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NUMERICAL AND THEORETICAL STUDY OF THE PROPERTIES OF A LINEAR ELASTIC PERIDYNAMIC MATERIAL

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Abstract. *The peridynamic theory is a generalization of classical continuum mechanics and takes into account the interaction between material points separated by a finite distance within a peridynamic horizon δ. The δ parameter corresponds to a length scale and is treated as a material property related to the microstructure of the body. This work concerns a study of the properties of a linear elastic peridynamic material in the context of a three-dimensional state-based peridynamic theory, which considers both length and relative angle changes, and is based upon a free energy function that contains four material constants. Using convergence results of the peridynamic theory to the classical linear elasticity theory in the limit of vanishing sequences of δ and a correspondence argument between the proposed free energy function and the strain energy density function from the classical theory, expressions were obtained relating three peridynamic constants to the classical elasticity constants of an isotropic linearly elastic material. To evaluate the fourth peridynamic material constant, the correspondence argument is used together with the deformation field of an elastic beam subjected to pure bending. This work also concerns the validation of the proposed linearly elastic peridynamic model through numerical simulations of mechanical problems formulated in the context of both the classical linear elasticity and peridynamic theories. Simulation results will be presented at the meeting.*

Keywords: *Peridynamics, Length Scale, Elasticity, Nonlocal Theory, Free Energy Function*

1 INTRODUCTION

Peridynamics is a non-local theory of continuum mechanics that considers the interaction of material points due to forces acting at a finite distance. The interaction between particles is considered null when this distance exceeds a certain value called peridynamic horizon (δ). The elastic peridynamic theory is a generalization of the classical elasticity theory in the sense that the peridynamic operators converge to the corresponding operators of the classical elasticity on the small horizon limit. The motivation for developing this theory comes from the intention of modeling the behavior of a material in regions with singularities. In contrast with the classical approach, the balance of linear momentum is formulated as an integral equation that remains valid in the presence of discontinuities, such as in the case of Griffith cracks.

Silling et al. (2007) propose a strain energy density function for isotropic materials using the extension scalar field, which is the length change between two particles due to deformation. The authors obtain an energy function for the simple elastic peridynamic material that, as in the classical elasticity theory for isotropic materials, depends only upon two elastic peridynamic constants.

Aguiar and Fosdick (2013) present a three-dimensional state-based linearly elastic peridynamic theory using the relative displacement field between particles. The authors propose a free energy function that depends on deformation measures that are analogous to the measures of strain in classical linear elasticity theory and contains four elastic peridynamic constants. Using vanishing sequences of the horizon δ, the authors find two relations between three peridynamic constants and the two Lamé constants of classical linear elasticity.

In order to obtain a third relation, Aguiar (2015) introduces a correspondence argument between the free energy peridynamic function and a weighted average of the energy density function of the classical theory. This argument provides the two relations mentioned above in the case of homogenous deformations, being, therefore, compatible with the theory presented by Aguiar and Fosdick (2013). Aguiar (2015) uses this argument along with the non-homogenous deformation of a circular shaft under uniform torsion to obtain the third relation. This relation along with those two relations mentioned above allow evaluating three of the four peridynamic constants.

In this paper, we have used the correspondence argument together with the non-homogenous deformation of a beam bent by terminal couples to obtain an expression for the fourth peridynamic constant in terms of the classical elasticity constants.

With the four peridynamic constants so determined, we wish to simulate numerically both the torsion of a circular shaft and the beam bent by terminal couples using the Aguiar and Fosdick (2013) model and to compare the obtained computational results with corresponding analytical results from classical theory. This numerical investigation is under development.

In Section 2, we introduce measures of angular distortion and length changes between material points in the peridynamic theory. These measures are used in Section 3 to present the free energy function proposed by Aguiar and Fosdick (2013). This function is used to obtain the linearized form of the vector force state, which is analogous to the stress tensor of the classical linear theory. Corresponding expressions for the strain energy function and the linearized force vector state proposed by Silling et al. (2007) are presented in Section 3.1. It is observed that these expressions correspond to particular cases of the expressions proposed by Aguiar and

Fosdick (2013). The correspondence argument mentioned above is introduced in Section 3.2, which is used in the determination of three peridynamic constants. The use of this argument along with the deformation field of a cylindrical beam bent by terminal couples allows to obtain a closed-form expression for the fourth peridynamic constant in Section 4, which is one of the main goals of this work. Another goal is to simulate numerically problems in mechanics to validate the peridynamic model proposed by Aguiar and Fosdick (2013). Simulation results will be presented at the event. Conclusions of this work are found in Section 5.

2 KINEMATICS OF SMALL DEFORMATION

Let $\mathcal{B} \in \mathbb{E}^3$ be the undistorted reference configuration of an elastic body and let $\mathbf{y} := \boldsymbol{\chi}(\mathbf{x}, t)$ be the position of the particle $\mathbf{x} \in \mathcal{B}$ at time $t \geq 0$. Here, a neighborhood of any point $\mathbf{x}_0$ is a sphere of radius δ centered at $\mathbf{x}_0$, which we denote by $N_\delta(\mathbf{x}_0) \subset \mathcal{B}$. The vector $\boldsymbol{\xi} := \mathbf{x} - \mathbf{x}_0$ is called a bond from $\mathbf{x}$ to $\mathbf{x}_0$, where $\mathbf{x} \in N_\delta$, as illustrated in Fig. 1. The set $\mathcal{H}_\delta(\mathbf{x}_0)$ is the collection of all bonds to $\mathbf{x}_0$.

