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ON A RECURSIVE METHODOLOGY FOR SEMI-ANALYTICAL SOLUTIONS OF SYMMETRIC AND UNSYMMETRIC LAMINATED THIN PLATES

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Abstract. *The present paper has as objective the introduction and analysis of a new procedure in order to derive semi-analytical solutions of symmetric and unsymmetrical laminated rectangular thin plates in a recursive manner. The methodology is based on three main characteristics: (a) decomposition of the differential operator into two or more components, (b) an infinite expansion of the differential equation solution and, (c) determination of each superposed solution by a relationship between the divided operators and previous solutions. The first expanded term concerns to the plate's isotropic solution and, in each step, orthorhombic laminae influence is inserted. In order to approximate the solutions, the pb-2 Rayleigh-Ritz Method is used. Obtained solutions are discussed and compared to those found in the literature.*

Keywords: *Recursive Methodology, Laminated Plates, Semi-analytical Solution, Rayleigh-Ritz Method*

1 INTRODUCTION

From automotive to aerospace industry, the use of laminated plates and shells is increasing in several applications due to its good strength/weight ratio. As a result, the mechanical behaviour of such components need to be deeply comprehended. This work has as objective an introduction of a recursive methodology, namely Adomian Decomposition Method (Adomian, 1994), applied to the analysis of sandwich and laminated plates. The method is based on three main concepts: the decomposition of the differential operator, the expansion of the problem's solution in an infinite series and, with the aid of a recursive procedure, the determination of each expanded term. This methodology is known as a special case of homotopy-like methods (Abbasbandy, 1996 and Li, 2009). Essentially, these methods recursively enhance simpler solutions with more complex terms of the differential equation by several small perturbation, which are taken into accont in a recursive fashion. The Rayleigh-Ritz Method (Liew & Wang, 1993) is used in order to approximate the solution space of thin plate's differential equation as well as an additive decomposition of the constitutive tensor (Browaeys & Chevrot, 2004) is utilised so as to manage the differential operator decomposition.

Classical Laminated Plate Theory (CLPT) is a counterpart of the Classical Plate Theory (CPT) for inhomogeneous materials which change discretely in the normal direction of the reference surface. Basically, it considers an equivalent anisotropic layer, whose constitutive properties are homogenised with respect to the laminae elastic properties (Altenbach, 1998). Consequently, the same differential equation of the CPT is obtained. The difference between both lies on the stress field: in the CLPT, the stress is a piecewise function instead of a linear function with respect to the thickness, as in the CPT. In consequence of having the same differential equation, procedures to determine the solutions of a thin homogeneous plate can be directly applied to inhomogeneous ones.

Rayleigh-Ritz Method (RRM) is used due to its excellence in solving plate problems (Kitipornchai et al., 1994, Singh & Elaghabash, 2003). Using weighted kinematically admissible interpolation functions, the solution space is approximate by the minimization of the total potential energy functional. The *pb-2* modification circumvents the interpolation basis requirement of being kinematically admissible by multiplications of specific functions, namely Bhat beam functions (Bhat, 1985), which impose this requisite. Consequently, the interpolation functions can be chosen freely.

Adomian Decomposition Method (ADM) (Adomian, 1994) is a homotopy-like method which constructs a solving routine by recursive means. The non-linear partial differential equation is decomposed into linear, remainder and non-linear terms. Furthermore, the solution is superposed in infinite series. The difference between the linear and the remainder is essentially the knowledge of its inverse. By a correlation among the decomposed terms, each part of the expanded solution is determined. The method has been applied by Li (2009) and Tabatabaei (2010) to solve non-linear differential equations. Cheniguel & Ayadi (2011) has made use of it so as to determine the solution of the heat equation as well as Rao (2010) has utilised the method in order to develop solutions for the Ricatti differential equation.

Summarising, with some algebraic manipulations, the ADM is applied to the linear system obtained by the minimization of the total potential energy (*pb-2* RRM). The decomposition of the differential operator is performed regarding an additive decomposition (Browayes & Chevrot, 2004) of the constitutive tensor, which is related to symmetry classes and its hierarchy

(Chadwick et Al., 2001). Essentially, this decomposition extracts from the original constitutive tensor an isotropic part and this operation is implemented in each lamina. The solution expansion is related to the expansion of the weighting constants. As a result, the recursive procedure is reduced to the solution of linear systems. The obtained solutions are discussed and compared to those found in the literature.

Both index and matrix notation are used. The work is followed by 5 sections: a small review of the CLPT, the *pb-2* Rayleigh-Ritz Method applied to CLPT, an introduction to the ADM and its use to the determination of laminated plates solution, some numerical results and the conclusions.

