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ADAPTIVE ITERATIVE BEM-FEM COUPLING PROCEDURES TO ANALYZE INELASTIC MODELS

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Abstract. *The analysis of complex systems may be more effectively handled considering the combination of different numerical methods, in a way that each numerical technique can be applied to deal with the particularities of the model that better fit its positive features. In this sense, the adaptive iterative coupling of the Boundary Element Method (BEM) and of the Finite Element Method (FEM) is discussed here, taking into account static nonlinear models. Optimal relaxation parameters are employed to speed up the convergence of the iterative coupling, and non-matching discretizations at common interfaces, as well as adaptive refinement within the FEM subdomains, are allowed, enabling more versatile and accurate approaches. A single unified iterative loop is considered in order to deal with all the focused iterative solutions simultaneously (i.e., the nonlinear analysis, the adaptive analysis and the coupling analysis), rendering a very efficient methodology. In this context, multiple sequential iterative loops, which represent a rather computationally demanding approach, can be avoided without significantly increase the number of the iterative steps of the dominant iterative process, considerably improving the performance of the method. At the end of the paper, numerical results are presented, illustrating the potentialities and the effectiveness of the proposed techniques.*

Keywords: *Iterative Coupling, Boundary Elements, Adaptive Finite Elements, Elastoplasticity*

1 INTRODUCTION

The analysis of complex systems may be more effectively handled considering the combination of different numerical methods. In this context, each numerical technique can be applied dealing with the particularities of the model that better fit its positive features. Taking into account combined formulations, a great amount of research works focuses on the coupling of the Boundary Element Method (BEM) and the Finite Element Method (FEM), which are very popular numerical techniques that contribute with complementary beneficial features. In fact, the BEM is quite suitable to handle infinite or semi-infinite media and high variations or discontinuous behaviour, while the FEM is very effectively applied to model complex configurations, in which heterogeneity, anisotropy, nonlinear behaviour etc. may occur.

Researchers have tested combinations of the BEM and the FEM in order to profit from their respective advantages, trying to evade their disadvantages, and nowadays several works dealing with BEM-FEM coupling are available. Classical BEM/FEM coupling procedures can pose several problems regarding efficiency, accuracy and/or flexibility. Indeed, the coupled system of equations has a banded symmetric structure only in the FEM part, while in the BEM part it is non-symmetric and fully populated; as a consequence, the optimized solvers usually employed with the FEM can no longer be used, and more expensive calculations (namely in computer time) are required. Moreover, in many cases very different physical properties exist in each part of the model, and this can lead to badly conditioned matrices if direct coupling procedures are used; consequently, numerically unstable systems may be obtained and lead to inaccurate results. Additionally, it should be mentioned that standard coupling methods require matching discretizations along the common interfaces between subdomains, which reduce the flexibility of and generality of the techniques. Finally, if nonlinear models are considered, standard coupling implies large and complex systems of equations to be dealt with several times within a single analysis, leading to excessive computational costs.

As an alternative, iterative coupling procedures have been developed by several authors. Initially, static problems were studied considering iterative coupling approaches, and linear and nonlinear behaviour have been simulated (Lin et al., 1996; Elleithy et al., 2001; Jahromi et al., 2009). Later on, dynamic problems were focused, and time (Soares et al., 2004; Soares, 2008; 2012) and frequency domain (Bendali et al., 2007; Soares and Godinho, 2012; Coulier et al., 2014) iterative analyses have been implemented (in this case, an overview is presented by Soares and Godinho (2014)). Nowadays, advanced techniques regarding the iterative coupling of the BEM and the FEM to analyze static nonlinear models can be encountered in the work of Soares and Godinho (2015).

When iterative coupling approaches are used, each sub-domain of the global model is analyzed independently, as an uncoupled model. A sequential renewal of the variables at the common interfaces is then performed, until convergence is achieved. Several advantages can be identified for these iterative methodologies, when compared to standard coupling schemes: (i) the sub-domains can be analysed separately, leading to smaller and better-conditioned systems of equations (different solvers, suitable for each sub-domain, may be employed); (ii) the coupling procedure only requires interface routines, allowing the simple reuse of existing codes (thus, coupled systems may be solved by separate program modules, taking full advantage of specialized features and disciplinary expertise); (iii) non-matching nodes at common interfaces can be used, improving the flexibility and versatility of the coupled analyses, especially when different discretization methods are considered; (iv) more efficient analyses can be obtained, once the global model can be reduced to several sub-domains with reduced size matrices; etc..

Here, adaptive iterative BEM-FEM coupling procedures are discussed for the analysis of nonlinear static models, with focus in elastoplastic problems. An adaptive FEM approach is applied to the regions where the inelastic behavior is expected to occur, while the BEM is applied to the regions with an elastic material behavior. In the past, BEM-FEM iterative coupling has already been reported in the literature for elastoplasticity (as, for example, in Elleithy et al. (2004), Boumaiza and Aour (2014), etc.). It should be noted that, in most of these references, two iterative loops are considered, with the complete iterative loop of the nonlinear FEM analysis occurring within an iterative step of the BEM-FEM iterative coupling. This approach becomes very demanding from the computational point of view. As an alternative, here the authors use a single iterative loop in which three iterative analyses are carried out together, at a common iterative step: (i) the FEM nonlinear analysis; (ii) the FEM adaptive discretization; (iii) the BEM-FEM coupling. As described later in this work, this unified iterative approach does not lead to a significant increase in the number of iterations required by the dominant iterative analysis of the model (if considered separately). The technique is thus quite efficient.