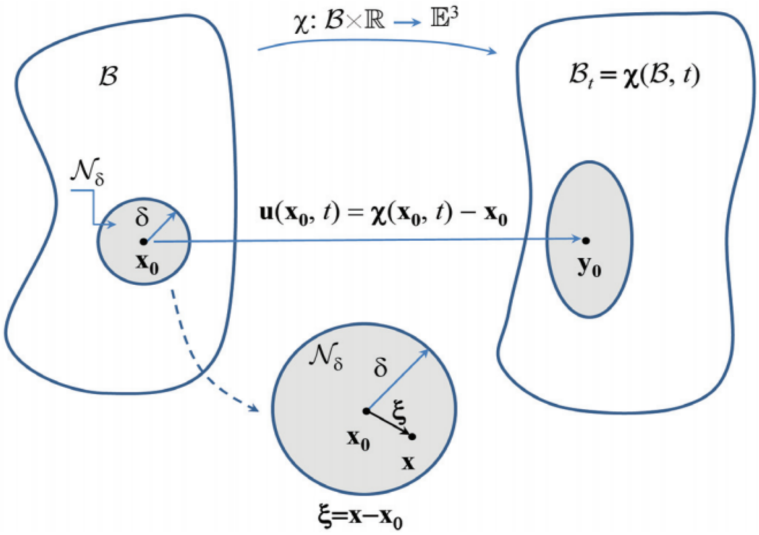

Figure 1: Reference and deformed configurations of a body. Source: Aguiar and Fosdick (2013).

A peridynamic state at $(\mathbf{x}_0, t)$ of order m is a function $\underline{\mathbf{A}}(\mathbf{x}_0, t)\langle\cdot\rangle : \mathcal{H}_\delta(\mathbf{x}_0) \rightarrow \mathcal{L}_m$, where $\mathcal{L}_m$ is the set of all tensors of order m. Thus, the image of a bond $\xi \in \mathcal{H}_\delta(\mathbf{x}_0)$ for the state $\underline{\mathbf{A}}(\mathbf{x}_0, t)\langle\cdot\rangle$ is the tensor of order m, $\underline{\mathbf{A}}(\mathbf{x}_0, t)\langle\boldsymbol{\xi}\rangle$. We denote by $\mathcal{A}_m$ the set of all states at $(\mathbf{x}_0, t)$ of order m. Similarly, we introduce the definition of a double state at $(\mathbf{x}_0, t)$ of order p, $\underline{\underline{\mathbf{D}}}(\mathbf{x}_0, t)\langle\cdot,\cdot\rangle : \mathcal{H}_\delta(\mathbf{x}_0) \times \mathcal{H}_\delta(\mathbf{x}_0) \rightarrow \mathcal{L}_p$. The dependency between two states $\underline{\mathbf{A}}(\mathbf{x}_0, t)\langle\cdot\rangle : \mathcal{H}_\delta(\mathbf{x}_0) \rightarrow \mathcal{L}_m$ and $\underline{\mathbf{u}}(\mathbf{x}_0, t)\langle\cdot\rangle : \mathcal{H}_\delta(\mathbf{x}_0) \rightarrow \mathcal{L}_p$ is denoted by $\underline{\mathbf{A}}(\mathbf{x}_0, t)\langle\boldsymbol{\xi}\rangle = \widehat{\underline{\mathbf{A}}}(\mathbf{x}_0, t)[\underline{\mathbf{u}}]\langle\boldsymbol{\xi}\rangle$. For notational convenience, we shall not exhibit the dependence on the time variable t and, when the meaning is clear, may also omit the dependence on the particle $\mathbf{x}_0$.

The difference deformation state $\underline{\boldsymbol{\chi}} \in \mathcal{A}_1$ at $\mathbf{x}_0 \in \mathcal{B}$ is defined through

$$\underline{\boldsymbol{\chi}}\langle\boldsymbol{\xi}\rangle := (\boldsymbol{\chi}(\mathbf{x}) - \boldsymbol{\chi}(\mathbf{x}_0)) \mid_{\mathbf{x}=\mathbf{x}_0+\boldsymbol{\xi}} .$$

With $\underline{\mathbf{u}} \in \mathcal{A}_1$ being the difference displacement state and $\underline{\mathbf{x}} \in \mathcal{A}_1$ the reference position vector state at $\mathbf{x}_0 \in \mathcal{B}$, we may write $\underline{\boldsymbol{\chi}} = \underline{\mathbf{u}} + \underline{\mathbf{x}}$. The difference deformation and displacement quotient states at $\mathbf{x}_0 \in \mathcal{B}$ are then defined by

$$\underline{\mathbf{f}} := \frac{\underline{\boldsymbol{\chi}}}{|\underline{\mathbf{x}}|} = \underline{\mathbf{h}} + \underline{\mathbf{e}}, \qquad \underline{\mathbf{h}} := \frac{\underline{\mathbf{u}}}{|\underline{\mathbf{x}}|}, \tag{1}$$

where $\underline{\mathbf{e}} := \underline{\mathbf{x}}/\,|\underline{\mathbf{x}}|$, and $|\underline{\mathbf{A}}|$ is the magnitude state of $\underline{\mathbf{A}}$, defined through

$$|\underline{\mathbf{A}}|\,\langle\boldsymbol{\xi}\rangle := \sqrt{\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle}, \tag{2}$$

with " $\cdot$ " being the scalar product in $\mathbb{E}^3$.

