GFEM STABILIZATION TECHNIQUES APPLIED TO TRANSIENT DYNAMIC ANALYSIS

Paulo de Oliveira Weinhardt

paulo.weinhardt@gmail.com

Marcos Arndt

arndt.marcos@gmail.com

Roberto Dalledone Machado

rdm@ufpr.br

Graduate Program in Numerical Methods in Engineering, Federal University of Paraná

Curitiba, 81.531-980, Paraná, Brasil

Abstract. *In the context of dynamic analysis of structures, one of the limitations of the Finite Element Method (FEM) is the difficulty of approaching the high frequencies. This lack of precision becomes more significant as the loading excite modes with higher frequencies. Aiming at address this problem one may use the Finite Element Method Generalized / Extended (GFEM /XFEM) to enrich the approximation space and better represent these high frequency modes. Despite the excellent properties of GFEM / XFEM as high accuracy, application versatility and excellent convergence rates, there are aspects that still limit its applicability as the numerical instability associated with this enrichment process even in well-placed boundary value problems. GFEM/XFEM matrices may be ill-conditioned, which may result in a accuracy loss, and even resulting in numerically singular matrices. In this work two proposals are presented to circumvent the GFEM sensitivity problem. Examples of one-dimensional transient analysis are presented and results are discussed analyzing the effects of adopting the preconditioning of enrichment functions strategy.*

Keywords: Dynamic Analysis, GFEM, Partition of Unity, Conditioning, SGFEM.

1   INTRODUCTION

The Generalized Finite Element Method (GFEM) is a Galerkin method applied to subdomains. In this method, some enrichment functions that reflect particular characteristics of the problem are added to improve the quality of approximation. To simulate the propagation of cracks, for example, it can be used functions that contain singularities or discontinuities. Similarly, for problems of structural dynamics it may be used functions containing trigonometric terms since this type of functions are usually associated with analytical solutions.

GFEM was applied to dynamic analysis is presented in Arndt (2009) and Torii (2012). Arndt (2009) addressed enrichment methodologies for modal analysis of framed structures. The approach's efficiency in obtaining excellent results in terms of frequency spectrum has been shown. However, applying successive layers of enrichment results in increased sensitivity of the numerical problem. Thereafter, Torii (2012) extended GFEM studies in dynamics, applying it to the two-dimensional domains and involving modal and transient analyzes.

Based on these studies, Shang (2014) extends the studies of GFEM to elastoplastic dynamic transient analysis proposing solutions for numerical difficulties present in the FEM approach. The results obtained are satisfactory but, nevertheless, the numerical sensitivity problem is still present.

Addressing numerical sensitivity problems that arise in the application of enrichment proposals, two alternatives are adopted such as presented in Weinhardt *et al.* (2015): application of the concepts of the Stable Generalized Finite Element Method and application of preconditioning changes in enrichment functions.

Stable GFEM was first proposed by Babuška & Banerjee (2012) to adress ill-conditioning issues associated to systems of equations arising from the GFEM application. This solution consists on a subtle modification of enrichment functions prior to its association to the Partition of Unity. Babuška & Banerjee (2012) still present the mathematical foundation that ensures the conditioning of these preconditioned systems are as good as those resulting from FEM. In this paper, we present an extension of this proposal to the transient analysis, continuing the work presented by Weinhardt *et al.* (2015).

Following the idea that small changes in enrichment functions can change the numerical conditioning of the problem without significant loss of accuracy properties of the GFEM, this article presents the effects of a subtle change in the functions used for one-dimensional analysis such as presented by Arndt (2009). This proposal has been proved to be very effective in improving the stability for the modal analysis and so it's presented for transient analysis.

2   METHODOLOGY

Trigonometric Enrichment

For free vibration problem has proposed by Arndt (2009) a block of enrichment functions to the problem of dynamic analysis with GFEM. This group of functions consists of building a couple of clouds, a sine and a cosine, subordinated cover of enriched node. These clouds are written in the domain of element as two pairs of sine and cosine functions. The basic domain is considered to $\xi \in [0, 1]$.

Sine cloud:

$$\gamma_{1j} = sin(\beta_j L_e \xi) \tag{1}$$
$$\gamma_{2j} = sin(\beta_j L_e (\xi - 1))$$

Cosine cloud:

$$\varphi_{1j} = cos(\beta_j L_e \xi) - 1 \tag{2}$$
$$\varphi_{2j} = cos(\beta_j L_e (\xi - 1)) - 1$$

Where L_e is the length of the element and $\beta_j = j\pi$ is a hierarchical enrichment parameter proposed by Arndt (2009) to j function levels.

2.1 SGFEM-based Stabilization

Stable Generalized Finite Element Method (SGFEM) was firstly proposed to address numeric conditioning issues of GFEM Babuška & Banerjee (2012). This methods consists in the application of a subtle modification of enrichment function prior to its inclusion in GFEM approximation space, see Gupta *et al.* (2013); Li (2014).

In the SGFEM, the enrichment function are modified as described in Eq. 3, as presented by Babuška & Banerjee (2012):

$$\widetilde{\varphi}_i(x) = \varphi_i(x) - I_\omega(\varphi_i(x)) \tag{3}$$

where,

- $\widetilde{\varphi}_i$: i-th stabilized enrichment function

- φ_i: i-th enrichment function

- $I_\omega(\varphi_i(x))$: linear interpolant of the i-th enrichment function subordinated to support ω

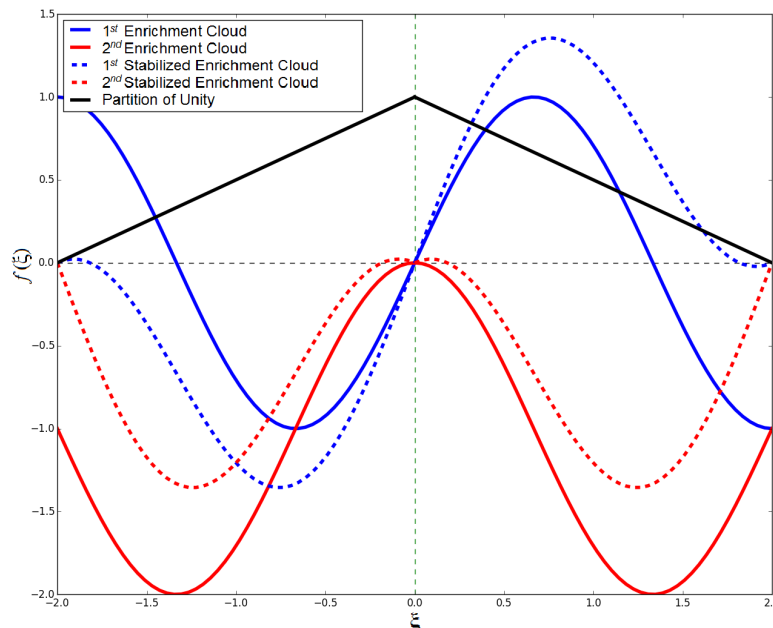The proposed stabilization of the first level of enrichment is shown in Fig.1.

Figure 1: Stabilization process for the first level of enrichment functions.

2.2 Heuristic Modification Stabilization

Note that the variation in the enrichment parameter β_1 implies different characteristics of approaching, as already pointed out by Arndt (2009) and Torii (2012). However, the gain in accuracy for certain frequencies does not appear to be associated with numerical stability gain. In fact, apparently there is a certain trade-off between accuracy and numerical stability, regarding the choice of the parameter β_1.

A second point of interest is the observation of β_j family of parameters. Parameters described by β_j are intrinsically related with enrichment functions and, consequently, with relations between the enriched functions present in the approximation space base. Thus, evolution of β_j influences the numerical characteristics of the approximation, such as stability. Therefore, we sought a change in the enrichment function group to stabilize its continuous application, avoiding the construction of approach spaces that tend to linear dependence.

The proposed modification is basically the change parameter formation rule β_j enrichment parameter, resulting in a subtle modification of each level of enrichment.