Differently from the model proposed by Elleithy et al. (2009; 2012), in which the FEM subdomains are expanded (and BEM subdomains are shrunk) as the plastic zones evolve, here the regions modelled by the BEM do not change along the analysis, allowing the matrices of the BEM subdomains to be computed only once. A coarse discretization is also initially adopted for the FEM subdomains, and then it is adaptively enriched using the information of the solution at each step. At the end of the process, a refined FEM discretization occurs at regions where the plastic zones occur, providing an optimal FEM simulation linked to a very efficient BEM analysis. As can be easily understood, this adaptive approach implies non-matching nodes at BEM-FEM common interfaces to be allowed, otherwise the BEM discretizations would also have to adapt, increasing the computational cost of the analysis.

In the present work, the authors discuss two different iterative coupling approaches. For each one, either displacements or tractions may be considered prescribed to the BEM common interface (and, complementarily, either tractions or displacements are prescribed to the FEM common interface). The use of each technique can be decided for each model to analyse in accordance to the characteristics of the model, avoiding, for instance, singular systems of equations to be obtained due to the lack of essential boundary conditions on a subdomain. In both cases, an optimal relaxation parameter is introduced to speed up and/or to ensure the convergence of the iterative coupling analysis. As previously referred in the literature, this is of great importance in order to guarantee the robustness and efficiency of the technique.

In what follows, first, the governing equations of the elastoplastic model are briefly presented, and general aspects of the BEM and FEM are described; then, the iterative BEM-FEM coupling algorithm is discussed, and some numerical examples are presented.

2 GOVERNING EQUATIONS

The basic governing equations related to elastoplastic materials are given by:

$$\sigma_{ij,j} = \gamma_i \tag{1}$$

$$d\sigma_{ij} = D^{ep}_{ijkl} d\varepsilon_{kl} \tag{2}$$

$$d\varepsilon_{ij} = \tfrac{1}{2}(du_{i,j} + du_{j,i}) \tag{3}$$

where equation (1) is the equilibrium equation and equations (2) and (3) stand for incremental relations. The Cauchy stress, using the usual indicial notation for Cartesian axes, is represented by σ_{ij}, and u_i and γ_i stand for displacement and body force distribution components, respectively (inferior commas indicate partial space derivatives). Equation (2) is the constitutive law, written incrementally. The incremental strain components $d\varepsilon_{ij}$ are defined in the usual way from the displacements, as described by equation (3). In equation (2), D^{ep}_{ijkl} is a tangential tensor defined by suitable state variables and the direction of the increment. Within the context of associated isotropic work hardening theory (Chen, 1988; Khan and Huang, 1995), the tangent constitutive tensor is defined as:

$$D^{ep}_{ijkl} = D_{ijkl} - (1/\psi) D_{ijmn} a_{mn} a_{op} D_{opkl} \tag{4}$$

where

$$D_{ijkl} = 2\mu\nu/(1-2\nu)\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{5a}$$

$$a_{kl} = \partial\bar{\sigma} / \partial\sigma_{kl} \tag{5b}$$

$$\psi = a_{ij} D_{ijkl} a_{kl} + H \tag{5c}$$

$$H = \partial\sigma_0 / \partial\bar{\varepsilon}^p \tag{5d}$$

In equations (5), $\bar{\sigma}$ and $\bar{\varepsilon}^p$ are the equivalent (or effective) stress and plastic strain, respectively; σ_0 is the uniaxial yield stress; H is the plastic-hardening modulus (for the case of a perfectly plastic material $H = 0$); μ and ν stand for the shear modulus and the Poisson ratio, respectively; and δ_{ik} is the Kronecker delta. In case of elastic analyses, the Cauchy stresses can be defined by $\sigma_{ij} = D_{ijkl}\varepsilon_{kl}$, where D_{ijkl} (see equation (5a)) is the elastic constitutive tensor (this linear relation is a particular case of equation (2)).

In addition to equations (1)-(5), boundary conditions are prescribed as follows, in order to completely define the problem:

$$u_i = \bar{u}_i \quad \text{at } \Gamma_u \tag{6a}$$

$$\tau_i = \sigma_{ij} n_j = \bar{\tau}_i \quad \text{at } \Gamma_\tau \tag{6b}$$

where the prescribed values are indicated by overbars and τ_i stands for traction components along the boundary whose unit outward normal vector is represented by n_i. Following equations (6), the boundary of the model (Γ) is divided into an essential (Γ_u) and a natural (Γ_τ) boundary, where $\Gamma_u \cup \Gamma_\tau = \Gamma$ and $\Gamma_u \cap \Gamma_\tau = 0$.

In the next section, boundary and finite element techniques are briefly described. Here, elastoplastic regions are treated by the FEM, whereas elastic subdomains may be analyzed by the BEM or by the FEM.