When the deformation of the body is small, we suppose that there is an ϵ, $|\epsilon| < 1$, such that, at $\mathbf{x}_0 \in \mathcal{B}$, $\underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle \equiv \underline{\mathbf{u}}\langle\boldsymbol{\xi}\rangle/|\boldsymbol{\xi}| = O(\epsilon)$, $|\boldsymbol{\xi}| < \delta$. Using Eq. (1) and Eq. (2), we get

$$|\underline{\mathbf{f}}\langle\boldsymbol{\xi}\rangle| \equiv \frac{|\underline{\boldsymbol{\chi}}\langle\boldsymbol{\xi}\rangle|}{|\boldsymbol{\xi}|} = 1 + \underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle + O(\epsilon^2). \tag{3}$$

The strain of a bond $\boldsymbol{\xi}$ at $\mathbf{x}_0$ is defined as the change of length of $\boldsymbol{\xi}$ per unit of length $\boldsymbol{\xi}$ as the result of a deformation, i.e., $\underline{e}\langle\boldsymbol{\xi}\rangle := (|\underline{\boldsymbol{\chi}}| - |\boldsymbol{\xi}|)/|\boldsymbol{\xi}|$. Thus, the infinitesimal normal strain state $\underline{\varepsilon}\langle\cdot\rangle : \mathcal{H}_\delta \to \mathbb{R}$ at $\mathbf{x}_0$ is given by the second term on the right-hand side of Eq. (3), viz.,

$$\underline{\varepsilon}\langle\boldsymbol{\xi}\rangle := \widehat{\underline{\varepsilon}}[\underline{\mathbf{h}}]\langle\boldsymbol{\xi}\rangle \equiv \underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle. \tag{4}$$

From Eq. (4) we see that $\widehat{\underline{\varepsilon}}[\cdot]\langle\boldsymbol{\xi}\rangle : \mathcal{A}_1 \to \mathbb{R}$ is a linear function.

Next, we define the infinitesimal shear strain state $\underline{\underline{\gamma}}\langle\cdot,\cdot\rangle : \mathcal{H}_\delta(\mathbf{x}_0) \times \mathcal{H}_\delta(\mathbf{x}_0) \to \mathbb{R}$ through the expression

$$\underline{\underline{\gamma}}\langle\boldsymbol{\xi},\boldsymbol{\eta}\rangle := \widehat{\underline{\underline{\gamma}}}[\mathrm{h}]\langle\boldsymbol{\xi},\boldsymbol{\eta}\rangle \equiv \frac{1}{2}(\underline{\underline{\mathbf{e}}}\langle\boldsymbol{\xi},\boldsymbol{\eta}\rangle \cdot \underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle + \underline{\underline{\mathbf{e}}}\langle\boldsymbol{\eta},\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{h}}\langle\boldsymbol{\eta}\rangle), \quad |\boldsymbol{\xi}| < \delta,\ |\boldsymbol{\eta}| < \delta, \tag{5}$$

where we have used the notation

$$\underline{\underline{\mathbf{e}}}\langle\boldsymbol{\xi},\boldsymbol{\eta}\rangle := \frac{(\mathbf{1} - \underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle \otimes \underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle)\underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle}{\sin\alpha} = \frac{\underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle - \underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle\cos\alpha}{\sin\alpha}, \tag{6}$$

in which $\mathbf{1}$ is the identity tensor in $\mathcal{L}_2$ and α is the smallest included angle between $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. From Eq. (5) and Eq. (6) we see that $\widehat{\underline{\underline{\gamma}}}[\cdot]\langle\boldsymbol{\xi},\boldsymbol{\eta}\rangle : \mathcal{A}_1 \to \mathbb{R}$ is a linear function.

The difference displacement quotient state at $\mathbf{x}_0 \in \mathcal{B}$, defined in Eq. (1), can be decomposed as

$$\underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle = \underline{\varphi}\langle\boldsymbol{\xi}\rangle\underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle + \underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle, \tag{7}$$

where $\underline{\varphi}$ is a scalar state that yields the radial component of $\underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle$ and $\underline{\mathbf{h}}_d$ is a vector state that satisfies $\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle = 0$.

3 SIMPLE PERIDYNAMIC MATERIAL

Aguiar and Fosdick (2013) use the infinitesimal strains $\widehat{\underline{\varepsilon}}[\mathbf{h}]$ and $\widehat{\underline{\underline{\gamma}}}[\mathbf{h}]$ defined in Eq. (4) and Eq. (5), respectively, to propose a quadratic free energy function for a simple elastic material. The reduced form of this function is given by

