A GENERAL MESH SMOOTHING METHOD USING SINGULAR VALUE DECOMPOSITION

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Abstract. *The elements geometry in finite element meshes can be enhanced by means of mesh smoothing procedures. This paper present a smoothing technique based on the concept of ideal shapes for finite elements and the use of finite element deformation analyses. The ideal shape for a particular element corresponds to a regular shape with the same area or volume. This ideal element is optimally placed over the original one. Assuming a given mechanical stiffness for an element, the resulting nodal points distances between the original and new shapes are used to calculate the forces required to deform the original shape into the ideal one. The forces collected from all elements using this procedure are applied as boundary conditions in a conventional finite element analysis. The results provide a deformed mesh where the elements show improved quality. The whole process can be applied iteratively to get better improvements, however, this technique shows that few iterations are enough to obtain significant enhancement.*

Keywords: *Mesh Smoothing, Singular Value Decomposition, Finite Element Method*

1 INTRODUCTION

The accuracy of finite element analyses is highly dependent on the quality of the mesh in use. In fact, very thin or skewed elements may lead to poor results. Several authors pointed the effect on results due to distorted elements (Lee et al., 1993; Babuška et al., 1976; Oh et al., 1996; Shewchuk, 2002; Kim et al., 2012). Mesh improving techniques aim to enhace the shape of elements in order to increase the value of a given quality metric. Park et al. (2011) pointed the existence of three quality improvement techniques: adaptivity, smoothing and swapping.

Several mesh improvement methods were proposed after the invention of the finite element method (Field, 1988; Canann et al., 1998b; Bank et al., 1997; Freitag, 1997; Knupp, 2000; Freitag et al., 2002; Díaz et al., 2005; Yilmaz et al., 2009; Jiao et al., 2011; Park et al., 2011; Sun et al., 2012; Vartziotis et al., 2013; Renka, 2015; Wei et al., 2015). In particular, smoothing methods aim to improve the quality of elements by moving individual nodes while keeping the original mesh topology. Canann et al. (1998a) described three main approaches for mesh smoothing: Laplacian, optimization-based and physics-based approaches.

This paper proposes a physics-based mesh smoothing method for two-dimensional finite element meshes. In a similar way as the method proposed by Wei et al. (2015), this method is composed by two main steps: Local regularization and Global optimization. The local regularization step is performed by finding an ideal shape for each element in the mesh; later each ideal shape is adjusted at the location of the corresponding element using a least-squares fitting technique. In the second step, a global mesh optimization is performed by solving a finite element static analysis on the mesh. The boundary conditions are given by all nodal forces needed to deform the original elements into the ideal ones. Significant enhacement was obtained by applying the proposed method on meshes composed by triangular and quadrilateral elements.

2 SMOOTHING PROCEDURE

The proposed smoothing procedure is composed by two main steps: Local regularization and Global optimization. These steps can be executed iteratively to continue improving the quality of a mesh.

2.1 Local regularization of elements

In this step, for each element in the mesh, an ideal shape with ideal position should be found. Given an element, the idea is to get the geometry of an ideal shape, usually a regular polygon for triangular and quadrilateral elements, with the same area A as the original element. Initially, for each type of element, a matrix of initial coordinates that provide an area equal to one is considered. For example, for any quadrilateral element, the matrix with the initial coordinates is given by:

$$\mathbf{C}_0 = \begin{bmatrix} 0. & 0. \\ 1. & 0. \\ 1. & 1. \\ 0. & 1. \end{bmatrix}. \tag{1}$$

To get an ideal element with the same area as the original element, these coordinates are scaled by $A^{1/2}$, thus:

$$\mathbf{C}_r = A^{1/2}\mathbf{C}_0. \tag{2}$$

Later the set of points whose coordinates are listed in matrix $\mathbf{C}_r$ is adjusted (moved and rotated) to the coordinates of the origininal element. This is performed by a least-squares fitting procedure.