3 SUBDOMAINS DISCRETIZATION

Assuming a boundary element discretization, elastostatic models may be analyzed by the following system of equations (Brebbia et al., 1984; Brebbia and Dominguez, 1998):

$$(\mathbf{C}+\mathbf{H})\mathbf{u}_{k+1}^{n+1} = \mathbf{G}\boldsymbol{\tau}_{k+1}^{n+1} + \mathbf{d}_{k+1}^{n+1} \tag{7}$$

where $\mathbf{u}$ and $\boldsymbol{\tau}$ stand for displacement and traction vectors, respectively. The superscript $n+1$ stands for the current incremental step of the analysis and the subscript $k+1$ stands for its iterative step (an iterative coupled nonlinear analysis is aimed in this work). $\mathbf{C}$ stands for the location matrix and $\mathbf{G}$ and $\mathbf{H}$ are influence matrices. Vector $\mathbf{d}$ accounts for domain terms, such as body force contributions. Matrices $\mathbf{G}$ and $\mathbf{H}$ can be computed taking into account each boundary element e of the model, and vector $\mathbf{d}$ can be evaluated considering each domain integration cell c:

$$\mathbf{H} = \bigcup_e \int_{\Gamma_e} \mathbf{T}^* \mathbf{N}\, d\Gamma \tag{8a}$$

$$\mathbf{G} = \bigcup_e \int_{\Gamma_e} \mathbf{U}^* \mathbf{N}\, d\Gamma \tag{8b}$$

$$\mathbf{d}_{k+1}^{n+1} = \bigcup_c \int_{\Omega_c} \mathbf{U}^* \boldsymbol{\gamma}_{k+1}^{n+1}\, d\Omega \tag{8c}$$

In equations (8), $\mathbf{T}^*$ and $\mathbf{U}^*$ stand for traction and displacement fundamental matrices, respectively, and **N** represents the adopted BEM interpolation matrix.

By re-arranging the system of equations (7), taking into account the boundary conditions of the problem, just known and unknown variables can be disposed at the right and at the left hand side of the system of equations, respectively, allowing its solution. Equations (7-8) only intend to summarily describe the boundary element formulation considered here; for further details on the topic, the books of Brebbia et al. (1984) and Brebbia and Dominguez (1998) are suggested.

Considering a finite element discretization, the basic equation describing a nonlinear model is given by:

$$f(\mathbf{u}_{k+1}^{n+1}) = \mathbf{f}_{k+1}^{n+1} \tag{9}$$

where $f(\mathbf{u})$ and $\mathbf{f}$ stand for internal and external forces, respectively. Taking into account a linearized incremental approach, equation (9) can be rewritten as (Simo and Hughes, 1998; Belytschko et al., 2000):

$$\mathbf{K}_k \delta\mathbf{u}_{k+1} = \mathbf{f}_k^{n+1} - f(\mathbf{u}_k^{n+1}) \tag{10a}$$

$$\mathbf{u}_{k+1}^{n+1} = \mathbf{u}_k^{n+1} + \delta\mathbf{u}_{k+1} \tag{10b}$$

where $\mathbf{K}_k$ stands for the tangent nonlinear stiffness matrix, $\mathbf{f}_k^{n+1} - f(\mathbf{u}_k^{n+1})$ represents the nonlinear residual vector and $\delta\mathbf{u}_{k+1}$ is the variation of the incremental displacements, calculated at each iterative step.

For elastoplastic models, the vectors and matrices represented in equation (10a) may be described as follows, considering each finite element e:

$$\mathbf{f}_k^{n+1} = \bigcup_e \left(\int_{\Omega_e} \mathbf{N}^T \boldsymbol{\gamma}_k^{n+1} d\Omega + \int_{\Gamma_{\tau e}} \mathbf{N}^T \bar{\boldsymbol{\tau}}_k^{n+1} d\Gamma \right) \tag{11a}$$

$$f(\mathbf{u}_k^{n+1}) = \bigcup_e \int_{\Omega_e} \mathbf{B}^T \boldsymbol{\sigma}_k^{n+1} d\Omega \tag{11b}$$

$$\mathbf{K}_k = \bigcup_e \int_{\Omega_e} \mathbf{B}^T \mathbf{D}_k^{ep} \mathbf{B} d\Omega \tag{11c}$$

where **B** stands for the strain matrix and $\mathbf{D}_k^{ep}$ is the nonlinear constitutive matrix (as described by equation (4)). **N** stands for the adopted FEM interpolation matrix. The stress state in equation (11b) can be evaluated following equations (2) and (3); i.e., by considering $\boldsymbol{\sigma}_k^{n+1} = \boldsymbol{\sigma}^n + d\boldsymbol{\sigma}_k^{n+1} = \boldsymbol{\sigma}^n + \mathbf{D}_k^{ep} d\boldsymbol{\varepsilon}_k^{n+1} = \boldsymbol{\sigma}^n + \mathbf{D}_k^{ep} \mathbf{B} d\mathbf{u}_k^{n+1}$, where $d\mathbf{u}_k^{n+1} = \mathbf{u}_k^{n+1} - \mathbf{u}^n$.

Equations (9-11) briefly describe the finite element formulation considered here; for further details on the topic, the books of Simo and Hughes (1998) and Belytschko et al. (2000) are suggested. Once equations (7-11) are considered, subdomains modelled by the BEM or by the FEM can be analyzed. In the next section, the iterative coupling of these subdomains is discussed.

4 COUPLED ANALYSIS

To perform the coupled BEM-FEM analysis, continuity and equilibrium equations must hold at the common interfaces between subdomains, namelly:

$${}^F u_i = {}^B u_i \tag{12a}$$

$${}^F \tau_i + {}^B \tau_i = 0 \tag{12b}$$

where the superscripts *B* and *F* indicate if a variable is related to the BEM or to the FEM subdomain, respectively.