2 CLASSICAL LAMINATED PLATE THEORY: A REVIEW

The displacement field of the Classical Laminated Plates Theory (CLPT) is written as (Reddy, 1993):

$$\begin{aligned} U_1\left(x_1, x_2, x_3\right) &= u_1\left(x_1, x_2\right) - x_3\, u_{3,1}\left(x_1, x_2\right) \\ U_2\left(x_1, x_2, x_3\right) &= u_2\left(x_1, x_2\right) - x_3\, u_{3,2}\left(x_1, x_2\right) \\ U_3\left(x_1, x_2\right) &= u_3\left(x_1, x_2\right) \end{aligned} \tag{1}$$

where U_i are the plate's displacement in $\hat{e}_i$ direction, u_α and u_3 correspond to the membrane and transverse displacements, respectively. As one can observe, the displacements u_i on the right side of eq. (1) are dependent only on the reference plane coordinates $x_1 \times x_2$. In describing the displacement field in such fashion, all the subsequent variables refer only to those coordinates while the dependence on x_3 is explicit. Thus, from now on, parameters x_1 and x_2 are omitted, for sake of simplicity. Moreover, the displacement is homogeneous by the CLPT, independently of the plate's inhomogeneity.

By the linear kinematic relation, the plate's strains can be described as:

$$\boldsymbol{\epsilon} = \boldsymbol{\varepsilon} - x_3 \boldsymbol{\kappa} \tag{2}$$

which:

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \epsilon_1 & \epsilon_2 & \epsilon_6 \end{Bmatrix}^{\mathrm{T}}$$

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} u_{1,1} & u_{2,2} & u_{1,2} + u_{2,1} \end{Bmatrix}^{\mathrm{T}} \qquad \boldsymbol{\kappa} = \begin{Bmatrix} u_{3,11} & u_{3,22} & 2u_{3,12} \end{Bmatrix}^{\mathrm{T}}$$

The strains in eq. (2) are decomposed into membrane $(\boldsymbol{\varepsilon})$ and curvature $(\boldsymbol{\kappa})$ terms. In inhomogeneous plates, for instance, sandwich and laminated plates, these entities are coupled.

The constitutive relation is written as:

$$\boldsymbol{\sigma}^{(k)} = \mathbf{Q}^{(k)} \boldsymbol{\epsilon} = \mathbf{Q}^{(k)} \boldsymbol{\varepsilon} - x_3 \mathbf{Q}^{(k)} \boldsymbol{\kappa} \tag{3}$$

where (Reis & Albuquerque, 2010):

$$\mathbf{Q}^{(k)} = \mathbf{R}^{(k)^{-1}} \hat{\mathbf{C}} \mathbf{R}^{(k)} \tag{4}$$

$\mathbf{Q}^{(k)}$ is related to the stiffness of the lamina k with respect to the global coordinates. $\hat{\mathbf{C}}$ is the reduced constitutive tensor, for the plane stress state, written as:

$$\hat{\mathbf{C}} = \begin{bmatrix} \hat{C_{11}} & \hat{C_{12}} & 0 \\ & \hat{C_{22}} & 0 \\ sym & & \hat{C_{66}} \end{bmatrix} \tag{5}$$

where:

$$\begin{aligned} &\hat{C}_{11} = \frac{E_{11}}{1-\nu_{12}\nu_{21}} \qquad \hat{C}_{22} = \frac{E_{22}}{1-\nu_{12}\nu_{21}} \\ &\hat{C}_{12} = \nu_{12}E_{22} \qquad \hat{C}_{66} = G_{12} \qquad \nu_{21}E_{11} = \nu_{12}E_{22} \end{aligned} \tag{6}$$

which is related to an orthorhombic symmetry. In eq. (4), $\mathbf{R}^{(\mathrm{k})}$ transforms an orthorhombic reduced constitutive tensor and it is explicit as:

$$\mathbf{R}^{(k)} = \begin{bmatrix} \cos^2(\theta_k) & \sin^2(\theta_k) & \sin(\theta_k)\cos(\theta_k) \\ \sin^2(\theta_k) & \cos^2(\theta_k) & -\sin(\theta_k)\cos(\theta_k) \\ 2\sin(\theta_k)\cos(\theta_k) & -2\sin(\theta_k)\cos(\theta_k) & \sin^2(\theta_k)-\cos^2(\theta_k) \end{bmatrix} \tag{7}$$

which θ_k is the directive angle between the local, in which $\hat{\mathbf{C}}$ is defined, and the global coordinates. This matrix is the result from the fourth-order reduced constitutive tensor $(2\times 2\times 2\times 2)$ transformation and the contraction $\Re^4 \rightarrow \Re^2$. Moreover, $\mathbf{R}^{(k)^{\mathrm{T}}} = \mathbf{R}^{(k)^{-1}}$. Normally, equation (4) is presented explicitely (Liew & Lam, 1991).