2.2 Least-squares fitting of two point sets

Given two point sets, $\mathbf{P}_i$ and $\mathbf{P}'_i$ with $i = 1, 2, 3, ..., n$, where $\mathbf{P}_i$ needs to be moved and rotated in order to get its points as close as possible to the points of $\mathbf{P}'_i$. For a rotation matrix $\mathbf{R}$ and translation matrix $\mathbf{T}$ we could write:

$$\mathbf{P}'_i = \mathbf{R}\mathbf{P}_i + \mathbf{T} + \mathbf{N}_i. \tag{3}$$

where $\mathbf{N}_i$ is a set of noise vectors. Then the problem is to find $\mathbf{R}$ and $\mathbf{T}$ that minimizes the following norm:

$$\sum_{i=1}^{n} \left\| \mathbf{P}'_i - (\mathbf{R}\mathbf{P}_i + \mathbf{T}) \right\|^2. \tag{4}$$

Arun et al. (1987) presented a procedure for least-squares fitting of two point sets. The procedure is applicable in two and three dimensions. According to the authors, a 3×3 matrix is calculated as a function of the coordinates of both sets as:

$$\mathbf{H} = \sum_{i=1}^{n} \mathbf{q}_i \mathbf{q}'^T_i. \tag{5}$$

where:

$$\mathbf{q}_i = \mathbf{P}_i - \overline{\mathbf{P}}. \tag{6}$$

$$\mathbf{q}'_i = \mathbf{P}'_i - \overline{\mathbf{P}}'. \tag{7}$$

$$\overline{\mathbf{P}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{P}_i. \tag{8}$$

$$\overline{\mathbf{P}}' = \frac{1}{n}\sum_{i=1}^{n} \mathbf{P}'_i. \tag{9}$$

From the singular value decomposition of $\mathbf{H}$ we get:

$$\mathbf{H} = \mathbf{U}\mathbf{A}\mathbf{V}. \tag{10}$$

For most cases of fitting point sets, the rotation matrix is found by:

$$\mathbf{R} = \mathbf{V}\mathbf{U}. \tag{11}$$

Finally, the translation matrix $\mathbf{T}$ is calculated by:

$$\mathbf{T} = \bar{\mathbf{P}}' - \mathbf{R}\bar{\mathbf{P}}. \quad (12)$$

This procedure is used to adjust the nodal coordinates of the ideal element (or reference element) to the nodes of the original element. In this sense, $\mathbf{P}$ represents the set with the nodal positions of the original element and $\mathbf{P}'$ the set with the nodal positions of the ideal element prior to fitting.

To guaranty a proper adjustment, an initial rotation is performed to the coordinates of the scaled ideal element. Thus, the optimally placed nodal coordinates of the ideal element can be obtained trough the rigid transform:

$$\mathbf{C} = \mathbf{C}_r\mathbf{R}^T + \mathbf{T}. \quad (13)$$

where $\mathbf{R}$ represents the rotation matrix between coordinate systems (x, y) and (r, s) as shown in Figure 1.