Here, two different configurations can be considered, regarding the BEM-FEM interface: (i) in the first configuration (configuration 1), the common interface is considered as an essential boundary for the BEM, with prescribed displacements, and as a natural boundary for the FEM, with prescribed tractions; (ii) in the second configuration (configuration 2), the common interface is considered as a natural boundary for the BEM, with prescribed tractions, and as an essential boundary for the FEM, with prescribed displacements. For both cases, an iterative renewal of the variables at the common interface is carried out, taking into account equations (12).

For the BEM-FEM iterative coupling algorithm considered here, the analysis starts with the FEM subdomain, and the FEM displacements (${}^F\mathbf{u}_{k+1}^{n+1}$) or tractions (${}^F\boldsymbol{\tau}_{k+1}^{n+1}$) at the common interface are evaluated (in this case, tractions are computed as projections of the stress state, as indicated by equation (6b)). After this, the corresponding values are applied to the BEM as prescribed boundary conditions, following equations (12) and the configuration in focus (i.e., configuration 1 or 2) for the common interface.

In this process a relaxation parameter, α, is considered, which is introduced here in order to ensure and/or to speed up convergence. Thus, the following variables may be computed:

$${}^B\bar{\mathbf{u}}_{k+1}^{n+1} = (\alpha)\, I_u^{BF}({}^F\mathbf{u}_{k+1}^{n+1}) + (1-\alpha)\ {}^B\bar{\mathbf{u}}_k^{n+1} \tag{13a}$$

$$ {}^{B}\bar{\boldsymbol{\tau}}_{k+1}^{n+1} = (\alpha)\, I_{\tau}^{BF}\,(-{}^{F}\boldsymbol{\tau}_{k+1}^{n+1}) + (1-\alpha)\; {}^{B}\bar{\boldsymbol{\tau}}_{k}^{n+1} \tag{13b} $$

where I stands for spatial interpolation functions; equations (13a) and (13b) are related to configuration 1 and 2, respectively.

It is important to remark that non-matching BEM and FEM nodes can easily be considered (which is particularly important if adaptive discretizations are used). Thus, a routine to spatially link the FEM and BEM computed values must be taken into account. In equations (13), I_u^{BF} and I_τ^{BF} link the displacements and tractions computed at the finite element nodes and faces, respectively, to their respective values at the boundary element functional nodes. These interpolation routines can be carried out based either on the interpolation functions of the elements, or considering enriched approaches.

Once the prescribed variables at the BEM common interface are computed, the BEM subdomains can be analyzed, allowing to compute displacements (${}^{B}\mathbf{u}_{k+1}^{n+1}$, configuration 2) or tractions (${}^{B}\boldsymbol{\tau}_{k+1}^{n+1}$, configuration 1) at the common interface. These values are applied to the FEM as prescribed boundary conditions, following equations (12). Thus, in order to obtain the FEM prescribed values, the following variables may be computed:

$$ {}^{F}\bar{\boldsymbol{\tau}}_{k+1}^{n+1} = I_{\tau}^{FB}(-{}^{B}\boldsymbol{\tau}_{k+1}^{n+1}) \tag{14a} $$

$$ {}^{F}\bar{\mathbf{u}}_{k+1}^{n+1} = I_{u}^{FB}({}^{B}\mathbf{u}_{k+1}^{n+1}) \tag{14b} $$

where, once again, I stands for spatial interpolation functions and equations (14a) and (14b) are related to configuration 1 and 2, respectively. In equations (14), I_u^{FB} and I_τ^{FB} link the displacements and tractions computed at the boundary element functional nodes, respectively, to their respective values at the finite element nodes.

After computing the FEM prescribed boundary conditions, the iterative cycle is reinitiated, and all the above described procedures are repeated until achieving convergence. It is important to highlight that just one iterative loop is considered here to deal with several types of iterative approaches, namely: (i) FEM nonlinear analysis; (ii) FEM adaptive discretization; (iii) BEM-FEM coupling. As illustrated in the next section, this approach seems to be very appropriate, since the number of iterative steps required by the isolated dominant iterative procedure is not significantly increased by considering all the iterating procedures together, in the same iterative loop. The usual alternative to the single iterative approach is a multiple iterative algorithm. In this case, an entire iterative loop is carried out within each iterative step of the "host" iterative loop. Thus, the multiple iterative procedure may considerably increase the computational effort of the analysis, once too many iterative steps occur and/or iterative processes are considered.

It is important to observe that the effectiveness of the present iterative coupling algorithm is intimately related to the relaxation parameter selection. An inappropriate selection for α can drastically increase the number of iterations in the analysis or, even worse, make convergence unfeasible. In this work, the following expression for an optimal relaxation parameter is considered:

$$ \alpha = ({}^{B}\mathbf{w}, {}^{B}\mathbf{w} - {}^{F}\mathbf{w}) \,/\, \| {}^{B}\mathbf{w} - {}^{F}\mathbf{w} \|^2 \tag{15} $$

where

$$ {}^{F}\mathbf{w} = I_{v}^{BF}(\pm {}^{F}\mathbf{v}_{k+1}^{n+1}) - I_{v}^{BF}(\pm {}^{F}\mathbf{v}_{k}^{n+1}) \tag{16a} $$

$$ {}^{B}\mathbf{w} = {}^{B}\overline{\mathbf{v}}_{k}^{n+1} - {}^{B}\overline{\mathbf{v}}_{k-1}^{n+1} \tag{16b} $$

and vector **v** stands for the traction or displacement vector, according to the configuration in focus.