$$\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \int_{N_\delta}\int_{N_\delta} \omega(|\boldsymbol{\xi}|,|\boldsymbol{\eta}|)\{\frac{\widehat{\alpha}_{11}}{2}(\widehat{\underline{\varepsilon}}[\mathbf{h}]\langle\boldsymbol{\xi}\rangle)^2 + \alpha_{12}\,\widehat{\underline{\varepsilon}}[\mathbf{h}]\langle\boldsymbol{\xi}\rangle\,\widehat{\underline{\varepsilon}}[\mathbf{h}]\langle\boldsymbol{\eta}\rangle + \frac{\alpha_{33}}{2}(\widehat{\underline{\underline{\gamma}}}[\mathbf{h}]\langle\boldsymbol{\xi},\boldsymbol{\eta}\rangle)^2 + \widehat{\alpha}_{13}\,\widehat{\underline{\underline{\gamma}}}[\mathbf{h}]\langle\boldsymbol{\xi},\boldsymbol{\eta}\rangle\,\widehat{\underline{\varepsilon}}[\mathbf{h}]\langle\boldsymbol{\xi}\rangle\}dv_\eta dv_\xi, \tag{8}$$

where $\omega(\cdot,\cdot)$ is a given symmetric weighting function and $\widehat{\alpha}_{11}$, α_{12}, $\widehat{\alpha}_{13}$, and α_{33} are elastic peridynamic constants.

Substituting Eq. (7) into Eq. (8) and using the equalities $\underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle\cdot\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle \equiv 0$, $\underline{\underline{\mathbf{e}}}\langle\boldsymbol{\xi},\boldsymbol{\eta}\rangle\cdot\underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle \equiv 0$ and $\underline{\underline{\mathbf{e}}}\langle\boldsymbol{\xi},\boldsymbol{\eta}\rangle\cdot\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle = \underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle\cdot\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle/\sin\alpha$, we get the alternative form

$$\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \widehat{W}_{\mathbf{x}_0}[\underline{\varphi}\,\underline{\mathbf{e}}] + \widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}_d] + \frac{\widehat{\alpha}_{13}}{2}\int_{N_\delta}\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle\cdot\int_{N_\delta}\frac{\omega(|\boldsymbol{\xi}|,|\boldsymbol{\eta}|)}{\sin\alpha}(\underline{\varphi}\langle\boldsymbol{\xi}\rangle + \underline{\varphi}\langle\boldsymbol{\eta}\rangle)\underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle dv_\eta dv_\xi, \tag{9}$$

where it follows from Eq. (8) that

$$\widehat{W}_{\mathbf{x}_0}[\underline{\varphi}\,\underline{\mathbf{e}}] = \frac{1}{2}\int_{N_\delta}\underline{\varphi}\langle\boldsymbol{\xi}\rangle\cdot\int_{N_\delta}\omega(|\boldsymbol{\xi}|,|\boldsymbol{\eta}|)[\widehat{\alpha}_{11}\underline{\varphi}\langle\boldsymbol{\xi}\rangle + 2\alpha_{12}\underline{\varphi}\langle\boldsymbol{\eta}\rangle]dv_\eta dv_\xi, \tag{10}$$

$$\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}_d] = \frac{\alpha_{33}}{4}\int_{N_\delta}\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle\cdot\int_{N_\delta}\frac{\omega(|\boldsymbol{\xi}|,|\boldsymbol{\eta}|)}{(\sin\alpha)^2}[\underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle\cdot\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle + \underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle\cdot\underline{\mathbf{h}}_d\langle\boldsymbol{\eta}\rangle]\underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle dv_\eta dv_\xi. \tag{11}$$

Near the natural state, the peridynamic equation of motion is given by

$$\rho(\mathbf{x}_0)\ddot{\mathbf{u}}(\mathbf{x}_0) = \int_{N_\delta}\{\underline{\mathbf{L}}(\mathbf{x}_0)\langle\mathbf{x}-\mathbf{x}_0\rangle - \underline{\mathbf{L}}(\mathbf{x})\langle\mathbf{x}_0-\mathbf{x}\rangle\}dv_x + \mathbf{b}(\mathbf{x}_0), \tag{12}$$

where ρ is the mass density, $\mathbf{u}$ is the displacement field, $\mathbf{b}$ is a prescribed density body force and $\underline{\mathbf{L}}(\mathbf{x}_0)\langle\cdot\rangle$ is the linearized form of the force vector state evaluated on bonds in $\mathbf{x}_0$. Equation (12) is analogous to the differential equation of balance of linear momentum from the classical linear theory.

For a simple elastic material, Aguiar and Fosdick (2013) show that

$$\underline{\mathbf{L}}(\mathbf{x}_0) \equiv \widehat{\underline{\mathbf{L}}}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \frac{\delta_{\underline{\mathbf{h}}}\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}]}{|\underline{\mathbf{x}}|}, \tag{13}$$

where $\delta_{\underline{\mathbf{h}}}$ is the Fréchet derivative with respect to $\underline{\mathbf{h}}$. Substituting Eq. (9) thru Eq. (11) into Eq. (13), we get