Recalling that parameter β_j is calculated by $\beta_j = j\alpha\pi$, it was proposed a modification, creating new stabilized parameters $\bar{\beta}_n$ given by:

$$\bar{\beta}_j = \left[2\left(j-1\right) + \frac{\beta_1}{\pi} \right]\pi \qquad j \geq 1 \tag{4}$$

It's interesting to note that $\bar{\beta}_1 = \beta_1$, since:

$$\bar{\beta}_1 = \left[2\left(1-1\right) + \frac{\beta_1}{\pi} \right]\pi = \left[0 + \frac{\beta_1}{\pi} \right]\pi = \beta_1 \tag{5}$$

This implies that there is no difference between approximation taken by this approach and the standard trigonometric enrichment first level of enrichment.

2.3 Transient Bar Problem

The trigonometric enrichment in the context of GFEM applied to one-dimensional transient analysis was discussed earlier by Torii (2012) and Shang (2014) covering several examples. This work will continue the discussion by presenting the application of p refinement with stabilization proposals set out in three enlightening examples. Stabilization alternatives in transient analysis, such as HHT used by Shang (2014), were avoided in order to keep focus on the analysis of the interactions of the enrichment process and the stabilization proposals of this work.

For the following examples, transient analysis arises in the application of the Newmark Method, as described in Bathe (1996), using mass and stiffness matrices generated by the application of GFEM with different approaches. Parameters were setted such as $\sqrt{\frac{E}{\rho}} = c = 1$, neglecting damping, and adopting a uniform mesh of 20 finite elements. For the time discretization, the 20 seconds analysis interval was divided in 2000 steps of 10^{-2} seconds.

The model considered for the three examples consists of a bar with a clamped end and the other free end where the load is applied as shown in Fig. 2. Trigonometric enrichment was adopted using $\beta_1 = \frac{3\pi}{4}$ due to its performance in modal analysis as presented by Weinhardt *et al.* (2015).
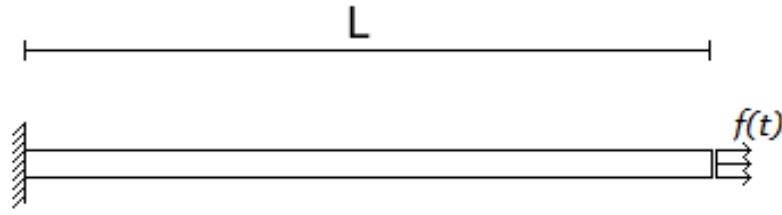
Figure 2: Clamped-free bar subject to time dependent force applied in the end.

2.4 Loading Cases

Two loading cases were chosen to test stabilization techniques. Those are as follow:

- Heaviside loading - abrupt loading that remains over time (example 3.1). Due this type of loading, discontinuities on the velocity field wil occur, see Monteiro (2009)

- Impulse loading - short time loading (example 3.2). Due this type of loading, hi high displacements gradients will be present, see Monteiro (2009)

Loading cases are exemplified in Fig. 3.
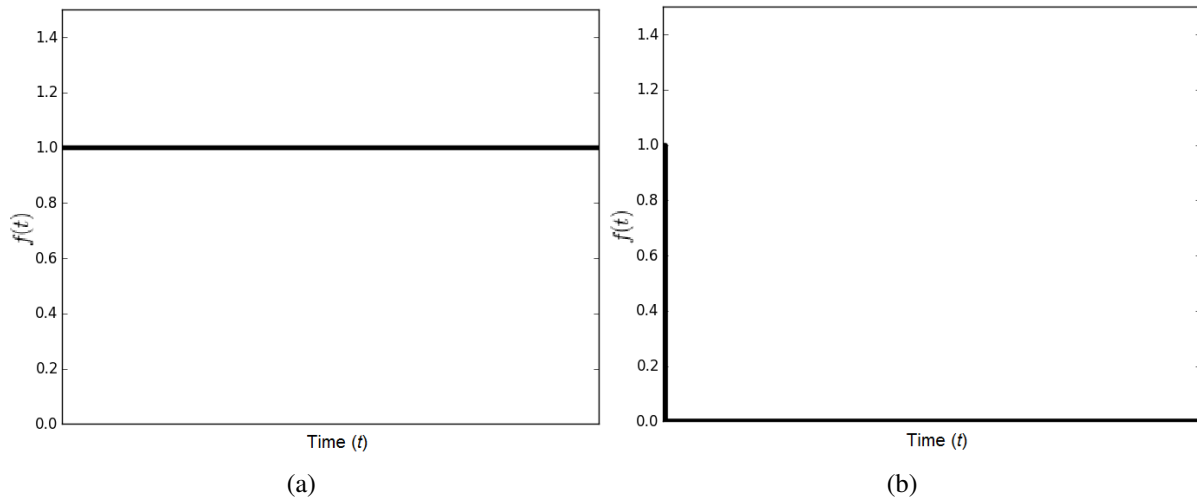
Figure 3: (a) Heaviside Loading, (b) Impulse Loading

3 RESULTS

Displacements, velocities and accelerations are presented for both loading cases in the following examples.

3.1 Heaviside Loading

The external load is applied abruptly with a value of $1N$ and maintained until the end of analysis, featuring a Heaviside force. The graphs of displacement and velocity, over time, of the free end are shown below for different approximation strategies.

Figures 4 and 5 presents transient response due to Linear FEM, one enrichment layer GFEM and SGFEM.
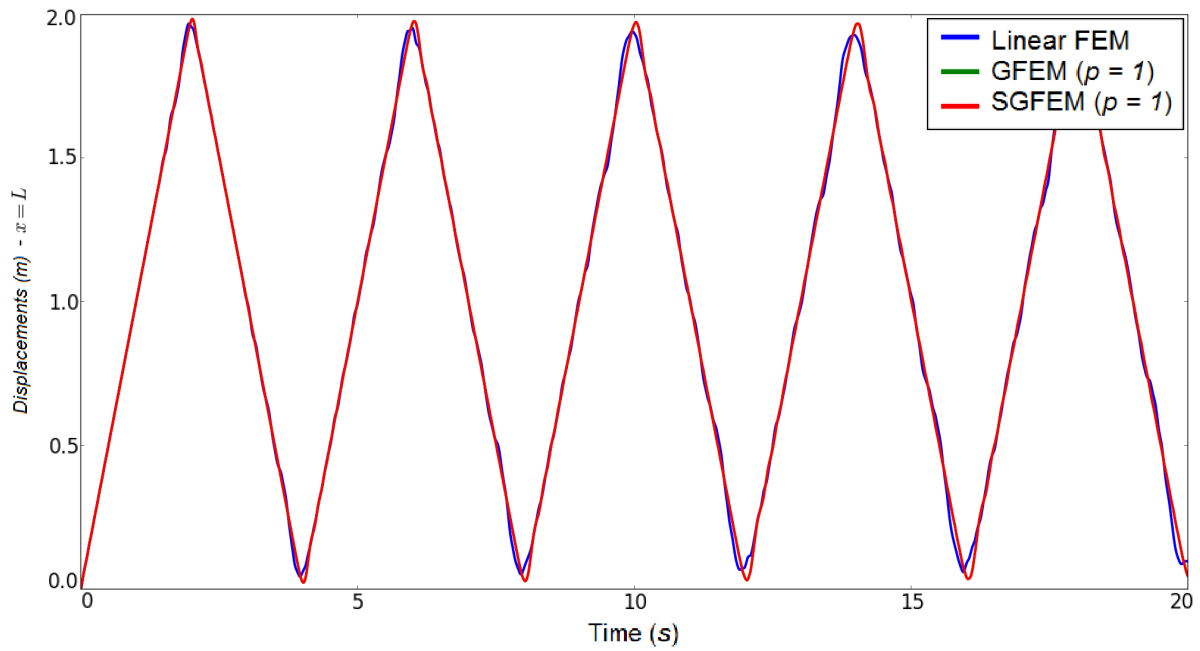