5 NUMERICAL EXAMPLES

To exemplify the application of the described procedure, two numerical examples are presented, illustrating the performance and potentialities of the discussed techniques. In the first example, a half-space model is analysed. In this case, the Drucker-Prager yield criterion is considered and adaptive and fixed BEM-FEM discretizations are carried out. In the second example, a cantilever beam is studied, following the von Mises yield criterion. Here, adaptive BEM-FEM coupled results are compared to those provided by an adaptive and fixed FEM approach.

In the first example, an infinite domain model is focused. In this case, the so-called configuration 2 is considered, and prescribed tractions are applied at the BEM common interface, whereas prescribed displacements are applied at the FEM common interface. In the second application, a finite domain model is analyzed and configuration 1 is followed, considering prescribed displacement and tractions applied to the BEM and FEM common interfaces, respectively. For this second application, linear boundary elements are considered, whereas, for the first application, constant boundary elements are employed. For all the analyses that follow, linear triangular finite elements are adopted, since discretizations considering this type of finite element are easier to adaptively refine. The adaptive procedure implemented here is based on the package provided by Chen and Zhang (2006) and the FEM discretization is adapted just at the first iterative step of each incremental step. A Newton-Raphson initial stress configuration is considered for the nonlinear analysis, and a tight relative error tolerance (displacement and force residual) of 10^{-5} is adopted for the convergence of the iterative process.

5.1 Example 1

In this first example, a half-space is analyzed, with the region closer to the loaded area being discretized by the FEM, and the remaining domain being discretized by the BEM. A schematic representation of the model is depicted in Fig.1. The geometry of the problem is defined by $L = 6.4\,m$ and $H = 4.0\,m$.

The physical properties of the model are $E=10^9 N/m^2$ and $\nu = 0.3$. A perfectly plastic material obeying the Drucker-Prager yield criterion is assumed, where $c=1.7\cdot 10^2 N/m^2$ and $\phi=10^0$. Two different BEM-FEM discretizations are considered here. In the first discretization, a fixed FEM mesh is employed, which is composed of 5120 elements. In the second discretization, an adaptive FEM mesh is considered, being its initial configuration depicted in Fig.1 (64 elements). For the first BEM-FEM discretization, 20 boundary elements of length 0.20 m are employed to discretize each vertical common interface, whereas 16 boundary elements of length 0.25 m are considered for the second discretization. For both discretizations, boundary elements of length 0.25 m and 0.20 m are applied to discretize the half-space (which is sufficiently extended up to 25 m at each side of the model) and the horizontal common interface, respectively. In this case, 12 incremental steps are considered.

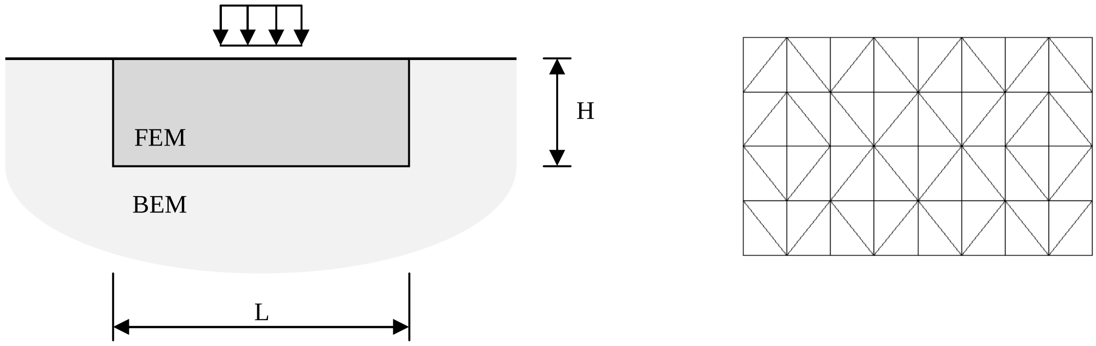


Figure 1. Sketch of the half-space model and initial FEM discretization for the adaptive analysis.

In Fig.2, the vertical displacements computed along the symmetrical vertical axis of the half-space model are depicted, considering the elastoplastic analyses. In this case, results obtained from a multiple iterative analysis are also depicted in the figure, for comparison. As one can observe, good agreement is observed among the results. In Fig.3, the evolution of the equivalent plastic strains, at the last three incremental steps, is described, considering the fixed and the adaptive BEM-FEM discretizations. Once again, good agreement is observed among the results. The evolution of the FEM discretization, taking into account the adaptive analysis, is also illustrated in Fig.3, considering the last three incremental steps. As one can observe, refinement is properly introduced into the analysis, being the region under the applied load progressively refined.