$$\widehat{\underline{\mathbf{L}}}_{\mathbf{x}_0}[\underline{\mathbf{h}}]\langle\boldsymbol{\xi}\rangle = \int_{N_\delta}\frac{\omega(|\boldsymbol{\xi}|,|\boldsymbol{\eta}|)}{|\boldsymbol{\xi}|}\Big\{\widehat{\alpha}_{11}\underline{\varphi}\langle\boldsymbol{\xi}\rangle\underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle + 2\alpha_{12}\underline{\varphi}\langle\boldsymbol{\eta}\rangle\underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle + \frac{\alpha_{33}}{2\sin\alpha}\left(\underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle\cdot\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle + \underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle\cdot\underline{\mathbf{h}}_d\langle\boldsymbol{\eta}\rangle\right)\underline{\underline{\mathbf{e}}}\langle\boldsymbol{\xi},\boldsymbol{\eta}\rangle + \frac{\widehat{\alpha}_{13}}{2}\left[(\underline{\varphi}\langle\boldsymbol{\xi}\rangle + \underline{\varphi}\langle\boldsymbol{\eta}\rangle)\underline{\underline{\mathbf{e}}}\langle\boldsymbol{\xi}\boldsymbol{\eta}\rangle + \frac{1}{\sin\alpha}(\underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle\cdot\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle + \underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle\cdot\underline{\mathbf{h}}_d\langle\boldsymbol{\eta}\rangle)\underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle\right]\Big\}dv_\eta. \tag{14}$$

3.1 Relation with the model proposed by Silling et al. (2007)

Aguiar (2015) shows that, near the natural state, the free energy function proposed by Silling et al. (2007) for a simple elastic peridynamic material can be approximated by

$$\underline{\widetilde{\mathbf{W}}_{\mathbf{x}_0}}[\underline{\mathbf{h}}] := \frac{\tilde{\kappa}\overline{\vartheta}[\underline{\varphi}]^2}{2} + \frac{\tilde{\alpha}}{2}\int_{N_\delta} \tilde{\omega}(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^2 \left(\underline{\varphi}\langle\boldsymbol{\xi}\rangle - \frac{\overline{\vartheta}[\underline{\varphi}]}{3}\right)^2 dv_{\boldsymbol{\xi}}, \tag{15}$$

where $\tilde{\omega} : \mathbb{R} \rightarrow \mathbb{R}$ is a known weighting function, $\tilde{\kappa}$ and $\tilde{\alpha}$ are peridynamic material constants, and

$$m := \int_{N_\delta} \tilde{\omega}(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^2 dv_{\boldsymbol{\xi}}, \quad \overline{\vartheta}[\underline{\varphi}] := \frac{3}{m}\int_{N_\delta} \tilde{\omega}(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^2 \underline{\varphi}\langle\boldsymbol{\xi}\rangle dv_{\boldsymbol{\xi}}. \tag{16}$$

Notice from Eq. (4) thru Eq. (7) that $\underline{\varepsilon}\langle\boldsymbol{\xi}\rangle \equiv \underline{\varphi}\langle\boldsymbol{\xi}\rangle$ and that, therefore, Eq. (16.b) is a weighted average of the infinitesimal normal strain in a δ-neighborhood of $\mathbf{x}_0$. It follows from Eq. (15) and Eq. (16.b) that distortions caused by angle changes between bonds are not considered in the energy function proposed by Silling et al. (2007).

The linearized force response function state, obtained from Eq. (13) together with Eq. (15) and Eq. (16), is given by

$$\underline{\widetilde{\mathbf{L}}_{\mathbf{x}_0}}[\underline{\mathbf{h}}]\langle\boldsymbol{\xi}\rangle = \tilde{\omega}(|\boldsymbol{\xi}|)|\boldsymbol{\xi}| \left[\left(\tilde{\kappa} - \frac{\tilde{\alpha}m}{3^2}\right)\frac{3}{m}\overline{\vartheta}[\underline{\varphi}] + \tilde{\alpha}\underline{\varphi}\langle\boldsymbol{\xi}\rangle\right] \underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle. \tag{17}$$

Comparing Eq. (14) with Eq. (17), it is observed that the expressions are equivalent if

$$\widehat{\alpha}_{11} = \frac{\tilde{\alpha}}{m}, \quad 2\alpha_{12} = \left(\frac{3}{m}\right)^2 \tilde{\kappa} - \frac{\tilde{\alpha}}{m}, \quad \alpha_{33} = \widehat{\alpha}_{13} = 0,$$

and

$$\omega(|\boldsymbol{\xi}|, |\boldsymbol{\eta}|) = \widetilde{\omega}(|\boldsymbol{\xi}|)\widetilde{\omega}(|\boldsymbol{\eta}|)|\boldsymbol{\xi}|^2|\boldsymbol{\eta}|^2. \tag{18}$$

The linearized model proposed by Silling et al. (2007) is, therefore, a particular case of the linear model proposed by Aguiar and Fosdick (2013).

3.2 Determination of three peridynamic constants

Using convergence results obtained by Silling and Lehoucq (2008), Aguiar and Fosdick (2013) show that, for a homogeneous deformation corresponding to an infinitesimal displacement gradient $\mathbf{H}_0$,

$$\frac{1}{2}\mathbf{H}_0 \cdot (\mathbb{C}_{\mathbf{x}_0}\mathbf{H}_0) = \widehat{W}_{\mathbf{x}_0}[\mathbf{H}_0\underline{\mathbf{e}}], \tag{19}$$

where $\mathbb{C}_{\mathbf{x}_0}$ is the elasticity tensor of the classical linear theory. They further show that the left-hand side of Eq. (19) reduces to the strain energy function of an isotropic classical linear elastic material, which is given by

$$\widehat{W}^L_{\mathbf{x}_0}[\mathbf{E}] = \frac{1}{2}[\lambda(tr\mathbf{E})^2 + 2\mu\mathbf{E}\cdot\mathbf{E}], \tag{20}$$