As the *pb-2* Rayleigh-Ritz Method (*pb-2* RRM) is applied to determine the stationary point through the variational principle of minimal potential energy (PMPE), the strain energy V_{int} and the external potential V_{ext} are defined in terms of the displacements. V_{int} can be written as:

$$\begin{aligned} V_{\text{int}} &= \frac{1}{2}\sum_{k=1}^{L}\left[\int_V \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{Q}^{(k)}\boldsymbol{\epsilon}\,\mathrm{d}V\right] \\ &= \frac{1}{2}\sum_{k=1}^{L}\left[\int_V \boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{Q}^{(k)}\boldsymbol{\varepsilon} - 2x_3\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{Q}^{(k)}\boldsymbol{\kappa} + x_3^2\boldsymbol{\kappa}^{\mathrm{T}}\mathbf{Q}^{(k)}\boldsymbol{\kappa}\,\mathrm{d}V\right] \end{aligned} \tag{8}$$

which L corresponds to the number of the layers in the plate. By integrating over the thickness, one can find:

$$\begin{aligned} V_{\text{int}} = \frac{1}{2}\int_\Lambda \boldsymbol{\varepsilon}^{\mathrm{T}}\left[\sum_{k=1}^{L} x_3|_{z_k}^{z_{k+1}}\mathbf{Q}^{(k)}\right]\boldsymbol{\varepsilon} + \\ -2\boldsymbol{\varepsilon}^{\mathrm{T}}\left[\sum_{k=1}^{L}\frac{1}{2}x_3^2|_{z_k}^{z_{k+1}}\mathbf{Q}^{(k)}\right]\boldsymbol{\kappa} + \boldsymbol{\kappa}^{\mathrm{T}}\left[\sum_{k=1}^{L}\frac{1}{3}x_3^3|_{z_k}^{z_{k+1}}\mathbf{Q}^{(k)}\right]\boldsymbol{\kappa}\,\mathrm{d}\Lambda \end{aligned} \tag{9}$$

where the summation is related to the plate's inhomogeneity and z_k is the lateral distance between the bottom of the lamina k and the reference surface. The eq. (9) is well-known in the

CLPT and it is written using three stiffness as:

$$V_{\text{int}} = \frac{1}{2}\int_{\Lambda} \boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{A}\boldsymbol{\varepsilon} - 2\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{B}\boldsymbol{\kappa} + \boldsymbol{\kappa}^{\mathrm{T}}\mathbf{D}\boldsymbol{\kappa}\, \mathrm{d}\Lambda = \frac{1}{2}\int_{\Lambda} \boldsymbol{\chi}^{\mathrm{T}}\mathbf{T}\boldsymbol{\chi}\, \mathrm{d}\Lambda \tag{10}$$

with:

$$\mathbf{A} = \sum_{k=1}^{L} x_3|_{z_k}^{z_{k+1}}\mathbf{Q}^{(k)} \qquad \mathbf{B} = \frac{1}{2}\sum_{k=1}^{L} x_3^2|_{z_k}^{z_{k+1}}\mathbf{Q}^{(k)} \qquad \mathbf{D} = \frac{1}{3}\sum_{k=1}^{L} x_3^3|_{z_k}^{z_{k+1}}\mathbf{Q}^{(k)} \tag{11}$$

and:

$$\boldsymbol{\chi}^{\mathrm{T}} = \left\{u_{1,1} \quad u_{2,2} \quad u_{1,2}+u_{2,1} \quad u_{3,11} \quad u_{3,22} \quad 2u_{3,12}\right\} \qquad \mathbf{T} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{D} \end{bmatrix} \tag{12}$$

$\mathbf{A}$, $\mathbf{D}$ and $\mathbf{B}$ matrices are the extensional, bending and bending-extensional coupling stiffness (Zhang & Kim, 2004). The vector $\boldsymbol{\chi}$ can be described by a differential operator and a displacement vector as:

$$\boldsymbol{\chi} = \boldsymbol{\partial}\mathbf{u} \tag{13}$$

where:

$$\boldsymbol{\partial}^{\mathbf{T}} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & \frac{\partial}{\partial x_2} & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_2^2} & 2\frac{\partial^2}{\partial x_1 \partial x_2} \end{bmatrix} \qquad \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{14}$$

The External Potential, V_{ext} is determined as:

$$V_{\text{ext}} = -\int_{\Lambda} \mathbf{q}^{\mathrm{T}}\mathbf{u}\, \mathrm{d}\Lambda \tag{15}$$

where $\mathbf{q}$ describes the applied loading over the mid-surface with respect to $\hat{e}_i$ direction.

3 *pb-2* RAYLEIGH-RITZ METHOD

The Rayleigh-Ritz Method approximate the solutions of the differential equations by weighted kinematically admissible functions. This condition is imposed to the basis by the method's modification, namely *pb-2*, based on the multiplication of the set by the Bhat's beam functions (Bhat, 1985).