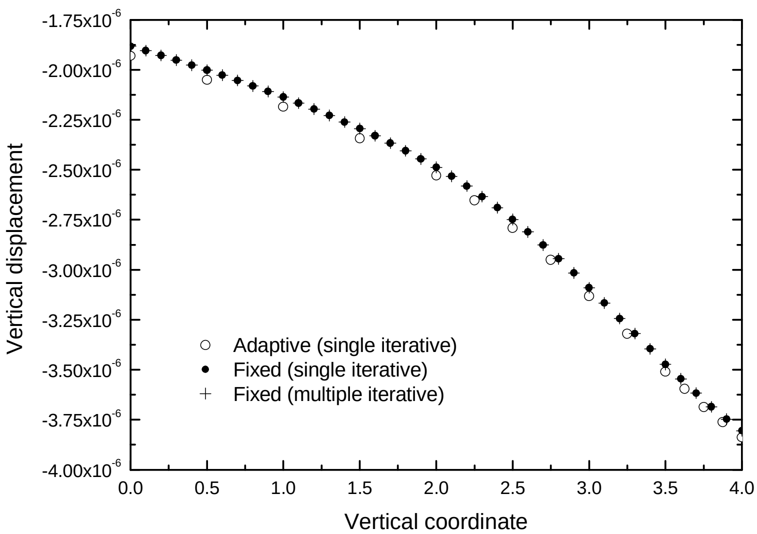
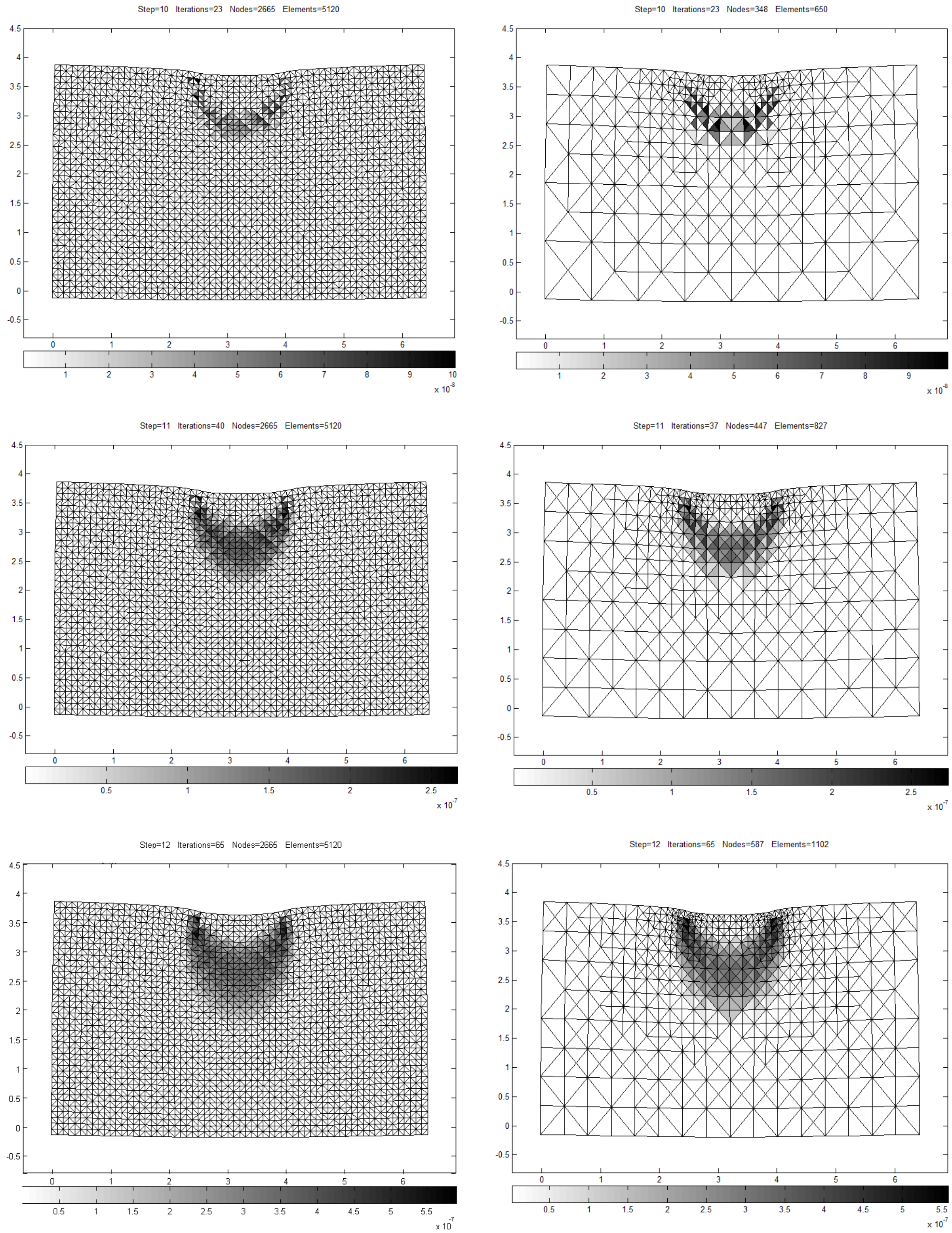


Figure 2. Vertical displacements along the symmetrical vertical axis of the half-space model considering BEM-FEM elastoplastic analyses.

In Tab.1, the number of iterations per incremental step for the half-space model is presented. As one can observe, the number of iterations does not significantly increase by considering several iterative procedures at a unique iterative loop, illustrating the good performance of the adopted technique.



(a) (b)

Figure 3. Evolution of the equivalent plastic strains at the last three incremental steps: (a) fixed FEM discretization; (b) adaptive FEM discretization.

Table 1. Number of iterations per incremental step for the half-space model

		Incremental Step											
		1	2	3	4	5	6	7	8	9	10	11	12
Elastic	Fixed	18	17	20	18	18	18	18	18	18	21	18	18
	Adaptive	23	27	26	20	21	20	21	20	25	16	21	21
Elastoplastic	Fixed	18	17	20	18	18	18	18	18	18	23	40	65
	Adaptive	23	27	26	20	21	20	21	20	25	23	37	65

For the present nonlinear model, the CPU time obtained for the single iterative adaptive BEM-FEM coupled analysis is approximately 19% of the CPU time obtained for the single iterative fixed BEM-FEM coupled analysis. In addition, the CPU time obtained for the single iterative fixed BEM-FEM coupled analysis is approximately 14% of the CPU time obtained for the multiple iterative fixed BEM-FEM coupled analysis. Thus, the proposed technique exhibits a considerably superior performance.