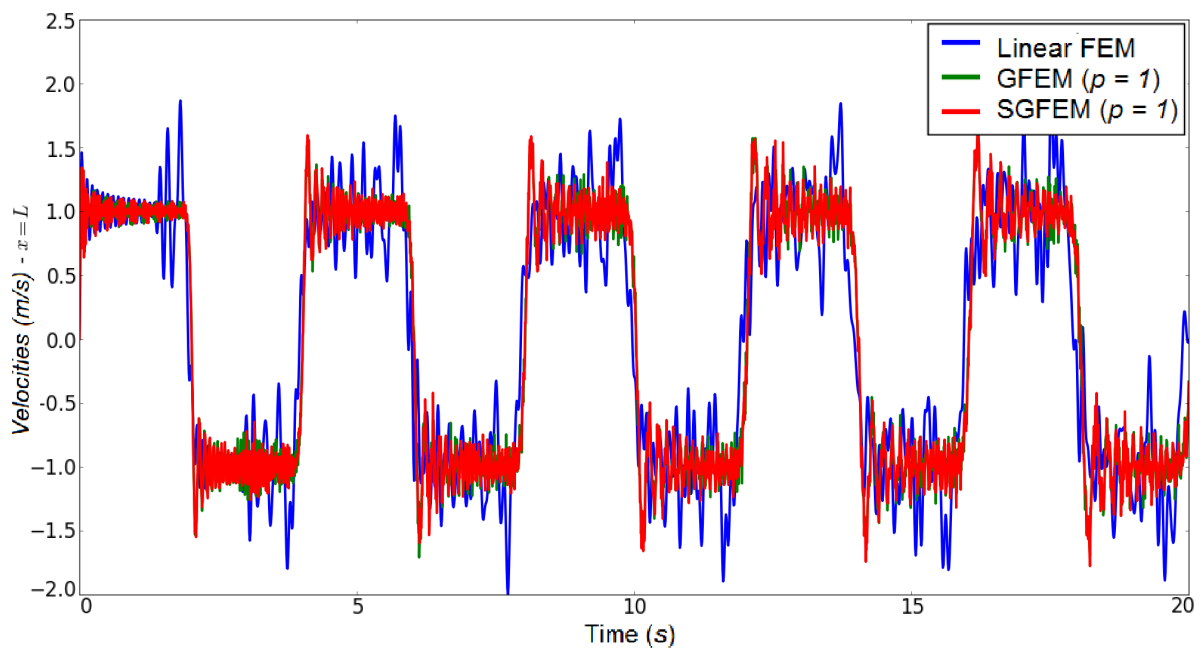**Figure 4: Transient Analysis - Displacements at the free end.**

Figure 5: Transient Analysis - Velocities at the free end.

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In order to complement the analysis the acceleration results are shown in Fig. 6 and Fig. 7.
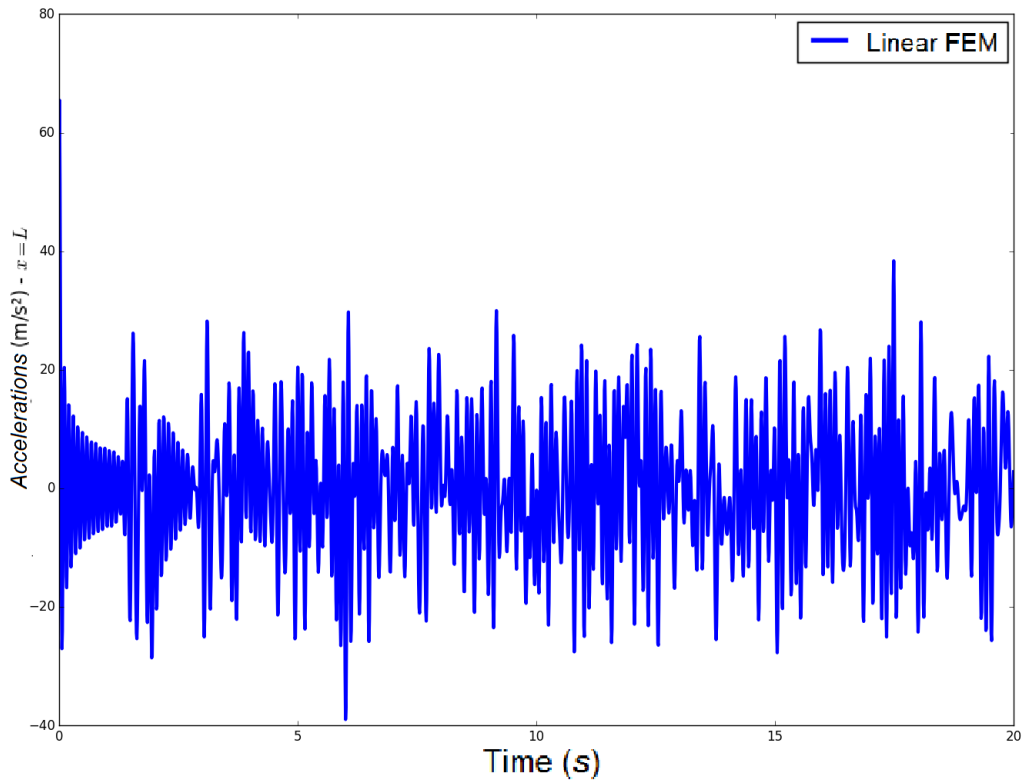
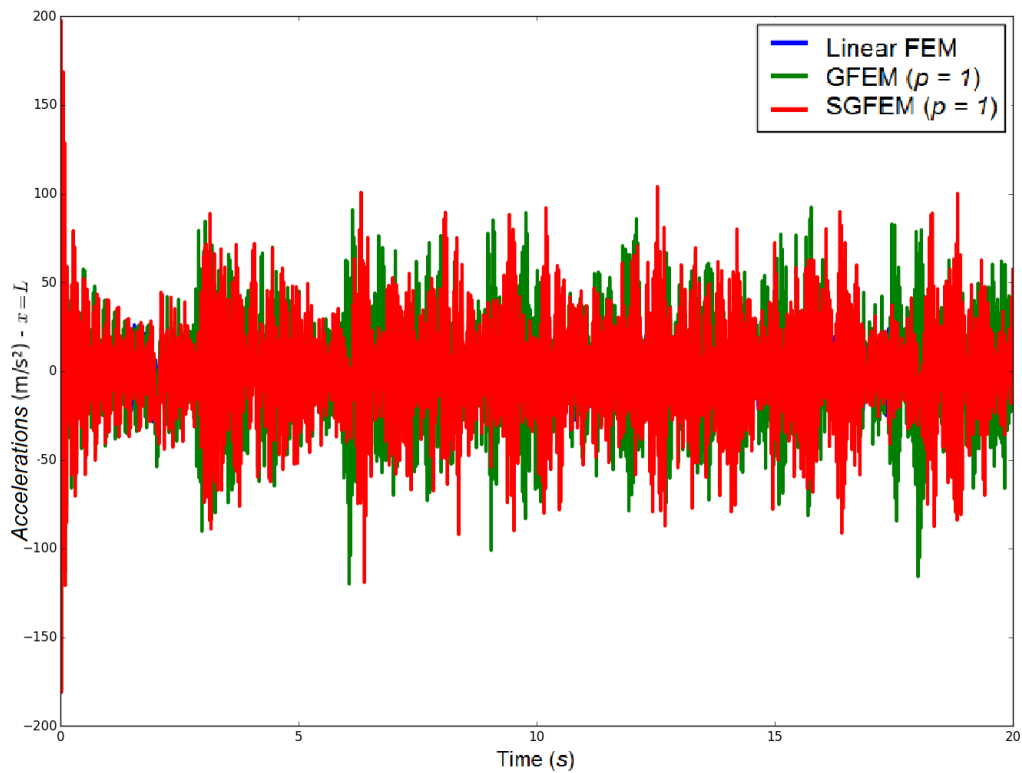**Figure 6: Transient Analysis (p-refinement) - Accelerations at free end (apenas FEM)**

Figure 7: Transient Analysis (p-refinement) - Accelerations at free end (FEM, GFEM e SGFEM).