The degrees of freedom on the right-site of eq. (1) are approximate by (McGee et al., 1996):

$$u_i = g^{(i)}\boldsymbol{\phi}^{\mathrm{T}}\mathbf{c}^{(i)} \tag{16}$$

where i does not sum in, $g^{(i)}$ correspond to the Bhat's beam functions multiplication with respect to each degree of freedom, $\boldsymbol{\phi}$ is the approximation basis (which are equal to all degrees of freedom) and $\mathbf{c}^{(i)}$ is the weighting constants vector. In a matrix notation, the eq. (16) is:

$$\mathbf{u} = \boldsymbol{\Phi}\boldsymbol{\lambda} \tag{17}$$

where:

$$\boldsymbol{\Phi} = \begin{bmatrix} g^{(1)}\boldsymbol{\phi}^{\mathrm{T}} & 0 & 0 \\ 0 & g^{(2)}\boldsymbol{\phi}^{\mathrm{T}} & 0 \\ 0 & 0 & g^{(3)}\boldsymbol{\phi}^{\mathrm{T}} \end{bmatrix}_{3\times 3n} \qquad \boldsymbol{\lambda} = \begin{Bmatrix} \mathbf{c}^{(1)} \\ \mathbf{c}^{(2)} \\ \mathbf{c}^{(3)} \end{Bmatrix}_{3n} \tag{18}$$

which n denotes the span of each interpolation basis and $\boldsymbol{\lambda}$ groups the constants vectors for all degrees of freedom. The Bhat's beam functions are simply line equations which describe mathematically the edges of the plate. Being the plate's geometry rectangular, $g^{(i)}$ is written as:

$$g^{(i)} = x_2^{bc_1}(x_1 - a)^{bc_2}(x_2 - b)^{bc_3}x_1^{bc_4} \tag{19}$$

where bc_i takes integer values and denotes the boundary condition type: free edge $bc_i = 0$, simply-supported $bc_i = 1$ and clamped $bc_i = 2$ (only for the transverse displacement degree of freedom). As a result of using *pb-2* modification, the boundary conditions imposing is straightforward and it enables the consideration of any set of conditions. Furthermore, in order to ease the quadrature process, the family of the approximation functions is polynomial.

The strain vector $\boldsymbol{\chi}$ (eq. (12)) is rewritten, in terms of $\boldsymbol{\Phi}$ and λ, as:

$$\boldsymbol{\chi} = \mathbf{N}\boldsymbol{\lambda} \qquad \mathbf{N} = \boldsymbol{\partial}\boldsymbol{\Phi} \tag{20}$$

and, with eq. (20), the eq. (10) is rewritten as:

$$V_{\text{int}} = \frac{1}{2}\boldsymbol{\lambda}^{\mathrm{T}} \int_{\Lambda} \mathbf{N}^{\mathrm{T}}\mathbf{T}\mathbf{N}\, \mathrm{d}\Lambda\ \boldsymbol{\lambda} \tag{21}$$

In order to simplify the quadratures, one can write:

$$\int_{\Lambda} \mathbf{N}^{\mathrm{T}}\ \mathbf{T}\ \mathbf{N}\ \mathrm{d}\Lambda = \mathbf{T} : \mathbb{M} \qquad (\mathbf{T} : \mathbb{M})_{IJ} = T_{KS}M_{KSIJ} \tag{22}$$

where $[I, J] = \{1, 2, \ldots, 3n\}$, $[K, S] = \{1, 2, 6\}$ and $\mathbb{M}$ is a fourth-order tensor determined as:

$$\mathbb{M} = \int_{\Lambda} \mathbf{N}^{\mathrm{T}} \otimes \mathbf{N}\mathrm{d}\Lambda \tag{23}$$

which $\otimes$ denotes the dyadic product. As a result:

$$V_{\text{int}} = \frac{1}{2}\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{T} : \mathbb{M}\ \boldsymbol{\lambda} \tag{24}$$

where : corresponds to the double dot product. The external potential is given by:

$$V_{\text{ext}} = \int_{\Lambda} \mathbf{q}^{\mathrm{T}}\mathbf{u}\ \mathrm{d}\Lambda = \int_{\Lambda} \mathbf{q}^{\mathrm{T}}\boldsymbol{\Phi}\ \mathrm{d}\Lambda\ \boldsymbol{\lambda} \tag{25}$$