where λ and μ are the Lamé constants and $\mathbf{E} := (\mathbf{H}_0 + \mathbf{H}_0^T)/2$ is the infinitesimal strain tensor. Using the resulting relation between $\widehat{W}^L_{\mathbf{x}_0}[\underline{\mathbf{h}}]$, given by Eq. (20), and $\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}]$, given by Eq. (9) thru Eq. (11), the autors obtain two relations between the first three peridynamic material constants appearing in Eq. (8) and classical elasticity constants. These relations are given by

$$2\widehat{\alpha}_{11} + \alpha_{33} = \frac{15E}{16(1+\nu)\omega_\delta}, \quad \widehat{\alpha}_{11} + 2\alpha_{12} = \frac{3E}{16(1-2\nu)\omega_\delta}, \tag{21}$$

where ν is the Poisson's ratio, E the Young's modulus, and $\omega_\delta \equiv \pi^2 \int_0^\delta \int_0^\delta \omega(\check{\rho}, \hat{\rho}) \check{\rho}^2 \hat{\rho}^2 d\hat{\rho} d\check{\rho}$.

To obtain a third relation, Aguiar (2015) introduces a correspondence argument according to which the free energy function of the peridynamic material at $\mathbf{x}_0$ near the natural state is equal to the weighted average of the strain energy density function from classical elasticity theory in a δ-neighborhood of $\mathbf{x}_0$. Considering that Eq. (18) holds, the correspondence argument yields

$$\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \overline{W}^L_{\mathbf{x}_0}[\underline{\mathbf{h}}] := \frac{1}{m} \int_{N_\delta} \widetilde{\omega}(|\boldsymbol{\xi}|)\, |\boldsymbol{\xi}|^2\, \widehat{W}^L_{\mathbf{x}_0}[\widehat{\mathbf{E}}[\underline{\mathbf{h}}]] dv_\xi, \tag{22}$$

where $\widehat{\mathbf{E}}[\underline{\mathbf{h}}]$ is the infinitesimal strain tensor obtained from the vector state $\underline{\mathbf{h}}$ and m is given by Eq. (16.a). Observe from Eq. (22) with $\widehat{\mathbf{E}}[\underline{\mathbf{h}}]$ constant that $\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \widehat{W}^L_{\mathbf{x}_0}[\underline{\mathbf{h}}]$. Therefore, Eq. (19) with its left-hand side replaced by Eq. (20) and $\omega(\cdot,\cdot)$ given by Eq. (18) is a particular case of Eq. (22).

In addiction, the autor considers the infinitesimal strain tensor field from a non-homogeneous deformation, corresponding to the experiment of uniform torsion of a circular shaft. For $\mathbf{x}_0$ located on the axis of the shaft, $\underline{\varphi}\langle\boldsymbol{\xi}\rangle \equiv 0$ for all bonds to point $\mathbf{x}_0$.

Using the correspondence argument, given by Eq. (22), and considering uniform torsion of a circular shaft, Aguiar (2015) obtains the relation

$$\alpha_{33} = \frac{20\mu}{m^2}. \tag{23}$$

Replacing Eq. (23) and the expressions $\mu = E/(2(1+\nu))$ and $\kappa = E/(3(1-2\nu))$ in Eq. (21), the other two contants can also be determined, being given by

$$\widehat{\alpha}_{11} = \frac{5\mu}{m^2}, \quad \alpha_{12} = \frac{1}{2m^2}(9\kappa - 5\mu)\ . \tag{24}$$

4 DETERMINATION OF THE CONSTANT $\widehat{\alpha}_{13}$

The determination of the constant $\widehat{\alpha}_{13}$ is one of the objectives of this work. This constant represents non-local effects of the peridynamic material and can not be determined from the approach leading expressions in Eq. (21). To determinate $\widehat{\alpha}_{13}$, we use a simple experiment in mechanics that provides a deformation field for which both radial and non-radial components of $\underline{\mathbf{h}}$ in Eq. (7) do not vanish. The experiment consists of a beam bent by terminal couples, as illustrated in Fig. 2. The solution of this problem is given by (SOKOLNIKOFF, 1956)

$$\underline{\mathbf{u}}(\xi_1, \xi_2, \xi_3) = \frac{M}{2EI}[(\xi_3{}^2 + \nu\xi_1{}^2 - \nu\xi_2{}^2)\mathbf{e}_1 + 2\nu\xi_1\xi_2\mathbf{e}_2 - 2\xi_1\xi_3\mathbf{e}_3], \tag{25}$$

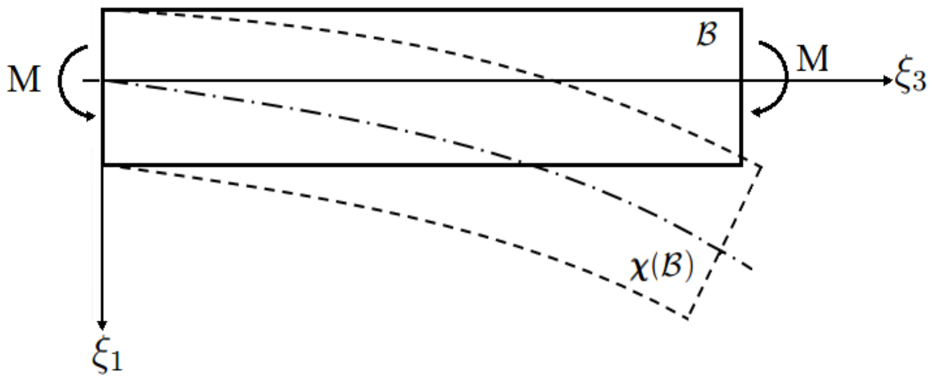

Figure 2: Beam bent by terminal couples.