Analytical refinement is applied, reaching 4 enrichment levels for the three approaches: GFEM, SGFEM e Heuristic Modification. Results are presented in Fig. 8 and Fig. 9.
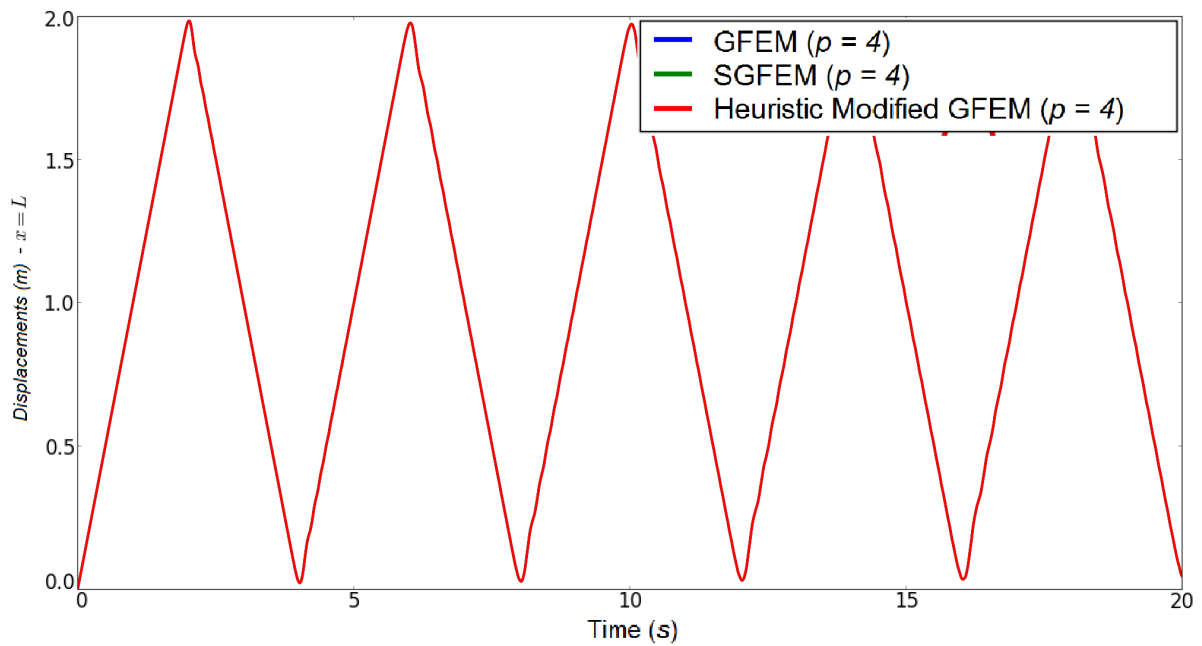

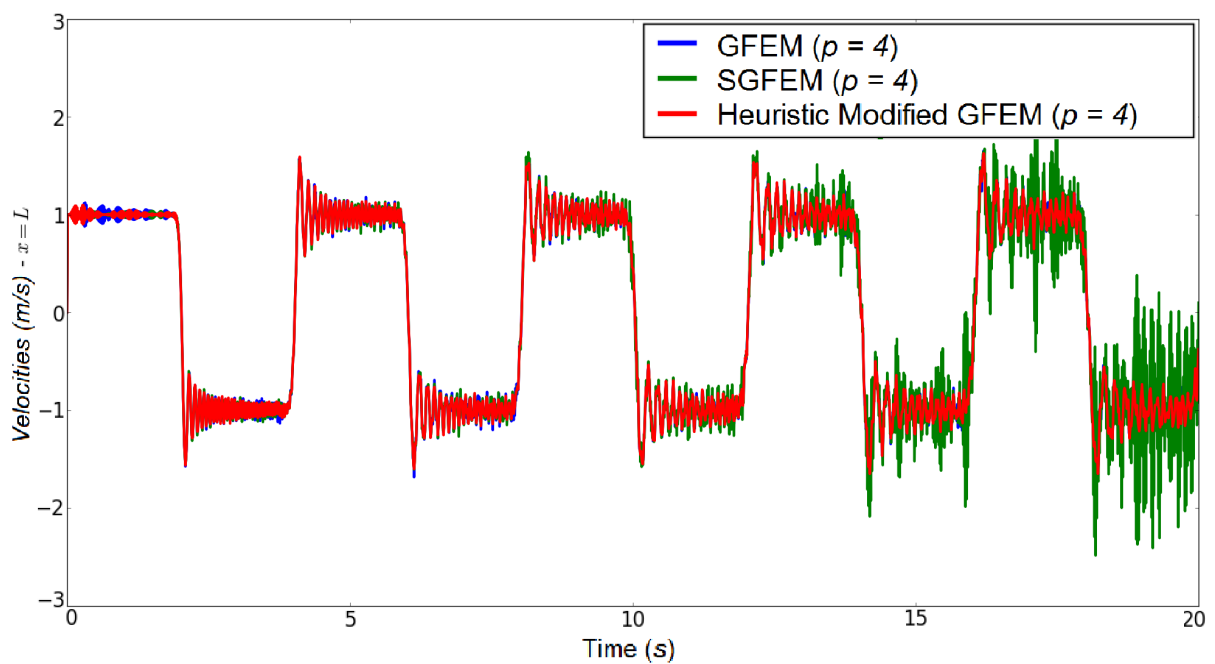**Figure 8: Transient Analysis (p-refinement) - Displacements at the free end.**

Figure 9: Transient Analysis (p-refinement) - Velocities at the free end.

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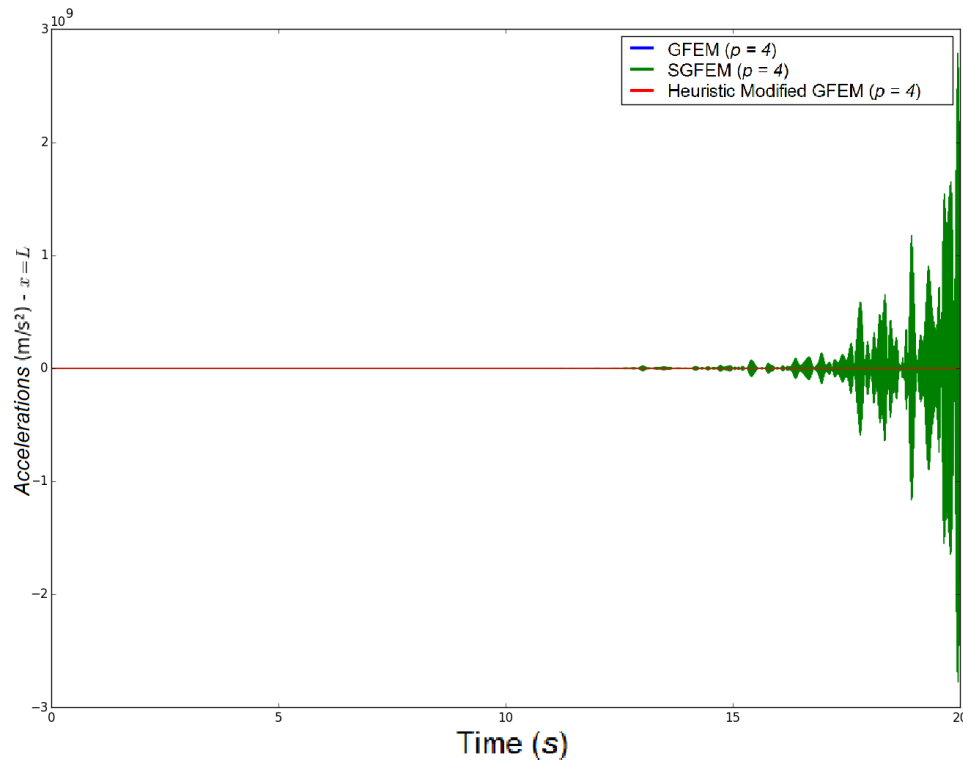Figures 10 and 11 presents results corresponding to acceleration field over time.