The stationary point is determined by the minimization of the Total Energy Potential. One finds:

$$\Pi = V_{\text{int}} + V_{\text{ext}} \qquad \delta\Pi = 0 \quad \therefore \quad \frac{\partial V_{\text{int}}}{\partial \boldsymbol{\lambda}} + \frac{\partial V_{\text{ext}}}{\partial \boldsymbol{\lambda}} = \mathbf{0} \tag{26}$$

where:

$$\mathbf{P}\boldsymbol{\lambda} = \frac{\partial V_{\text{int}}}{\partial \boldsymbol{\lambda}} = \mathbf{T} : \mathbb{M}\, \boldsymbol{\lambda} \qquad \mathbf{F} = \frac{\partial V_{\text{ext}}}{\partial \boldsymbol{\lambda}} = \int_{\Lambda} \boldsymbol{\Phi}^{\mathrm{T}} \mathbf{q}\, \mathrm{d}\Lambda \tag{27}$$

which results in the linear system:

$$\mathbf{P}\boldsymbol{\lambda} = \mathbf{F} \tag{28}$$

The matrix $\mathbf{T}$ is constructed by the stiffness matrix $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$. Thus, $\mathbf{P}$ represents a multi-laminar plate, having *a priori* a anisotropic behaviour. An additive decompostion (Browayes & Chevrot, 2004) on the reduced constitutive tensor (eq. 5) decomposes matrix $\mathbf{P}$ into an isotropic and anisotropic terms, in order to ease the obtaining of the anisotropic response.

4 ADOMIAN DECOMPOSITION METHOD

Adomian Decomposition Method (Adomian, 1994) has been devised in order to solve non-linear differential equations by the purpose of a recursive procedure, which is based on three points: (a) the differential operator decomposition, (b) a solution written as an infinite expansion and (c) the determination of each expanded term by a recursive procedure. When applied to eq. (26), it results in several linear systems, which requires to be solved in order to construct the expanded solution. Some steps, hewever, are necessary so as to arrange the equations for its employment.

The decomposition of the differential operator follows an additive decomposition of the reduced consitutive tensor (Browayes & Chevrot, 2004). Basically, an anisotropic tensor is projeted over an isotropic symmetry, resulting in:

$$\mathbf{C} = \mathbf{C}^{\text{iso}} + \mathbf{C}^{\text{ani}} \tag{29}$$

Furthermore, the eq. (4) modifies the construction of matrix $\mathbf{Q}^{(k)}$ as:

$$\mathbf{Q}^{(k)} = \mathbf{R}^{(k)^{-1}} \left(\mathbf{C}^{\text{iso}} + \mathbf{C}^{\text{ani}}\right) \mathbf{R}^{(k)} = \mathbf{C}^{\text{iso}} + \mathbf{R}^{(k)^{-1}} \mathbf{C}^{\text{ani}} \mathbf{R}^{(k)} = \mathbf{Q}^{\text{iso}^{(k)}} + \mathbf{Q}^{\text{ani}^{(k)}} \tag{30}$$

noting the invariability of isotropic materials with respect to transformations. One should note that the decomposition is applied in the lamina's level. In the same fashion and using eq. (30), matrix $\mathbf{T}$ (eq. (12)) is decomposed:

$$\mathbf{A} = \mathbf{A}^{\text{iso}} + \mathbf{A}^{\text{ani}} \quad \mathbf{B} = \mathbf{B}^{\text{iso}} + \mathbf{B}^{\text{ani}} \quad \mathbf{D} = \mathbf{D}^{\text{iso}} + \mathbf{D}^{\text{ani}} \quad \therefore \quad \mathbf{T} = \mathbf{T}^{\text{iso}} + \mathbf{T}^{\text{ani}} \tag{31}$$

$\mathbf{T}$ is determined in the plate's level. $\mathbf{T}^{\text{iso}}$ is exactly the same as an isotropic plate (without laminae) with decoupled extensional and bending stiffness. With the eq. (27) and eq. (31), the first point of ADM can be presented:

$$\mathbf{P} = \mathbf{T} : \mathbb{M} = \left(\mathbf{T}^{\text{iso}} + \mathbf{T}^{\text{ani}}\right) : \mathbb{M} = \mathbf{T}^{\text{iso}} : \mathbb{M} + \mathbf{T}^{\text{ani}} : \mathbb{M} = \mathbf{P}^{\text{iso}} + \mathbf{P}^{\text{ani}} \tag{32}$$

Basically, the matrix operator is decomposed into two parts, one relative to the isotropic symmetry and another with the remainder parts of the anisotropic tensor.

For the second one is given by:

$$u_3\left(x_1,x_2\right) = u_3^{(0)}\left(x_1,x_2\right) + u_3^{(1)}\left(x_1,x_2\right) + u_3^{(2)}\left(x_1,x_2\right) + \ldots + u_3^{(k)}\left(x_1,x_2\right) + \ldots \quad (33)$$

and with the eq. (16) and considering that all expanded terms use the same basis, it results in:

$$\boldsymbol{\lambda}^{\mathrm{T}} = \boldsymbol{\lambda}_{(0)}^{\mathrm{T}} + \boldsymbol{\lambda}_{(1)}^{\mathrm{T}} + \boldsymbol{\lambda}_{(2)}^{\mathrm{T}} + \ldots + \boldsymbol{\lambda}_{(k)}^{\mathrm{T}} + \ldots \quad (34)$$