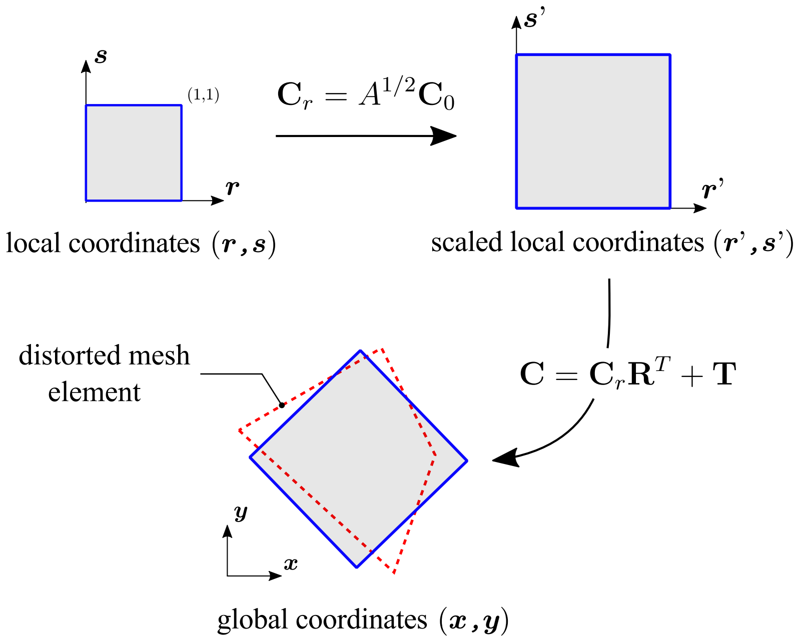


Figure 1: Least squares fitting of the regular element

2.3 Quality Metric

Acoording to Knupp (2001), an element quality metric is a scalar function of node positions that measures some geometric property of the element. There are several metrics proposed to represent the quality of elements in a finite element mesh (Robinson, 1987; Oddy A. Et al., 1988; Field, 2000). Usually those measures provide highest values to shapes that are close to the correspondig regular polygon.

In order to evalueate the improvement of quality of the elements after the application of the proposed algorithm, this work presents a simple but general quality metric. Thus, for a two-dimensional element $\boldsymbol{\tau}$ we have:

$$q(\boldsymbol{\tau}) = \frac{p^R}{p}. \quad (14)$$

where p is the perimenter of a mesh cell and p^R is the perimeter of a cell with ideal shape (e.g. regular polygon) with the same area. The isoperimetric inequality for regular polygons (see Osserman, 1978) ensures that $q(\boldsymbol{\tau})$ is limited to the real interval $(0, 1]$ for elements with linear sides. For the case of elements with quadratic sides, this value may rarely be greater than one.

Since the introduced quality measure is relative to the ideal quality of a regular polygon, it takes into account features as skew and elongation of elements. This simple measure can be easily implemented, with low computational cost, and extended to quadratic elements and even to three-dimensions.

2.4 Global regularization

After the ideal nodal positions for each element are already found following the local regularization procedure, the global regularization procedure uses these information to get a new mesh where elements present improved quality. This is achieved by solving a finite element deformation analysis assuming linear elastic behavior. For this analysis, the material parameters are given by the Young's modulus E and de Poisson ratio ν. In theory, any value could be taken for E but small values are preferred to avoid numerical errors, e.g. $E = 1$. As for the Poisson ratio, good result were obtained using $\nu = 0$.

For each element a displacements vector $\mathbf{U}$ is obtained subtracting the nodal coordinates from the ideal positions. Later, a vector of corresponding nodal forces $\mathbf{F}$ is determined with the aid of the element stiffness matrix $\mathbf{K}$, it is $\mathbf{F} = \mathbf{KU}$. The forces in $\mathbf{F}$ represent the necessary ones to turn the current element into its ideal counterpart after deformation. This procedure is repeated for each element in the mesh enabling to mount by superposition a global vector of forces $\mathbf{F}^G$.

Finally, a deformation analysis for the whole mesh is performed assuming the forces in $\mathbf{F}^G$ as natural boundary conditions. Essential boundary conditions are given by displacements restrictions according to the mesh boundary. The displacements obtained from this analysis provide the nodal shifts required to improve the mesh. After the mesh is updated the whole procedure can be repeated to get better improvements. Usually, the first iteration provides the greatest improvements. In most analyses, the authors observed that few iterations, less than five, were necessary to obtain significant enhancement.

2.5 Smoothing Algorithm

The smoothing algorithm used in this work can be summarized as:

1. Local regularization
   - 1.1. For each element in mesh get the ideal nodal positions.
   - 1.2. Scale, rotate and translate the ideal element and fit over the original element.
2. Global optimization
   - 2.1. For each element, get distances for each node to corresponding node of the ideal element to mount a displacement vector. Also calculate a corresponding force vector with the aid of the mechanical stiffness matrix from the element.
   - 2.2. Mount a global vector of forces that will represent natural boundary conditions.

2.3. Mount a global stiffness matrix.

2.4. Solve the finite element system using, as essentials boundary conditions, restrictions at nodes that are intended to be fixed.

2.5. Update the mesh with resulting displacements.

3. Evaluation of mesh improvement.

3 NUMERICAL EXAMPLES

In order to assess the applicability and effectiveness of the proposed mesh improvement method, four numerical examples are examined in this section. Triangular and quadrilateral meshes consisting of linear elements with different topologies are treated. The algorithm outlined in Section 2.4 was implemented in the *FemMesh* (Durand, 2015) software library, written in Julia language (Bezanson et al., 2012).

