \leq i \leq n\}$. This implies that $|S| = n$. Let $(a, b), (c, d) \in V(P_m \times P_n) \backslash S$. Suppose $m = 2$ and $n \geq 4$. That is, $a = c$ and $b \neq d$. Let $\{b_1, b_2, ..., b_t\}$ be a $b$-$d$ path, where $b_1 = b$ and $b_t = d$. Then $\{(a, b), (2, b_1), (2, b_2), ..., (2, b_t), (c, d)\}$ is an $S$-path. Suppose $m = 3$ and $n \geq 4$. If $a \neq c$ and $b \neq d$, then $\{(a, b), (2, b_1), (2, b_2), ..., (2, b_t), (c, d)\}$ is an $S$-path. Thus, $S$ is a hub set of $P_m \times P_n$. It follows that $h(P_m \times P_n) \leq n$.

Suppose there exists a hub set $S'$ such that $|S'| < n$. Let $A_i = \{(a, i) : 1 \leq a \leq m\}$. Consider the following claims:

1. If $S' \cap A_1 = \emptyset$, then $|S' \cap A_2| = m$.
2. If $S' \cap A_n = \emptyset$, then $|S' \cap A_{n-1}| = m$.
3. $|S' \cap A_i| \geq 1$, $1 < i < n$.

Suppose $S' \cap A_1 = \emptyset$. That is, $(a, 1) \notin S'$ for $1 \leq a \leq m$. Then $(a, 2) \in S'$ for $1 \leq a \leq m$ since $S'$ is a hub set. This implies that $|S' \cap A_2| = m$. The next claim is proved similarly with the previous claim. Now, if $|S' \cap A_2| = m$, then $(a, j) \in S'$ for some $a \in V(P_m)$ and $3 \leq j \leq n$ since $S'$ is a hub set. Thus, if $S' \cap A_1 = \emptyset$, we have $|S' \cap A_i| \geq 1$ for $1 < i < n$.

Now, if there exists $t$, with $1 < t < n$, such that $|S' \cap A_t| = 0$. Since $S'$ is a hub set, we have the following cases:

**Case 1**. If $|S' \cap A_1| = ... = |S' \cap A_{t-1}| = m$. Then $|S' \cap A_{t+1}| \geq 1$,..., $|S' \cap A_{n-1}| \geq 1$, $|S' \cap A_n| \geq 1$. Thus, $|S'| = |S' \cap A_1| + ... + |S' \cap A_n| \geq m(t-1) + (n-t) = mt - m + n - t$. If $m = 2$, then $|S'| \geq 2t - 2 + n - t = t + n - 2 \geq n$ which is a contradiction. If $m = 3$, then $|S'| \geq 3t - 3 + n - t = 2t + n - 3 \geq n$ which is a contradiction.

Suppose $|S' \cap A_{t+1}| \geq 1$,..., $|S' \cap A_{n-1}| = m$, $|S' \cap A_n| = 0$. Thus, $|S'| = |S' \cap A_1| + ... + |S' \cap A_n| \geq m(t-1) + (n-t-2) + m = mt + n - t - 2$. If $m = 2$ and $1 < t < n$, $2t + n - t - 2 = t + n - 2 \geq n$ since $2 \leq t < n$. If $m = 3$ and $1 < t < n$, $3t + n - t - 2 = 2t + n - 2 \geq n$ since $2 \leq t < n$.

**Case 2**. If $|S' \cap A_{t+1}| = ... = |S' \cap A_n| = m$. The proof is similar to the previous case.

Thus,

$$\begin{aligned}
|S'| &= \sum_{i=1}^{n} |S' \cap A_i| \\
&= |S' \cap A_1| + |S' \cap A_2| + |S' \cap A_3| + ... + |S' \cap A_{n-2}| + |S' \cap A_{n-1}| \\
&+ |S' \cap A_n| \\
&= 2 + (n-2) \geq n
\end{aligned}$$

Therefore, for $n \geq 4$, $m = 2$ or $m = 3$, then $h(P_m \times P_n) = n$. □

**Theorem 3.4.** Let $4 \le m \le n$, where $m$ and $n$ are positive integers. Then

$$h(P_m \times P_n) \le \begin{cases} \frac{mn+m}{3} & \text{, if } m \equiv 0 \pmod 3 \text{ ,} \\ \frac{mn+n}{3} & \text{, if } m \not\equiv 0 \pmod 3 \text{ , } n \equiv 0 \pmod 3 \text{ ,} \\ \frac{mn+m+n-3}{3} & \text{, if } m \equiv 1 \pmod 3 \text{ , } n \equiv 1 \pmod 3 \text{ ,} \\ \frac{mn+m+n-2}{3} & \text{, otherwise .} \end{cases}$$

*Proof.* Suppose $m \equiv 0 (\text{mod } 3)$ and $4 \le m \le n$. Suppose $m \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 3$. Let

$$S = \{(i,2) : 1 \le i \le m\} \cup \{(3i-1, j) : 1 \le i \le \frac{m}{3}, 3 \le j \le n\}.$$

This implies that $|S| = m + \frac{m}{3}(n-2) = \frac{mn+m}{3}$. Let $(a,b),(c,d) \in V(P_m \times P_n)\backslash S$. If $d((a,b),(c,d)) = 1$, the degenerate case, then the trivial $S$-path holds for this. Assume $d((a,b),(c,d)) \ge 2$. Consider the following subcases:

**Case 1**. Suppose $a = c$. Let $a = 3k - r$, where $r = 0, 2$. If $b = 1$ and $\{b_1, b_2, ..., b_t\}$ is a $b$-$d$ path in $P_n$, where $b_1 = b$ and $b_t = d$, then

$$\{(a,b),(a,b_2),(3k-1,b_2),...,(3k-1,b_t),(c,d)\}$$

is an $S$-path. If $2 < b < d \le n$, then

$$\{(a,b),(3k-1,b),(3k-1,b_2),...,(3k-1,b_t),(c,d)\}$$

is an $S$-path.

**Case 2**. Suppose $b = d$. If $b = 1$ and $\{a_1, a_2, ..., a_t\}$ is an $a$-$c$ path in $P_m$, where $a_1 = a$ and $a_t = c$, then

$$\{(a,b),(a,2),(a_2,2),...,(a_t,2),(c,d)\}$$

is an $S$-path. Let $a = 3k_1 - r_1$ where $r_1 = 0, 2$ and $b = 3k_2 - r_2$ where $r_2 = 0, 2$. If $b > 2$, then

$$\{(a,b),(3k_1-1,b),...,(3k_1-1,2),...,(3k_2-1,2),...,(3k_2-1,d),(c,d)\}$$

is an $S$-path.

**Case 3**. Suppose $a \ne c$ and $b \ne d$. Let $c = 3k - r$ where $r = 0, 2$. If $b = 1$, then

$$\{(a,b),(a,2),...,(3k-1,2),...,(3k-1,d),(c,d)\}$$

is an $S$-path. Suppose $2 < b < d \le n$. Let $a = 3k_1 - r_1$ and $c = 3k_2 - r_2$ for some $r_1, r_2 = 0, 2$. Then

$$\{(a,b),(3k_1-1,b),...,(3k_1-1,2),...,(3k_2-1,2),...,(3k_2-1,d),(c,d)\}$$

is an $S$-path.

Thus, $S$ is a hub set of $P_m \times P_n$. It follows that $h(P_m \times P_n) \le \frac{mn+m}{3}$.
The cases $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$ are proved similarly.
The proofs of the remaining cases are constructive and are proved similarly.

□

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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