5.2 Example 2

In this second example, a cantilever beam is analyzed, in which the first half of the model is discretized by the FEM, whereas its second half is discretized by the BEM. A sketch of the model is depicted in Fig.13. The geometry of the problem is defined by $L = 1.0\,m$ and $H = 0.5\,m$. The physical properties of the model are $E = 2\cdot 10^{11} N/m^2$ and $\nu = 0.3$. A perfectly plastic material obeying the von Mises yield criterion is assumed, where the uniaxial yield stress is $1.5\cdot 10^4 N/m^2$. For the adaptive BEM-FEM analysis, an initial FEM mesh with 400 elements is considered (as depicted in Fig.4), as well as 60 boundary elements of equal length (i.e., 0.05 m) are employed (double nodes are also considered at the corners of the BEM subdomain). Results provided by this BEM-FEM coupled configuration are compared to those provided by an adaptive FEM solution, in which an initial mesh of 800 elements is considered to discretize the entire model, and by a fixed FEM solution, in which a mesh of 5200 elements is employed. For all configurations, 6 incremental steps are considered.

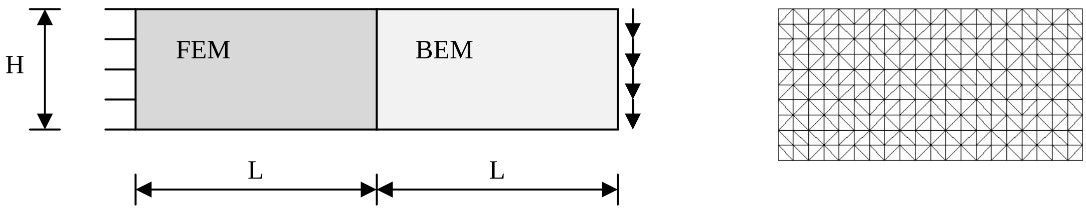


Figure 4. Sketch of the cantilever beam model and initial FEM discretization for the BEM-FEM adaptive analysis.

In Fig.5, vertical displacements calculated along the anti-symmetrical horizontal axis of the beam are depicted, considering elastoplastic analyses. As one can observe, good agreement is observed among the results. Here, it is important to mention that the considered fixed FEM mesh provides a poorer discretization in the regions were the nonlinear behaviour occurs, and thus is probably less accurate in describing the inelastic behaviour of the region. Fig.6 illustrates the equivalent plastic strains at the last incremental step of the analyses, when the adaptive FEM and BEM-FEM discretizations are considered. Once again, good agreement among the results is observed.

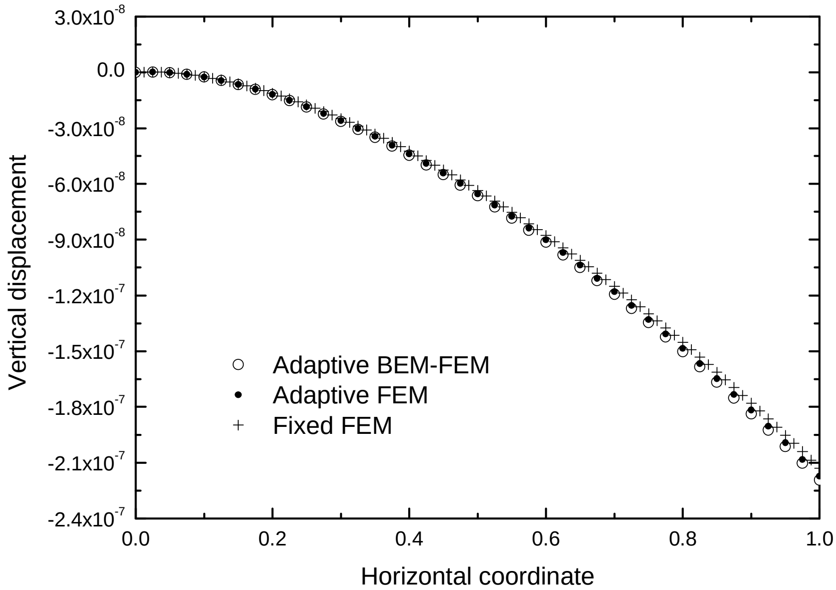
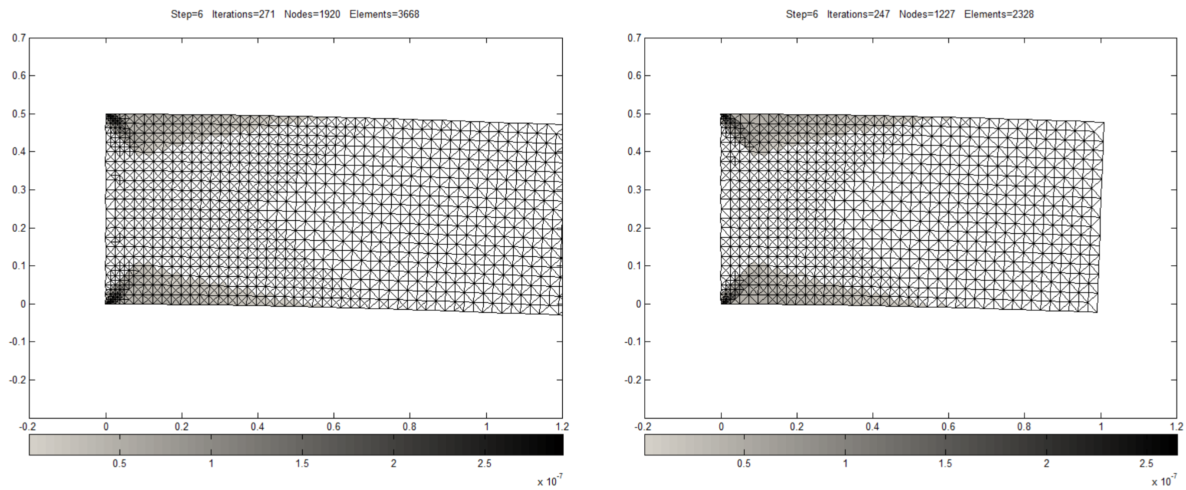