Figure 10: Transient Analysis (*p*-refinement) - Accelerations at free end (GFEM, SGFEM e Heuristic Modification)
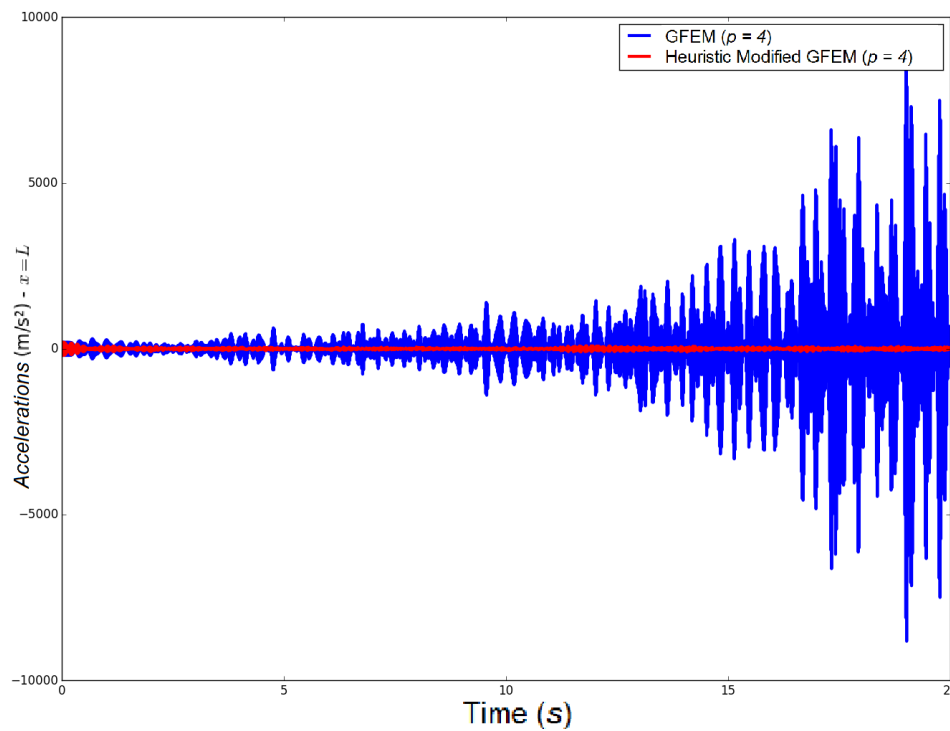

Figure 11: Transient Analysis (*p*-refinement) - Accelerations at free end (GFEM e Heuristic Modification).

Results aiming to highlight the stabilization proposal are presentend in Fig.12 and Fig.13, taking up to p-refinement to high-order.
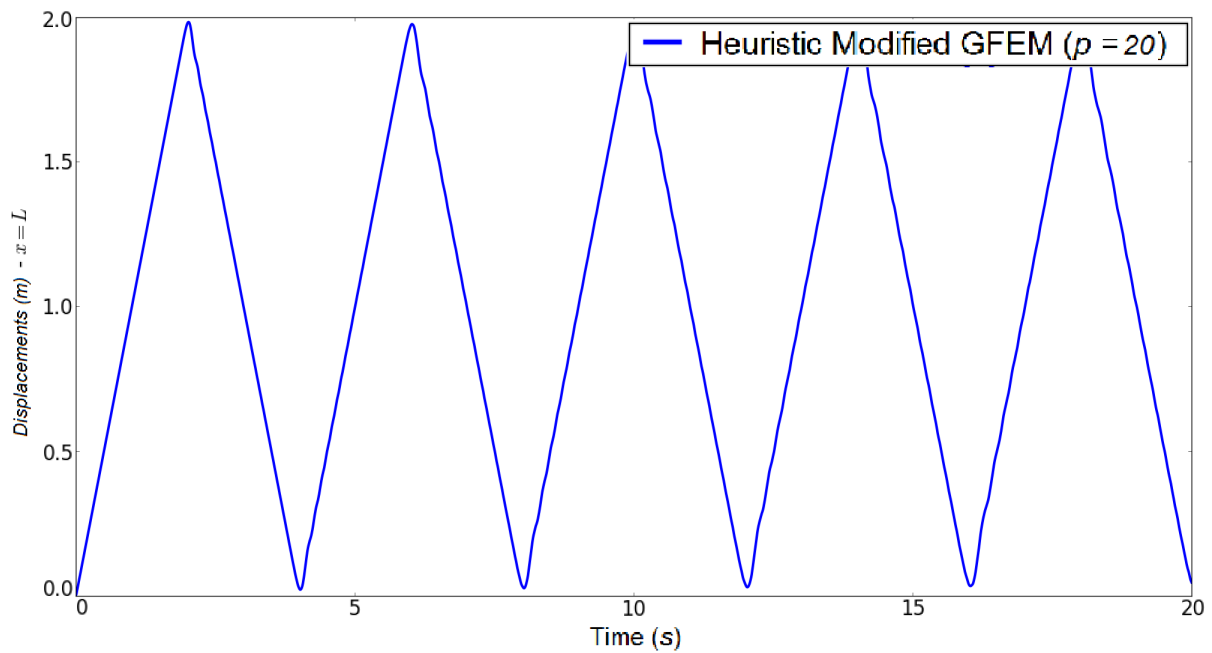

Figure 12: Transient Analysis (High-order p-refinement) - Displacements at the free end.
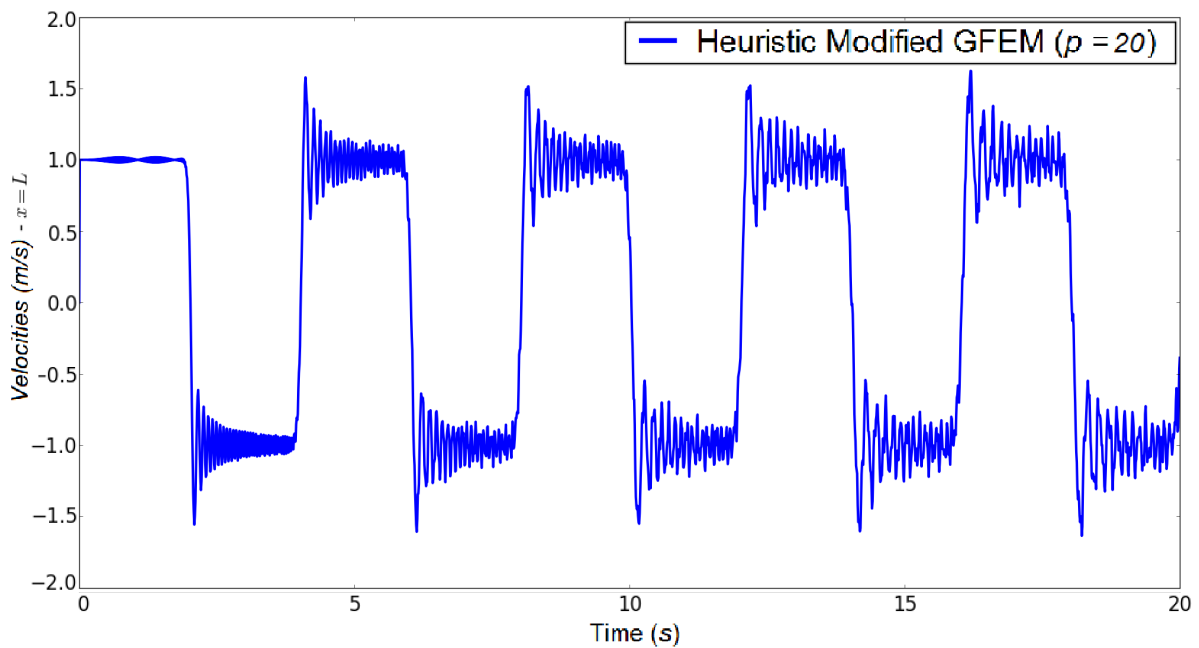

Figure 13: Transient Analysis (High-order p-refinement) - Velocities at the free end.