The third point is the recursive procedure, given by:

$$\begin{aligned}
\boldsymbol{\lambda}_{(0)} &= \mathbf{B}^{\mathrm{iso}^{-1}}\mathbf{W} \\
\boldsymbol{\lambda}_{(1)} &= -\mathbf{B}^{\mathrm{iso}^{-1}}\mathbf{B}^{\mathrm{ani}}\,\boldsymbol{\lambda}_{(0)} \\
\boldsymbol{\lambda}_{(2)} &= -\mathbf{B}^{\mathrm{iso}^{-1}}\mathbf{B}^{\mathrm{ani}}\,\boldsymbol{\lambda}_{(1)} \\
&\vdots \\
\boldsymbol{\lambda}_{(m)} &= -\mathbf{B}^{\mathrm{iso}^{-1}}\mathbf{B}^{\mathrm{ani}}\,\boldsymbol{\lambda}_{(m-1)}
\end{aligned} \quad (35)$$

An interesting point of the eq. (35) is that the first solution concerns to the isotropic solution of the problem, which is readily available. Recursively, the procedure enhances the solution if the laminae influences.

The recursive procedure stops when a tolerance is reached. The error between two sequential approaches can be determined by:

$$\begin{aligned}
e_x^{(k)} &= \left[\boldsymbol{c}_{(k)}^{(1)^{\mathrm{T}}}\left(\int_\Lambda g^{(1)}\,\boldsymbol{\phi}\otimes\boldsymbol{\phi}\,g^{(1)}d\,\Lambda\right)\boldsymbol{c}_{(k)}^{(1)}\right]^{1/2} \\
e_y^{(k)} &= \left[\boldsymbol{c}_{(k)}^{(2)^{\mathrm{T}}}\left(\int_\Lambda g^{(2)}\,\boldsymbol{\phi}\otimes\boldsymbol{\phi}\,g^{(2)}d\,\Lambda\right)\boldsymbol{c}_{(k)}^{(2)}\right]^{1/2} \\
e_z^{(k)} &= \left[\boldsymbol{c}_{(k)}^{(3)^{\mathrm{T}}}\left(\int_\Lambda g^{(3)}\boldsymbol{\phi}\,\otimes\boldsymbol{\phi}\,g^{(3)}d\,\Lambda\right)\boldsymbol{c}_{(k)}^{(3)}\right]^{1/2}
\end{aligned} \quad (36)$$

where $\boldsymbol{c}_{(k)}^{(I)}$ refers to the actual constants set. The quadratures can be produced in pre-processor routine, while the products are evalued in each step.

5 NUMERICAL RESULTS

Some numerical results are produced in order to compare the obtained solutions with some available in the literature. The constitutive properties used in the numerical analysis are (Bhaskar & Kaushik, 2004, Bhaskar & Dhaoya, 2009, Ungbhakorn & Wattanasakulpong, 2006):

$$E_{11} = 25E_{22} \qquad \nu_{12} = 0.3 \qquad G_{12} = 0.5E_{22} \quad (37)$$

The results are presented in non-dimensional parameters as:

$$\hat{w} = \frac{w(a/2,b/2)E_{22}}{q_3a^4} \qquad \hat{M}_x^{(I)} = \frac{M_x(x_I,y_I)}{q_3a^2} \qquad \hat{M}_y^{(I)} = \frac{M_y(x_I,y_I)}{q_3a^2} \quad (38)$$

where I corresponds to the assessing position: $N=(a/2,b)$, $W=(0,b/2)$ or $C=(a/2,b/2)$, denoting North, West and Centre, respectively. Three sets of analysis were produced so as to fully test the presented methodology:

n	$\hat{w}$	$\hat{M}_y^{(N)}$	$\hat{M}_x^{(W)}$	$\hat{M}_x^{(C)}$	$\hat{M}_y^{(C)}$
6	0.0066609983	-0.00041541048	0.00016644796	0.12998022	0.0084897971
8	0.0066599885	-0.00021406058	0.00049449374	0.12982307	0.0084583114
10	0.0066601812	0.00002267297	-0.00018349308	0.12985910	0.0084690416
12	0.0066601241	0.00006844937	-0.00009240182	0.12984045	0.0084657734
14	0.0066601499	0.00001459438	0.00008161516	0.12984753	0.0084679967
⋆	0.006660	-	-	0.1298	0.008467
†	0.0066607732	-	-	0.12980955	0.0084689597

⋆ Bhaskar & Kaushik (2004)
† FEM (Ansys® Inc., 2016)

Table 1: SSSS Square laminated plates [0°/90°/0°] under uniform loading

n	$\hat{w}$	$M_y^{(N)}$	$M_x^{(W)}$	$M_x^{(C)}$	$M_y^{(C)}$
6	0.0013705555	-0.015705369	-0.088328621	0.044358238	0.0018178576
8	0.0013710241	-0.015580564	-0.088276433	0.044481147	0.0018417416
10	0.0013709210	-0.015648329	-0.088009855	0.044440178	0.0018356330
12	0.0013709571	-0.015691986	-0.088101850	0.044457825	0.0018384285
14	0.0013709443	-0.015686079	-0.088139820	0.044450349	0.0018371945
⋆	0.001371	-0.01569	-0.08809	0.04441	0.001838
†	0.0013712142	-0.015074004	-0.086476484	0.044448498	0.018381428