For a given general mesh composed of N elements, one can define the set $\{q_1, q_2, ..., q_N\}$ in which q_i is the computed quality metric, given by Eq. (12), for the element i. Three parameters are considered to evaluate the global mesh quality, namely, q_{min}, q_{avg} and q_{max}, defined as:

$$q_{min} = \mathbf{min}\left\{q_1, q_2, ..., q_N\right\}, \qquad q_{avg} = \frac{1}{N}\sum_{i=1}^{N} q_i\,, \qquad q_{max} = \mathbf{max}\left\{q_1, q_2, ..., q_N\right\}. \quad (15)$$

As stated previously, the smoothing procedure can be applied iteratively to increasingly improve the overall mesh quality. For the purpose of this paper, a convergence criterion was defined as:

$$\left|q^{k}_{min} - q^{k+1}_{min}\right| < \boldsymbol{\epsilon} \quad \wedge \quad \left|q^{k}_{avg} - q^{k+1}_{avg}\right| < \boldsymbol{\epsilon}. \quad (16)$$

where k indicate the calculation step and $\boldsymbol{\epsilon}$ is the required tolerance. Such condition enforces the iterative processes to terminate when the simple average and minimum element quality no longer improve considering two consecutive steps. For the problems regarded in this section, $\boldsymbol{\epsilon} = 10^{-2}$ was used.

3.1 Triangular Mesh

The interior nodes of a regular 1-by-1 mesh composed by near equilateral triangles were randomly displaced resulting in the distorted mesh depicted in Fig. 2. The smoothing process converges within 4 iterations and successfully improves the mesh quality while restoring the original mesh regular pattern. Figure 3 shows the initial distorted mesh and subsequent resulting meshes obtained along the iterations.

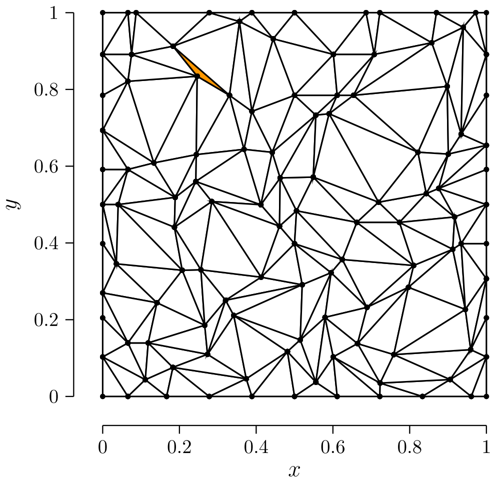
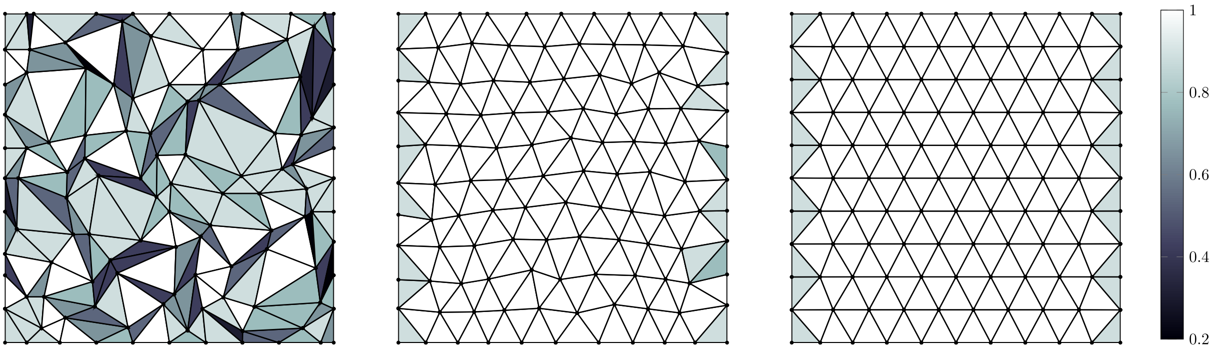


Figure 2: First Example - Triangular Mesh



(a) Initial Mesh (b) 1^{st} Iteration (c) 4^{th} Iteration

Figure 3: Element Quality Plot

Table 1 summarizes the quality parameters for each computed iteration. A drastic mesh improvement is verified in the first iteration. The enhancement obtained from the subsequent iterations provide marginal improvement until the process terminates.