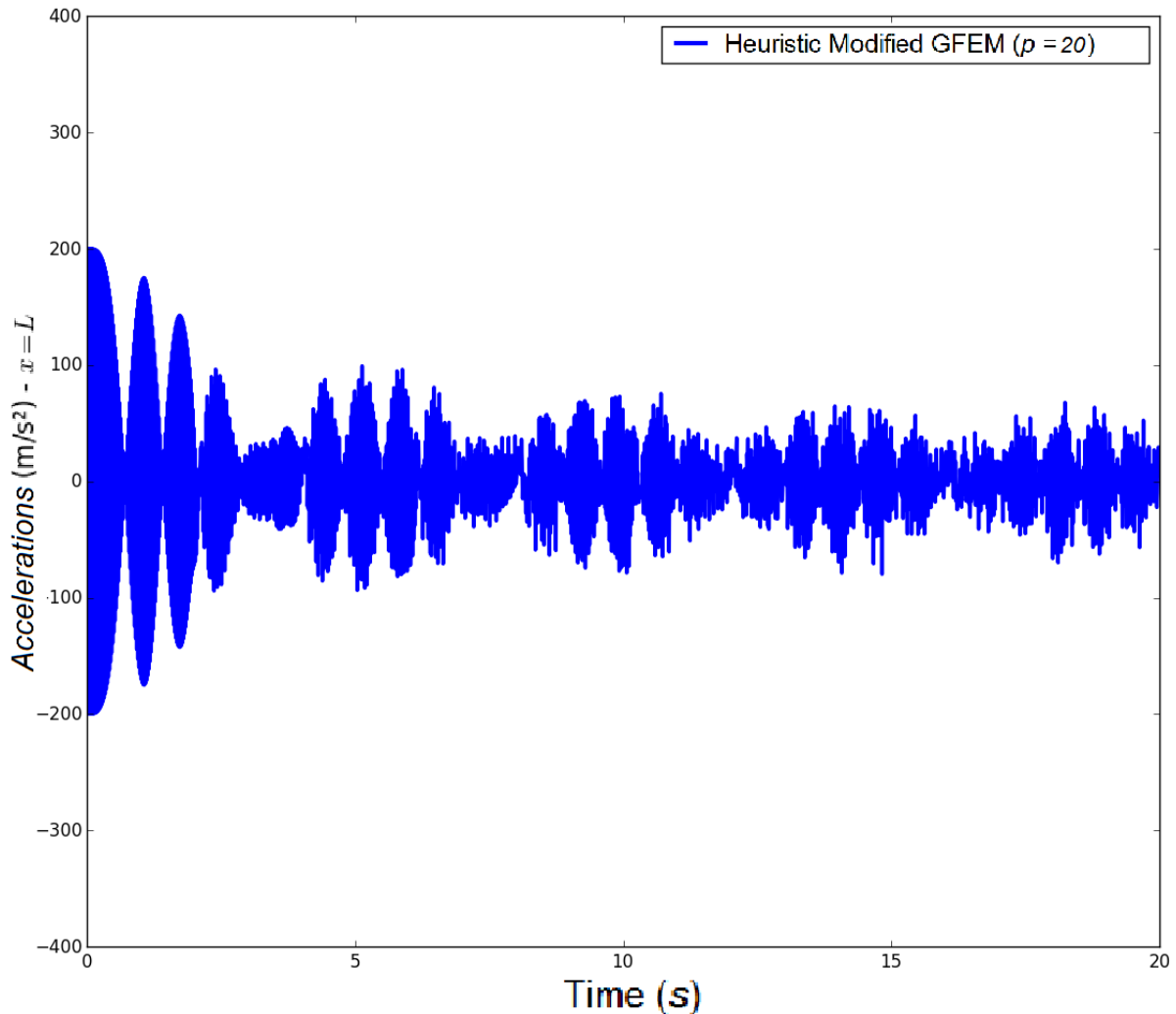Acceleration results are presented in Fig. 14.

Figure 14: Transient Analysis (p-refinement) - Accelerations at the free end.

The first loading case was studied and responses were presented for displacements, velocities and acceleration. Firstly, the approximation was made by one level of enrichment, concerning both GFEM and SGFEM, and Linear FEM. Figure 4 show that displacements are relatively accurated, even for Linear FEM approximation. However, velocity field presents reasonable pertubation as shown in Fig. 5. This results were expected since the loading provoke a velocity discontinuity over time, see Monteiro (2009). One may note that GFEM presents the most stable solution among this three alternatives. In acceleration terms, the response is predominantly spurious for the three approaches since its approximation is significantly difficult using continuous functions.

Applying 4 levels of enrichment, it was compared GFEM, SGFEM and the Heuristic Modification. Displacements responses for the three alternatives are pretty close, as shown in Fig. 8. However, velocities presented considerably different behaviours as Heuristic Modification resulted in better approximation and SGFEM's response rapidly deteriorated over time.

Numerical stability problems are highlighted in the approximation of acceleration field

shown in Fig. 10 and Fig. 11 where one may note that SGFEM has its answer drastically deteriorated as the analysis advances in time. GFEM also suffers significant deterioration, but more limited. Meanwhile, the Heuristic Modification demonstrated a more stable behaviour, although having a response with some disturbance.

Testing the stability of the proposed Heuristic Modification, Fig.12 and Fig.13 show the results of application of 20 levels of enrichment. Apparently that high order p-refinement did not result in significant gains in accuracy for both displacements and velocities. However, it is worth noting that the application that many enrichment levels did not compromised the stability of the numerical approach, following the trend shown in modal analysis as presented by Weinhardt *et al.* (2015).

Applying a modal superposition method can take advantage of these characteristics to extract eigenvalues generated by modal analysis to build the transient response with less effort. The acceleration field of this example corresponds to the second derivative of the displacement field which is of class C^0 and hence is discontinuous. Thus, the numerical approach is considerable difficulty, since the problem model is based on displacements.

3.2 Impulse Loading

The external load is applied with a value of $1N$ in the time range from 0 to 10^{-2} seconds featuring a impulse loading. The graphs of displacements and velocities of the free end are given below for different approximation strategies.

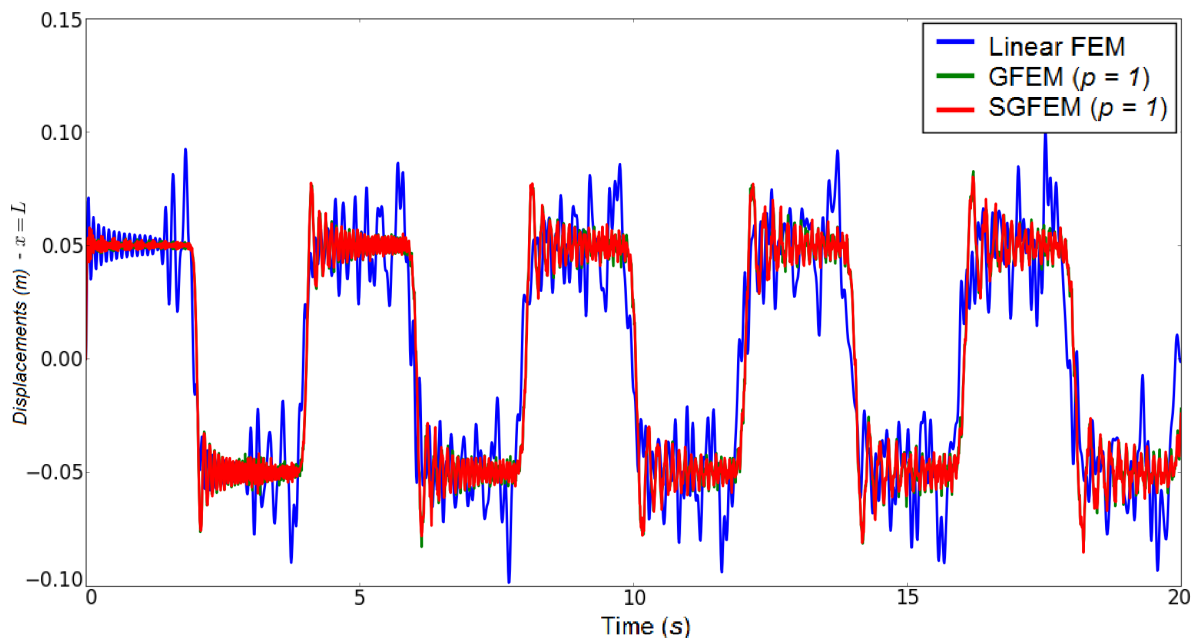Figures 15 and 16 presents transient response due to Linear FEM, one enrichment layer GFEM and Adapted SGFEM.