Figure 5. Vertical displacements along the anti-symmetrical horizontal axis of the cantilever beam model considering elastoplastic analyses.



(a) (b)

Figure 6. Equivalent plastic strains along the FEM deformed meshes (scale factor of 10^5) for the cantilever beam model: (a) adaptive FEM; (b) adaptive BEM-FEM.

The computed displacements and tractions along the BEM discretization, at the last incremental step of the elastoplastic analysis, are illustrated in Fig.7. Since the so-called configuration 1 is considered here, at the common interface, the displacements plotted in Fig.7 are based on the FEM response, whereas the tractions depicted in the figure are calculated from the solution of the BEM system of equations. Tab.2 presents the number of iterations per incremental step for the beam model. It can be seen that, for this application, convergence is obtained very quickly for the elastic model. Additionally, as it is described in the table, there is basically no additional iterative cost introduced into the analysis by the adopted BEM-FEM coupling approach (i.e., basically the same number of iterations are required by the FEM adaptive analysis and by the BEM-FEM adaptive analysis), once again highlighting the good performance of the adopted single iterative loop.

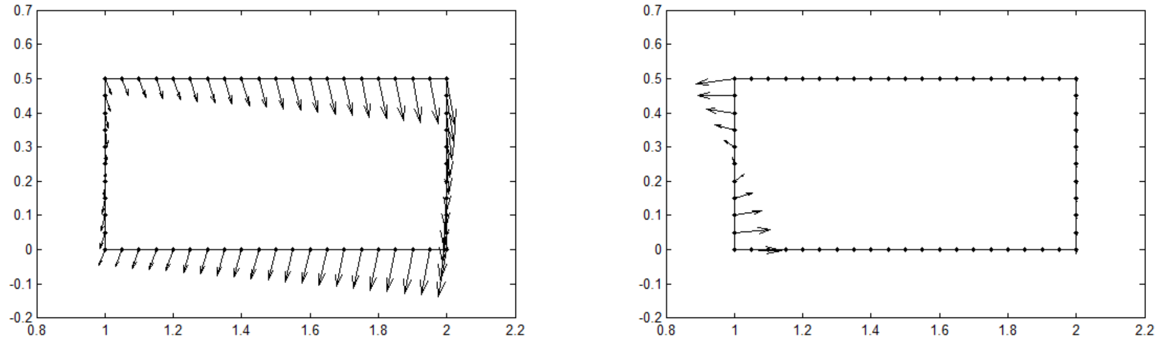


Figure 7. Displacements (left) and tractions (right) along the BEM mesh at the last incremental step of the adaptive elastoplastic analysis.

Table 2. Number of iterations per incremental step for the cantilever beam model

		Incremental Step					
		1	2	3	4	5	6
Elastic	FEM	4	4	4	4	4	4
	BEM-FEM	4	4	5	4	5	5
Elastoplastic	FEM	4	4	5	22	86	271
	BEM-FEM	4	4	5	4	24	247

In this second nonlinear application example, the CPU time for the adaptive BEM-FEM coupled analysis is around 55% of the CPU time obtained for the adaptive FEM analysis.

6 CONCLUSIONS

The present work addresses two iterative BEM-FEM coupling approaches, in which different prescribed boundary conditions at the BEM-FEM common interface may be used. Both coupling procedures allow adaptive nonlinear analyses in the FEM subdomains and independent discretizations at the common BEM-FEM interfaces. A quite powerful, versatile and generic numerical methodology is thus defined, which can be very useful in the solution of a wide range of engineering applications. Both techniques prove to be very efficient.

Optimal relaxation parameters are also used, which allows speeding up the convergence of the iterative coupling. A single iterative loop has been considered, enabling all iterative solutions to be carried out at once. The computational cost of the proposed analysis scheme is significantly reduced, and the proposed approach becomes very competitive. As it is illustrated in the previous section, the number of iterative steps related to the adopted unified iterative solution is not significantly higher than that of the dominant iterative approach acting isolated. Standard multiple iterative solutions can thus be avoided, eliminating overly excessive computational costs.

The two numerical examples presented here illustrated the versatility and effectiveness of the proposed techniques. In section 5, coupling configurations 1 and 2 are referred, finite and infinite models are analyzed, adaptive and fixed spatial discretizations are considered, constant and linear boundary element formulations are employed, linear and nonlinear solutions are carried out, different elastoplastic criteria are followed, etc. In fact, as previously stated, the present work enables a wide range of analyses, providing effective numerical tools to more properly deal with several complex problems.

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