Table 1: Mesh Quality Parameters for Each Iteration

Iteration	q_{min}	q_{avg}	q_{max}
Initial Mesh	0.233	0.750	0.998
1	0.791	0.967	1.000
2	0.865	0.982	0.999
3	0.882	0.982	0.998
4	0.883	0.982	0.998

The local regularization process of the element highlighted in Fig. 2 is represented in Fig. 4. The dotted line represents the ideal element optimally transformed according to the least squares fitting procedure described in Section 2.2. The displacement field required to regularize the element is represented as blue vectors. It can be noted that as the element's shape approximates to the ideal one, the quality metric approaches to unity.

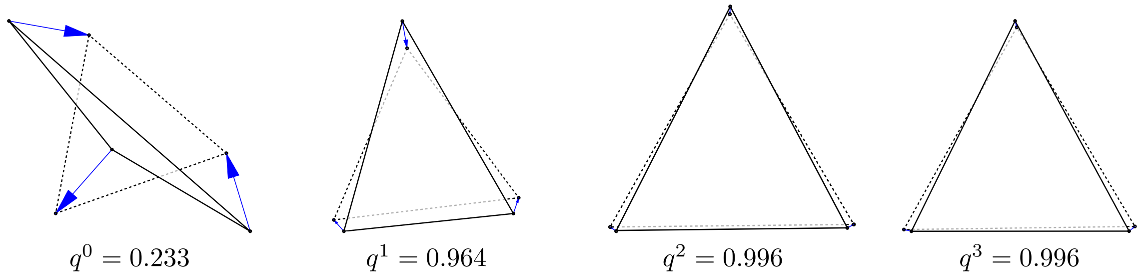
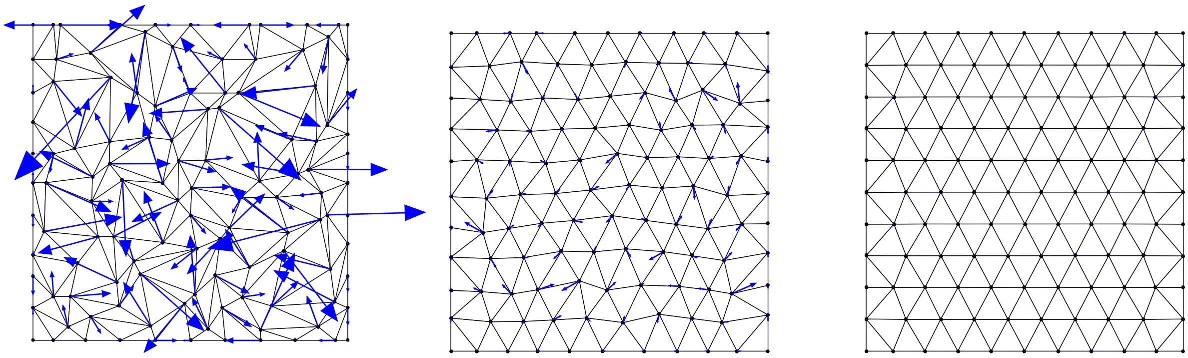


Figure 4: Local Regularization and Corresponding Displacement Field Along Iterations

Figure 5 shows the nodal forces resulting from the global optimization process. As the mesh quality improves the forces magnitude tend to decrease as expected.



(a) 1^{st} Iteration (b) 2^{nd} Iteration (c) 4^{th} Iteration

Figure 5: Nodal Force Field Resulting from the Global Optimization Process

3.2 Quadrilateral Mesh

As a second example, consider the distorted mesh displayed in Fig. 6, obtained by a random perturbation on the nodes coordinates of a regular structured mesh composed of 10-by-10 quadrilaterals.