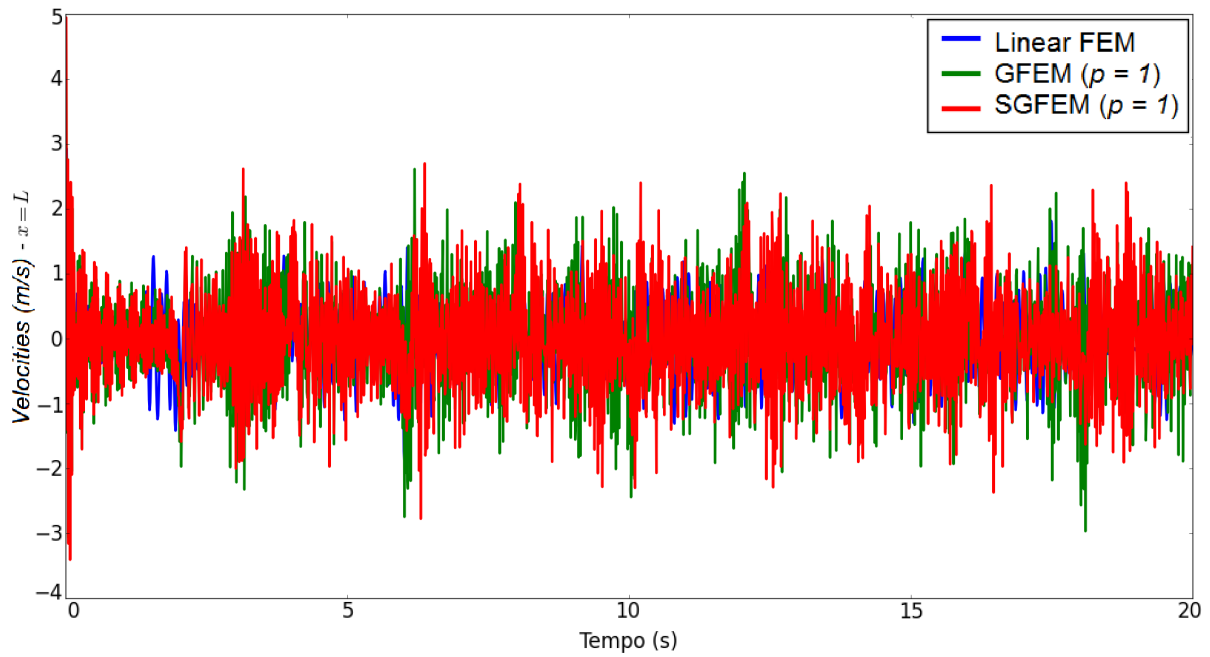Figure 15: Transient Analysis - Displacements at the free end.

Figure 16: Transient Analysis -Velocities at the free end.

Analytical refinement is applied, reaching 4 enrichment levels for the three approaches: GFEM, SGFEM e Heuristic Modification. Results are presented in Fig. 17 and Fig. 18.
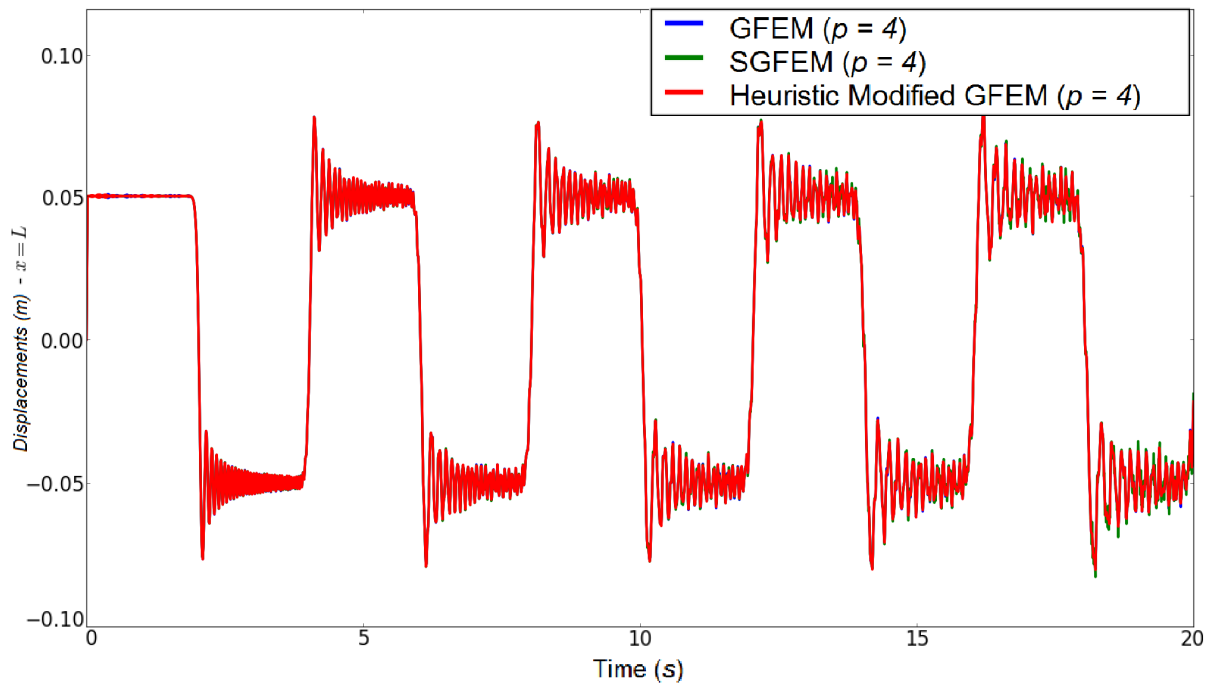

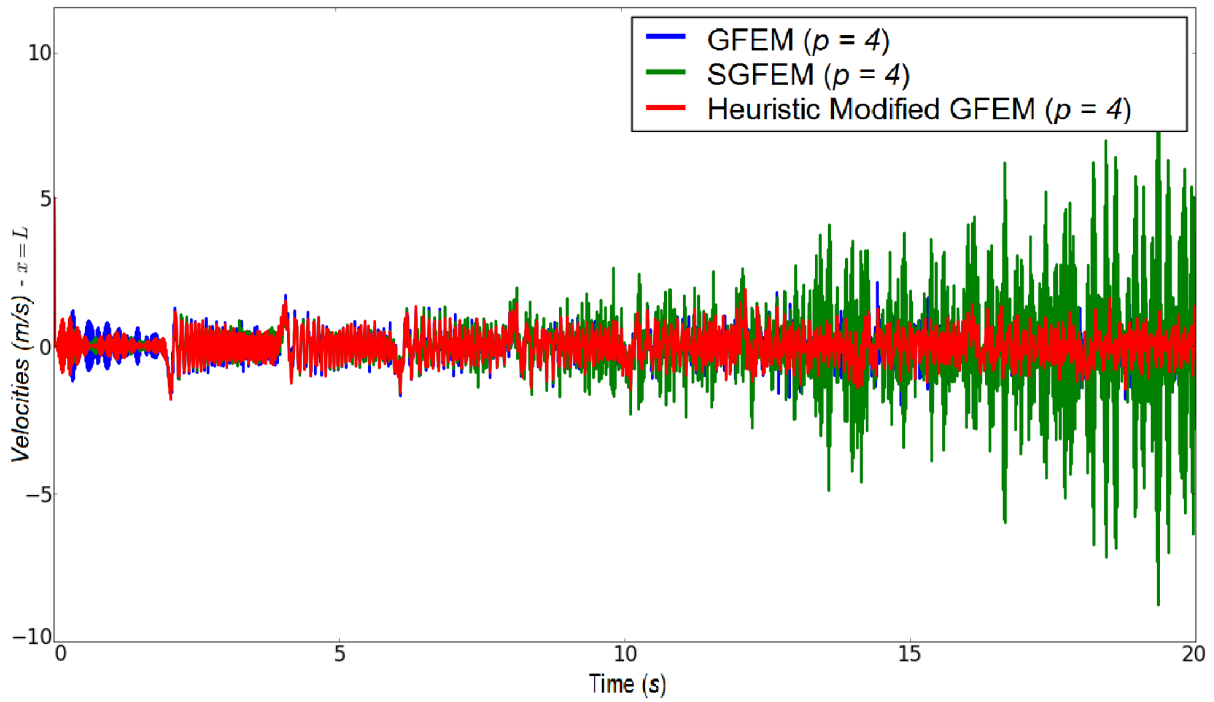**Figure 17: Transient Analysis (p-refinement) - Displacements at the free end.**

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Figure 18: Transient Analysis (*p*-refinement) - Velocities at the free end.