⋆ Bhaskar & Kaushik (2004)
† FEM (Ansys® Inc., 2016)

Table 2: CCCC Square laminated plates [0°/90°/0°] under uniform loading

1. SSSS and CCCC square symmetric plates with stacking configuration as [0°/90°/0°];
2. CCCC square unsymmetric plates with aleatory stacking configuration;
3. Rectangular symmetric plates with different sets of boundary conditions;

With this analysis, one can verify several aspects of the methodology. In the first item, a convergence study is provided and not only the displacements were consider but also the bending moments in three different positions. They are compared with the results of Bhaskar & Kaushik, 2004 and values obtained by the application of the Finite Element Method (FEM). For the second item, the convergence of the centre displacement are compared with the results of Bhaskar & Dhaoya, 2009 as well as results obtained by the FEM. In the third item, only the converged displacement are shown and compared with the results of reference (Ungbhakorn & Wattanasakulpong, 2006). This last analysis is made in order to verify the analysis of rectangu-

n	[0°/45°/-45°/90°]	[45°/0°/-45°/90°]	[45°/0°/90°/-45°]	[45°/-45°/0°/90°]
6	0.003420152	0.002925338	0.003534480	0.001783169
8	0.003421313	0.002930172	0.003536071	0.001785461
10	0.003421655	0.002930143	0.003536633	0.001784694
12	0.003421617	0.002930522	0.003536075	0.001785137
14	0.003421648	0.002930470	0.003536621	0.001784942
⋆	0.003422	0.002926	0.003536	0.001776
†	0.003426	0.002932	0.003543	0.001787

⋆ Bhaskar, K., & Dhaoya, J. (2009)
† FEM (Ansys® Inc., 2016)

Table 3: Centre displacement of CCCC square laminated plates with aleatory fibre orientation under uniform loading

lar plates with different sets of boundary condition. For all cases, the tolerance was set to 10^{-9} (eq. (36)). The recusive system stops when all tolerances of eq. (36) reaches this value.

In order to obtain the numeric results by FEM, the commercial software Ansys Multiphysics (Ansys® Inc., 2016) is used. A mesh with 40×40 quadratic shell elements (SHELL281) was used. A thickness of 0.001 [m] was considered in all analysis (due to the thick plate formulation of the element).

5.1 Symmetric [0°/90°/0°] square plates

The first analysed cases were square symmetric plates, with the stacking configuration described as [0°/90°/0°]. The convergence is studied by enhancing the span, i. e., increasing n. Tables 1 and 2 present the results obtained by the methodology and compare them with reference (Bhaskar & Kaushik, 2004) and values acquired by FEM (Ansys® Inc., 2016). One can observe that even for small values of n, the displacement was very close to the compared solutions. Naturally, the bending moments need a richer basis given the fact that they are obtained by the second derivative of the displacement. However, even with relatively small values of n, the agreement between the obtained and the reference results were good. In Table 1, $\hat{M}_y^{(N)}$ and $\hat{M}_x^{(W)}$ can be consider as the problem's natural boundary conditions. Given the variational nature of *pb-2* RRM, by the basis enrichment, these requirements are naturally satisfied. In the case of symmetric stacking, there is no coupling between the the membrane and bending effects.

5.2 Square clamped plates with aleatory stacking

For the aleatory stacking, the analysis took into account the coupling between the membrane and bending effects. As one can note, the convergence of the centre displacement was very good for all results, with both references (Bhaskar & Dhaoya, 2009 and Ansys® Inc., 2016). This shows that the recursive procedure can be used in both symmetric and unsymmetric laminated plate solutions.

b/a		0.5	1.0	1.5	2.0	2.5	3.0
SSSS	⋆	0.00203	0.00679	0.00767	0.00746	0.00720	0.00710
$(a/2, b/2)$	Present	0.0020341	0.0067966	0.0076783	0.007466	0.007205	0.007090
SCSS	⋆	0.00148	0.00303	0.00301	0.00287	0.00281	0.00281
$(a/2, b/2)$	Present	0.0014844	0.0030524	0.003031	0.002880	0.002829	0.002827
SCSC	⋆	0.00107	0.00156	0.00146	0.00141	0.00141	0.00142
$(a/2, b/2)$	Present	0.0010581	0.0015657	0.001464	0.001415	0.001413	0.001416
SSSF	⋆	0.00276	0.04879	0.21836	0.57838	1.16589	1.99063
$(0, b/2)$	Present	0.0027634	0.0048855	0.21828	0.57889	1.1668	1.9922
SCSF	⋆	0.00284	0.03050	0.05926	0.07078	0.07341	0.07261
$(0, b/2)$	Present	0.0028448	0.030578	0.059454	0.071045	0.073703	0.07290
CCCC	⋆	0.00044	0.00143	0.00149	0.00143	0.00141	0.00141
$(a/2, b/2)$	Present	0.00044101	0.0014365	0.0014928	0.001429	0.001414	0.001415
CCCS	⋆	0.00049	0.00247	0.00300	0.00293	0.00285	0.00283
$(a/2, b/2)$	Present	0.00048778	0.0024735	0.003008	0.002931	0.002849	0.002828
CCCF	⋆	0.00050	0.00926	0.03264	0.05490	0.06674	0.07094
$(0, b/2)$	Present	0.0005050	0.009298	0.032746	0.05508	0.06698	0.07122