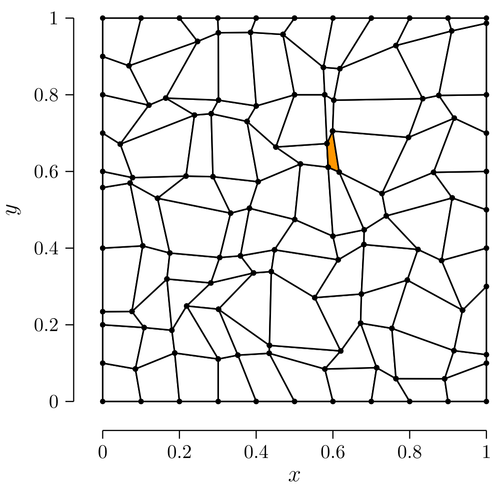
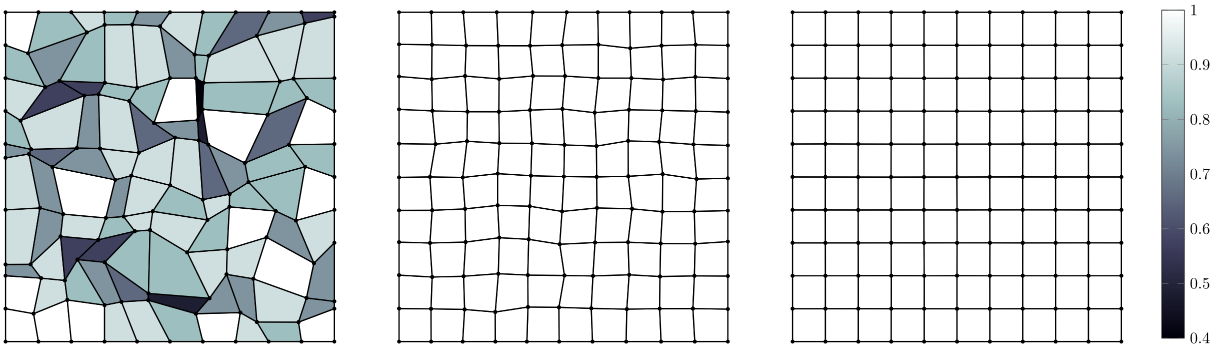


Figure 6: Second Example - Quadrilateral Mesh

Within 3 iterations the smothing process terminates, accordingly to the convergence criteria defined in Eq. (16). The structured mesh is recovered successfully as can be observed in Fig. 7.



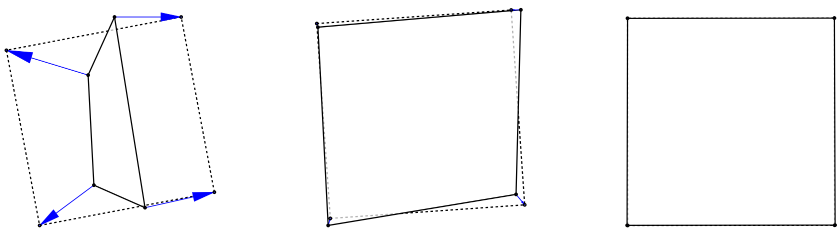
(a) Initial Mesh (b) 1^{st} Iteration (c) 3^{dr} Iteration

Figure 7: Element Quality Plot

Table 2: Mesh Quality Parameters for Each Iteration

Iteration	q_{min}	q_{avg}	q_{max}
Initial Mesh	0.437	0.827	0.992
1	0.977	0.996	1.000
2	1.000	1.000	1.000
3	1.000	1.000	1.000