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the orthonormal basis associated to the cartesian coordinates $\{\xi_1, \xi_2, \xi_3\}$ with origin at a point $\mathbf{x}_0$ on the axis of the shaft, which is aligned with the ξ_3-direction, M is the resulting moment applied at the ends of the beam, which is taken to be positive, I is the moment of inertia with respect to the ξ_2-direction, and we recall from section 3.2 that ν is the Poisson ratio and E is the Young's modulus. The non-zero components of the corresponding infinitesimal strain tensor $\mathbf{E}$ are given by

$$\epsilon_{11} = \epsilon_{22} = \frac{M}{EI}\nu\xi_1, \quad \epsilon_{33} = -\frac{M}{EI}\xi_1. \tag{26}$$

To apply the correspondence argument, in the form of Eq. (22), first, we use Eq. (20) together with $\widehat{\mathbf{E}}[\underline{\mathbf{h}}] \equiv \mathbf{E}(\xi_1, \xi_2, \xi_3)$ and the strain components given by Eq. (26) to obtain $\widehat{W}^L_{\mathbf{x}_0}[\widehat{\mathbf{E}}[\underline{\mathbf{h}}]] = (M^2/2EI^2)\xi_1^2$. Using the transformations $\xi_1 = \rho\, cos\theta\; sen\phi, \xi_2 = \rho\; sen\theta\; sen\phi, \xi_3 = \rho\; cos\phi$, where $(\rho,\ \phi,\ \theta)$ are spherical coordinates with origin at $\mathbf{x}_0$, and taking the limits of integration $\rho \in (0, \delta),\ \phi \in (0, \pi)$ and $\theta \in (0, 2\pi)$, we obtain

$$\overline{W}^L_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \frac{M^2 m_6}{6EI^2 m}, \tag{27}$$

where m_6 and $m = m_4$ are given by $m_n := 4\pi \int_0^\delta \widetilde{\omega}(\rho)\rho^n d\rho$.

Using the decomposition of Eq. (7), the expression $\underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle = \mathbf{u}(\rho, \phi, \theta)/\rho$, and the transformation from the cartesian coordinate system with base $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the spherical coordinate system with base $\{\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_\theta\}$, we get the radial and non-radial components

$$\underline{\varphi}\langle\boldsymbol{\xi}\rangle = \frac{M\rho}{2EI} cos\theta\; sen\phi(\nu\, sen^2\phi - cos^2\phi), \tag{28}$$
$$\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle = \frac{M\rho}{4EI}\{-cos\phi\; cos\theta[-3 - \nu + (1+\nu)cos(2\phi)]\mathbf{e}_\phi + 2sen\theta(\nu\, sen^2\phi - cos^2\phi)\mathbf{e}_\theta\},$$

respectively, of the vector state $\underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle(\rho, \phi, \theta)$.

Using Eq. (18), we rewrite Eq. (9) along with Eq. (10) and Eq. (11) in the form

$$\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \widehat{\alpha}_{11}\widehat{A}^{11}_{\mathbf{x}_0}[\underline{\mathbf{h}}] + \alpha_{12}\widehat{A}^{12}_{\mathbf{x}_0}[\underline{\mathbf{h}}] + \alpha_{33}(\widehat{A}^{33}_{\mathbf{x}_0}[\underline{\mathbf{h}}] + \widehat{B}^{33}_{\mathbf{x}_0}[\underline{\mathbf{h}}]) + \widehat{\alpha}_{13}(\widehat{A}^{13}_{\mathbf{x}_0}[\underline{\mathbf{h}}] + \widehat{B}^{13}_{\mathbf{x}_0}[\underline{\mathbf{h}}]), \tag{29}$$