Results aiming to highlight the stabilization proposal are presentend in Fig. 19 and Fig. 20, taking up to *p*-refinement to high-order.
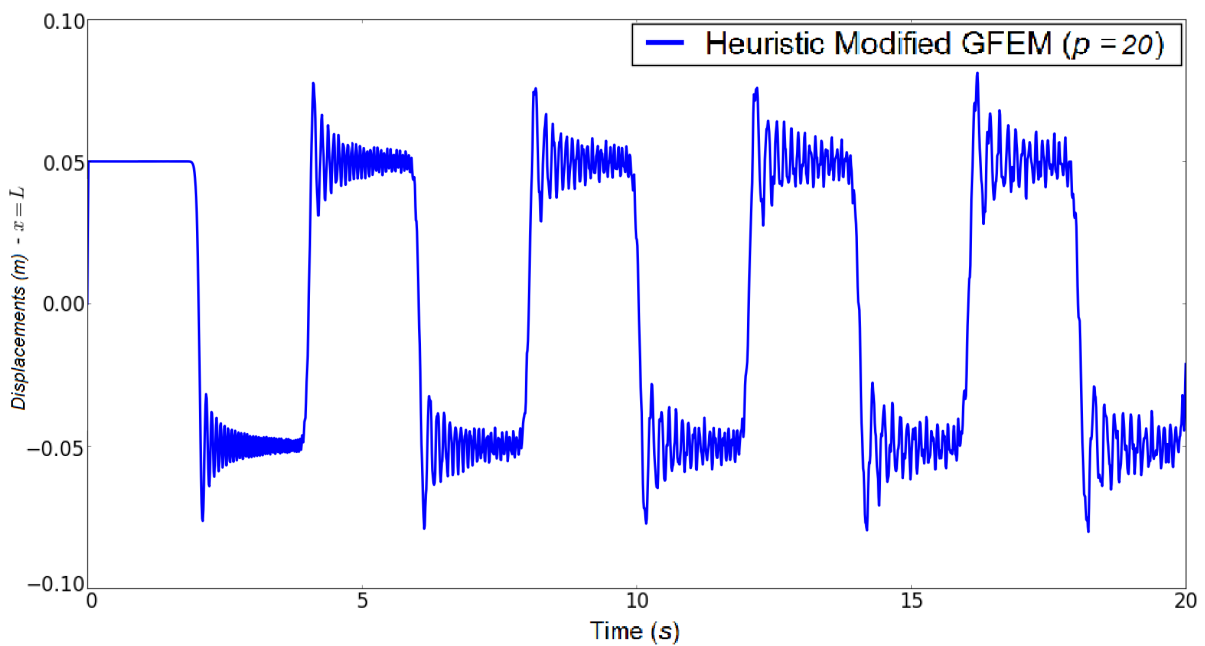

Figure 19: Transient Analysis (High-order *p*-refinement) - Displacements at the free end.
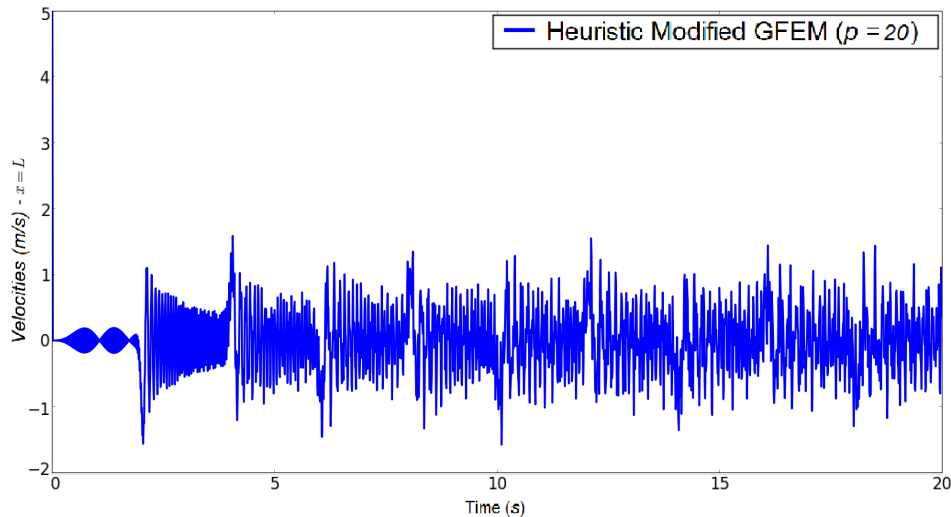
Figure 20: Transient Analysis (High-order *p*-refinement) - Velocities at the free end.

The second loading case was studied and responses were presented for displacements and velocities. Acceleration were omitted since its poorly approximated using continuous functions. Firstly, the approximation was made by one level of enrichment, concerning GFEM and SGFEM, and FEM. As shown in Fig.15 and Fig.16, displacements response has large oscillations for Linear FEM and a more accurate behavior for alternative enriched. On the other hand, the answer in terms of velocities present great disturbance for the three alternative approaches.

Applying 4 levels of enrichment, it was compared GFEM, SGFEM and the Heuristic Modification, as show in Fig.17 and Fig.18. The corresponding approximations for displacements are considerably close and more accurate than the previous results, and the Heuristic Modification resulted in a fairly accurate approximation for the first time steps. However, the velocities response for SGFEM presented a pretty deteriorated behaviour over time.

Testing the stability of the proposed Heuristic Modification, Fig.19 and Fig.20 show the results of application of 20 levels of enrichment. Apparently that high order *p*-refinement did not result in significant gains in accuracy for both displacements and velocities. However, it is worth noting that the application that many enrichment levels did not compromised the stability of the numerical approach, following the trend shown in modal analysis as presented by Weinhardt *et al.* (2015).

4   CONCLUDING REMARKS

This paper discussed issues relevant to the stability of the Generalized Method of Finite Elements applied to dynamic analysis. One-dimensional bar examples were presented contemplating transient analysis. The formulation of trigonometric enrichment was based on proposals of Arndt (2009) and Torii (2012).

Seeking to address the stability issue, were studied two stabilization alternatives. The first was based on an adaptation of the Finite Element Method Generalized Stabilized, initially proposed to problems falling within the system of equations for a resolution by Babuška & Banerjee (2012). The second proposal was the modification of the parameter β present in trigonometric functions GFEM enrichment proposed by Arndt (2009) and Torii (2012).

For transient analysis it was used the Newmark method with the taking advantage of the mass and stiffness matrices generated by GFEM. Although the results presented inherent disturbances of Newmark method, it was possible to compare the proposals for approximation among each other, and the second proposal stabilization stood out in most examples. The possibility of making high-order return fines with consequent improvement in response without affecting the CFL stability condition (Courant-Friedrichs-Lewy, see Moura & Kubrusly (2012)) was presented.

The results of this work, along with Weinhardt *et al.* (2015), point out that there are ways to overcome instability problems in GFEM applied to dynamic analysis, since simple proposals were able to positively impact the approaches. Therefore it is expected that further studies discuss GFEM stability issues in dynamic analysis, elucidating the nature of encountered problems and the construction of more accurate and stable approximations.

5 ACKNOWLEDGEMENTS

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