⋆ Ungbhakorn, V., & Wattanasakulpong, N. (2006)

Table 4: Transverse displacement of rectangular symmetric laminated plates [0°/90°]$_s$ with several sets of boundary conditions

5.3 Rectangular plates with different set of boundary conditions

These examples are taken into account in order to determine the the possibility of applying the methodology in non-symmetric boundary conditions and to rectangular plates. 48 cases were analysed, all of them with excellent agreement with reference (Ungbhakorn & Wattanasakulpong, 2006), even for cases with free edge. The obtained results, in Table 4, are produced with $n = 12$ and the positions of assessment – (x_1, x_2) – are described below the boundary conditions set.

5.4 ADM convergence

Tables 1 to 3 show the convergence of the *pb-2* Rayleigh-Ritz Method. The recursive procedure of eq. (35)must converge as well. Figure 1 shows the method's convergence for a SSSS square plate with $n = 8$. Most of the analysis have the same pattern, which changes sightly for other boundary conditions set and material properties. The large number of iterations are due to the small tolerance considered in all analysis.

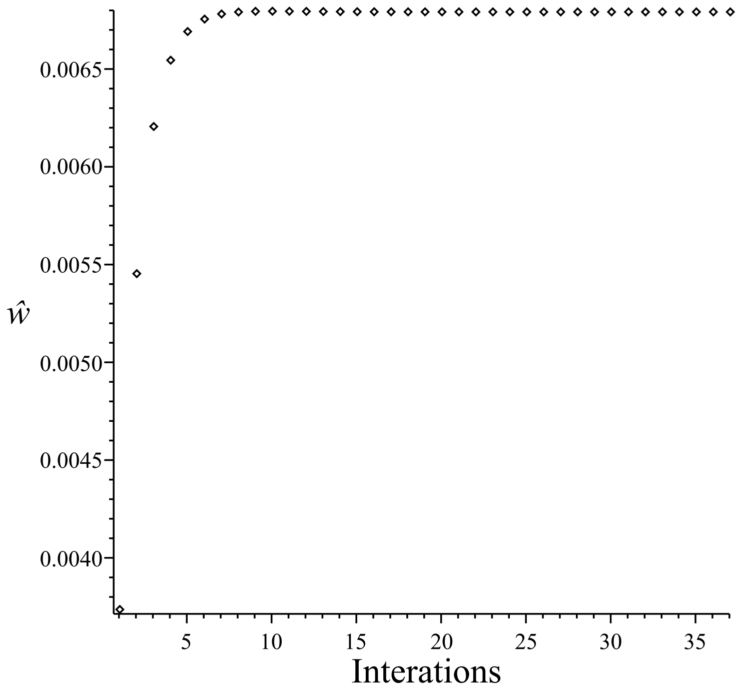

Figure 1: Convergence of ADM (SSSS square plate $n = 8$)

6 CONCLUSIONS

Adomian Decomposition Method was applied to determine the mechanical behaviour of laminated thin plates. The operator decomposition has produced with respect to an additive decomposition of the constitutive tensor, which extracts an isotropic part form an anisotropic tensor. This decomposition was operated in the lamina level and expanded to the plate level. The resultant was a superposition of the original plate stiffness into an isotropic and remainder stiffness. The plate displacement was expanded into infinite terms and they were recursivelly derived. *pb-2* Rayleigh-Ritz Method was employed in order to approximate the solution space. When applied to the recursive procedure, the *pb-2* RRM is expanded to several linear systems, in which each solution is part of the solution. The first step of the recursive system was the solution of an isotropic plate, with the same geometry and boundary conditions and readily available for most problems. From the next step until the convergence was reached, the anisotropic influence was inserted progressively into the solution.

As one can observe, the solutions of the methodology solutions excellent agreement with solutions found in the literature. The comparisons encompass symmetric and unsymmetric stacking, several sets of boundary conditions and aspect ratio of the plate. One of the most important properties of ADM is the Adomian Polynomials (Adomian, 1994), which approximate non-linearities of the differential equations by fast convergent polynomials. This is a subject to be address in a future work.

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