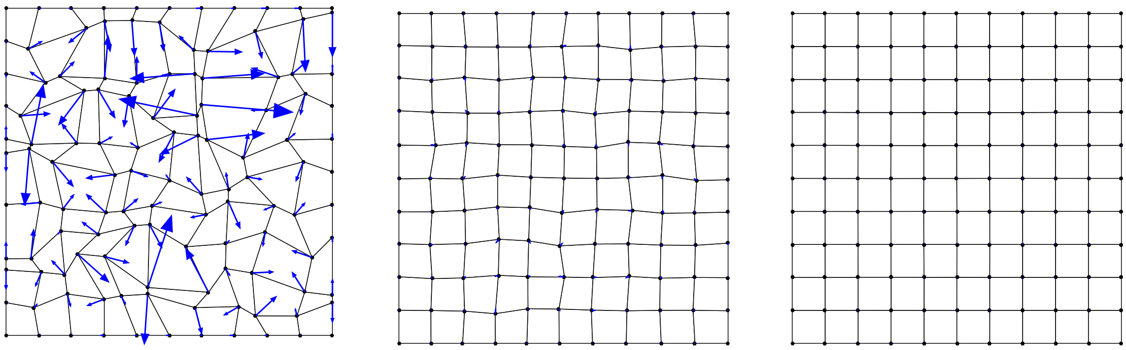
Table 2 summarizes the global mesh quality parameters, defined in Eq. (15), for each iteration along the smothing process of the mesh regarded in this example. An astounding overall quality improvement is verified at the very first iteration and a complete regularization at the second iteration. Numerical convergence is achieved in the third iteration due to the adopted precision ($\epsilon = 10^{-2}$).



$q^0 = 0.551$ $q^1 = 0.992$ $q^2 = 1.000$

Figure 8: Local Regularization and Corresponding Displacement Field Along Iterations

Figure 8 shows the local regularization process of the element highlighted in Fig. 6 in a orange color tone. The dotted lines represent the ideal element, transformed according to the rigid transform given by Eq. (13), with the same area as the mesh element represented in solid lines. The displacement field required to regularize the element, represented by blue vectors, are used to compute the nodal forces in the global optimization process. The resultant nodal forces from the global optimization process are plotted for each iteration in Fig. 9. It can be observed that the nodal resultant is proportional to the worst element quality that concurs at that node. Also, the total force magnitude decreases as the mesh quality improves, as exepected.



(a) 1^{st} Iteration (b) 2^{nd} Iteration (c) 3^{rd} Iteration

Figure 9: Nodal Force Field Resulting from the Global Optimization Process

4 CONCLUSIONS

An innovative mesh optimization method constituted by local and global regularizations is presented. The local regularization is based on the use of the singular value decomposition and the global regularization is performed using a physics-based approach. The proposed method is applied to two dimensional meshes of triangular and quadrilateral cells. Also, it does not present restrictions for meshes with concave boundary as in the Laplacian method. Results show rapid improvement in mesh quality in all examples. In most cases only three or four iterations were needed to get significant enhancement. The computation time for each iteration were mostly spent in the finite element solution used to get the nodal displacements used to update the mesh. The authors consider that the proposed method is applicable for three-dimensional meshes, however it will be subject of an upcoming paper.

REFERENCES

Arun, K Somani, Thomas S Huang, and Steven D Blostein (1987). "Least-squares fitting of two 3-D point sets". In: *IEEE Transactions on pattern analysis and machine intelligence* 5, pp. 698–700.

Babuška, Ivo and A Kadir Aziz (1976). "On the angle condition in the finite element method". In: *SIAM Journal on Numerical Analysis* 13.2, pp. 214–226.

Bank, Randolph E and R Kent Smith (1997). "Mesh smoothing using a posteriori error estimates". In: *SIAM Journal on Numerical Analysis* 34.3, pp. 979–997.

Bezanson, Jeff et al. (2012). "Julia: A Fast Dynamic Language for Technical Computing". In: arXiv:`1209.5145`.

Canann, Scott A, Joseph R Tristano, and Matthew L Staten (1998a). "An Approach to Combined Laplacian and Optimization-Based Smoothing for Triangular, Quadrilateral, and Quad-Dominant Meshes." In: *IMR*. Citeseer, pp. 479–494.

Canann, Scott A, SN Muthukrishnan, and RK Phillips (1998b). "Topological improvement procedures for quadrilateral finite element meshes". In: *Engineering with Computers* 14.2, pp. 168–177.

Díaz, Agustín, Celestino González-Nicieza, and Arturo Ernesto Álvarez-Vigil (2005). "Hexahedral mesh smoothing using a direct method". In: *Computers & geosciences* 31.4, pp. 453–463.