where

$$
\begin{aligned}
&\widehat{A}^{11}_{\mathbf{x}_0}[\underline{\mathbf{h}}] := \frac{1}{2}\int_{N_\delta} \widetilde{\omega}(|\boldsymbol{\xi}|)\, |\boldsymbol{\xi}|^2\, \underline{\varphi}\langle\boldsymbol{\xi}\rangle^2 \int_{N_\delta} \widetilde{\omega}(|\boldsymbol{\eta}|)\, |\boldsymbol{\eta}|^2\, dv_\eta dv_\xi,\\
&\widehat{A}^{12}_{\mathbf{x}_0}[\underline{\mathbf{h}}] := \int_{N_\delta} \widetilde{\omega}(|\boldsymbol{\xi}|)\, |\boldsymbol{\xi}|^2\, \underline{\varphi}\langle\boldsymbol{\xi}\rangle \int_{N_\delta} \widetilde{\omega}(|\boldsymbol{\eta}|)\, |\boldsymbol{\eta}|^2\, \underline{\varphi}\langle\boldsymbol{\eta}\rangle dv_\eta dv_\xi,\\
&\widehat{A}^{33}_{\mathbf{x}_0}[\underline{\mathbf{h}}] := \frac{1}{4}\int_{N_\delta} \widetilde{\omega}(|\boldsymbol{\xi}|)\, |\boldsymbol{\xi}|^2\, \underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle \cdot \int_{N_\delta} \frac{\widetilde{\omega}(|\boldsymbol{\eta}|)\, |\boldsymbol{\eta}|^2}{(sen\alpha)^2} (\underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle \cdot \underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle)\underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle dv_\eta dv_\xi,\\
&\widehat{B}^{33}_{\mathbf{x}_0}[\underline{\mathbf{h}}] := \frac{1}{4}\int_{N_\delta} \widetilde{\omega}(|\boldsymbol{\xi}|)\, |\boldsymbol{\xi}|^2\, \underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle \cdot \int_{N_\delta} \frac{\widetilde{\omega}(|\boldsymbol{\eta}|)\, |\boldsymbol{\eta}|^2}{(sen\alpha)^2} (\underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{h}}_d\langle\boldsymbol{\eta}\rangle)\underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle dv_\eta dv_\xi,\\
&\widehat{A}^{13}_{\mathbf{x}_0}[\underline{\mathbf{h}}] := \frac{1}{2}\int_{N_\delta} \widetilde{\omega}(|\boldsymbol{\xi}|)\, |\boldsymbol{\xi}|^2\, \underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle \cdot \int_{N_\delta} \frac{\widetilde{\omega}(|\boldsymbol{\eta}|)\, |\boldsymbol{\eta}|^2}{sen\alpha} (\underline{\varphi}\langle\boldsymbol{\xi}\rangle)\underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle dv_\eta dv_\xi,\\
&\widehat{B}^{13}_{\mathbf{x}_0}[\underline{\mathbf{h}}] := \frac{1}{2}\int_{N_\delta} \widetilde{\omega}(|\boldsymbol{\xi}|)\, |\boldsymbol{\xi}|^2\, \underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle \cdot \int_{N_\delta} \frac{\widetilde{\omega}(|\boldsymbol{\eta}|)\, |\boldsymbol{\eta}|^2}{sen\alpha} (\underline{\varphi}\langle\boldsymbol{\eta}\rangle)\underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle dv_\eta dv_\xi.
\end{aligned}
\tag{30}
$$

Next, using Eq. (28) and the limits of integration introduced above Eq. (27) in the first two expressions of Eq. (30), we get

$$
\widehat{A}^{11}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \frac{mm_6}{8}\frac{M^2}{(EI)^2}\frac{(24\nu^2 - 8\nu + 3)}{105} \quad , \quad \widehat{A}^{12}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = 0. \tag{31}
$$

The integrals in the other four expressions in Eq. (30) are evaluated numerically using MATHEMATICA 9 ©. Each one of them was divided in three parts that multiply either 1, ν, or, ν^2. The values obtained from numerical integration converge to rational numbers multiplied by π^2, or, π^3. The four resulting expressions are then given by

$$
\begin{aligned}
&\widehat{A}^{33}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \frac{mm_6}{64}\left(\frac{M}{EI}\right)^2 \frac{(64\nu^2 + 16\nu + 92)}{105}, \qquad \widehat{B}^{33}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = 0,\\
&\widehat{A}^{13}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = 0 \quad , \quad \widehat{B}^{13}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \frac{\pi {m_5}^2}{32}\left(\frac{M}{EI}\right)^2 \frac{(-11\nu^2 + 20\nu - 4)}{105}.
\end{aligned}
\tag{32}
$$

Substituting Eq. (31) and Eq. (32) back into Eq. (29) and equating the resulting expression to Eq. (27), we finally get that

$$
\widehat{\alpha}_{13} = \frac{140}{\pi}\frac{m_6\,\mu}{m\,m_5^2}\left(\frac{8\nu^2 - 8\nu - 1}{11\nu^2 - 20\nu + 4}\right). \tag{33}
$$

In this section we have used the correspondence argument given by Eq. (22) and considered the pure bending experiment defined by the displacement field given by Eq. (25) to obtain an expression for the constant $\widehat{\alpha}_{13}$, which is given by Eq. (33). This expression along with Eq. (23) and Eq. (24), corresponding to the other three peridynamic constants, can now be substituted back into Eq. (8), or, equivalently, into Eq. (9) thru Eq. (11) to describe the behavior of the elastic peridynamic material.

5 CONCLUSIONS

Three of the four peridynamic constants that appear in the free energy function defined by Eq. (8) were previously determined. From Eq. (9) it is noted that, to determinate the constant $\widehat{\alpha}_{13}$, we need to consider a deformation state in which both $\mathbf{h}_d$ and $\underline{\varphi}$ do not vanish. Considering the pure bending experiment and using the correspondence argument between the free energy function and the weighted average of the strain energy density function from classical elasticity theory, in the form of Eq. (22), we obtain a relation between $\widehat{\alpha}_{13}$ and the elastic constants from classical theory.

It is of great interest to numerically simulate the behavior of a body using the classical linear elasticity model and the peridynamic models proposed by Aguiar and Fosdick (2013) and Silling et al. (2007). Comparing the simulations results, we aim at validating the model proposed by Aguiar and Fosdick (2013) and studying the existing differences between this model and the one proposed by Silling et al. (2007). To achieve the goals, we are investigating numerically simple problems in mechanics such as, for example, the torsion of a circular shaft with no body force. The results of this investigation will be presented at the event.

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