Durand, Raul (2015). *FemMesh: Structured meshes for FEM analyses*. `https://github.com/RaulDurand/FemMesh`. [Online; accessed 01-July-2016].

Field, David A (1988). "Laplacian smoothing and Delaunay triangulations". In: *Communications in applied numerical methods* 4.6, pp. 709–712.

Field, David A (2000). "Qualitative measures for initial meshes". In: *International Journal for Numerical Methods in Engineering* 47.4, pp. 887–906.

Freitag, Lori A (1997). "On combining Laplacian and optimization-based mesh smoothing techniques". In: *ASME APPLIED MECHANICS DIVISION-PUBLICATIONS-AMD* 220, pp. 37–44.

Freitag, Lori A and Patrick M Knupp (2002). "Tetrahedral mesh improvement via optimization of the element condition number". In: *International Journal for Numerical Methods in Engineering* 53.6, pp. 1377–1391.

Jiao, Xiangmin, Duo Wang, and Hongyuan Zha (2011). "Simple and effective variational optimization of surface and volume triangulations". In: *Engineering with Computers* 27.1, pp. 81–94.

Kim, Jibum, Shankar Prasad Sastry, and Suzanne M Shontz (2012). "A numerical investigation on the interplay amongst geometry, meshes, and linear algebra in the finite element solution of elliptic PDEs". In: *Engineering with Computers* 28.4, pp. 431–450.

Knupp, Patrick M. (2000). "Achieving finite element mesh quality via optimization of the Jacobian matrix norm and associated quantities. Part Ia framework for surface mesh optimization". In: *International Journal for Numerical Methods in Engineering* 48.3, pp. 401–420.

Knupp, Patrick M. (2001). "Algebraic mesh quality metrics". In: *SIAM journal on scientific computing* 23.1, pp. 193–218.

Lee, N. and K.J. Bathe (1993). "Effects of element distortions on the performance of isoparametric elements". In: *International Journal for Numerical Methods in Engineering* 36.20, pp. 3553–3576.

Oddy A. Goldak J, McDill M and Bibby M. (1988). "A distortion metric for isoparametric elements". In: *Trans. Canad. Soc. Mech. Engrg.* 12, pp. 213–217.

Oh, HS and JK Lim (1996). "A simple error estimator for size and distortion of 2D isoparametric finite elements". In: *Computers & structures* 59.6, pp. 989–999.

Osserman, Robert (1978). "The isoperimetric inequality". In: *Bulletin of the American Mathematical Society* 84.6, pp. 1182–1238.

Park, Jeonghyung and Suzanne M Shontz (2011). "An alternating mesh quality metric scheme for efficient mesh quality improvement". In: *Procedia Computer Science* 4, pp. 292–301.

Renka, Robert J (2015). “Mesh improvement by minimizing a weighted sum of squared element volumes”. In: *International Journal for Numerical Methods in Engineering* 101.11, pp. 870–886.

Robinson, John (1987). “Some new distortion measures for quadrilaterals”. In: *Finite Elements in Analysis and Design* 3.3, pp. 183–197.

Shewchuk, J (2002). “What is a good linear finite element? interpolation, conditioning, anisotropy, and quality measures (preprint)”. In: *University of California at Berkeley* 73.

Sun, Lu, Guoqun Zhao, and Xinwu Ma (2012). “Quality improvement methods for hexahedral element meshes adaptively generated using grid-based algorithm”. In: *International Journal for Numerical Methods in Engineering* 89.6, pp. 726–761.

Vartziotis, Dimitris, Joachim Wipper, and Manolis Papadrakakis (2013). “Improving mesh quality and finite element solution accuracy by GETMe smoothing in solving the Poisson equation”. In: *Finite Elements in Analysis and Design* 66, pp. 36–52.

Wei, Peng et al. (2015). “Hexahedral mesh smoothing via local element regularization and global mesh optimization”. In: *Computer-Aided Design* 59, pp. 85–97.

Yilmaz, A Egemen and Mustafa Kuzuoglu (2009). “A particle swarm optimization approach for hexahedral mesh smoothing”. In: *International journal for numerical methods in fluids* 60.1, pp. 55–78.