CASE STUDY

The Transposition Project: origins, context and early findings

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Abstract

Within the Department of Mathematics at Cork Institute of Technology and at a wider level across the institute, the topic of transposition or rearranging equations has repeatedly been flagged as one of the most problematic. The transposition project aims to understand the reasons for students' difficulties and develop an effective intervention strategy.

Keywords: Transposition, algebra, rearranging equations, changing the subject.

1. Introduction

Transposition of formulae (also known as rearranging equations) is a skill vital in many industrial fields. It is hard to imagine a domain of science or engineering where a professional would not need to transpose a formula at some stage. The importance of the topic of transposition is reflected in Mathematics curricula at both second and third level throughout the world. In Ireland, the topic is covered at both the Junior and Leaving Certificate Examination level (the university matriculation examination) and yet it is apparent to lecturers at Cork Institute of Technology (CIT) that a large proportion of students that they encounter lack this key skill. CIT is a higher education college in the Republic of Ireland. The entry requirements at Institutes of Technology are often lower than those at the Universities and students attending CIT sometimes find the Mathematics content of their programmes challenging. The Mathematics Department at CIT is mainly a service department teaching on most programmes across the Faculties of Engineering and Science, and Business and Humanities. Approximately 75% of students within these Faculties take a mathematics module many of which have transposition of formulae on the syllabus. Remarkably, the mathematical deficiency seems to persist after the topic has been taught twice, in school and again at undergraduate level.

The difficulties surrounding the topic of transposition are frequently observed and discussed by lecturers from the Department of Mathematics, anecdotally in the staff room and more formally at staff meetings, seminars and programmatic reviews. Additionally, staff involved in the Mathematics Support Centre noted that many of the difficulties that students encounter in a wide variety of mathematical problems arise from a lack of proficiency in rearranging equations. A desire to understand and address this problem brought together twelve members of staff from the Department of Mathematics and over a series of meetings. As a result, the Transposition Project was formed. The main objectives of the project are summarised in Figure 1. This paper focuses on objectives 1 – 4.

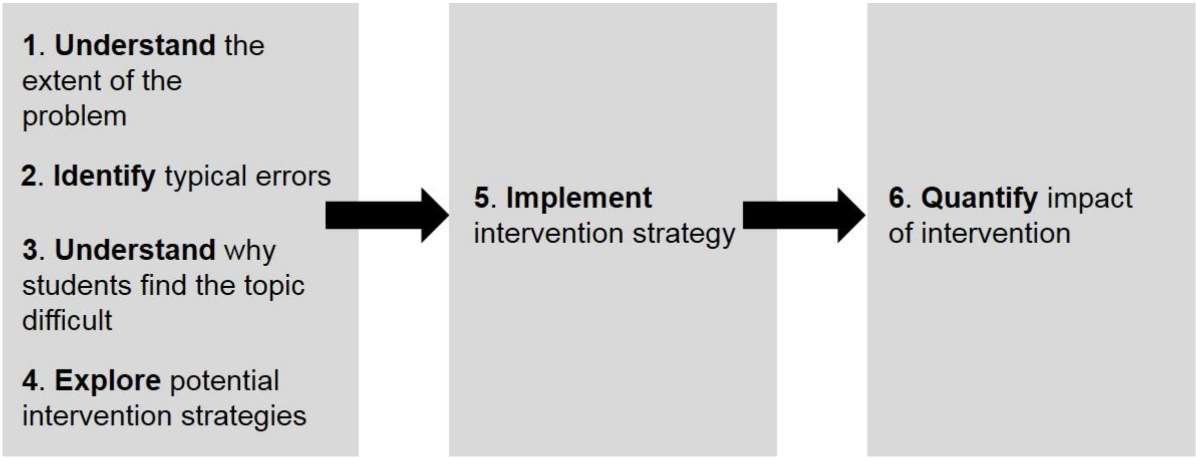


Figure 1: Summary of project objectives.

2. The extent of the problem

To understand the extent to which deficiency in the topic of transposition affects first year students at CIT, an online diagnostic test was given to over 350 students across the faculties of Science and Engineering and Business and Humanities. The test was designed to examine various algebra skills necessary in transposition as well as proficiency with transposition itself. A benchmark equation used to determine whether a student is proficient in the topic was as follows:

$$T = \frac{2v}{g} + 5$$

where the students were asked to isolate *g*. Alarmingly, over 75% of respondents were not able to obtain the correct answer.

In order to further assess the depth of the problem within the institute, two workshops for lecturing staff were held; one for Mathematics lectures and one for lecturers from applied disciplines (not Mathematics). Staff discussed the importance, context, teaching approach and understanding of the topic of transposition as well as potential interventions to improve proficiency. At the workshop for staff from applied disciplines, the discussion centred on the questions: '*Do students struggle with rearranging equations in applied courses*?' and '*How do these problems affect teaching applied topics*?' All participants agreed that students did struggle with the topic of transposition and this affected all the applied modules taught. Surprisingly, it was reported by workshop participants that in some modules the topic of transposition was so important to the understanding of applied concepts that it was taught (and assessed) again within the module, in the context of the subject material. Having to teach the topic again within a module reduces the amount of time available to cover the main (applied) material. Staff also reported that the assessment of key concepts related to their disciplines is confounded with the topic of transposition making it unclear whether a student is struggling due to lack of understanding of the concepts at hand or struggling to rearrange an equation.

CIT students are not unique in their aptitude for transposition of formulae. Our observations and subsequent interest in this echo efforts of mathematics teachers and educational researchers around the world. Multiple published works examining students' difficulties in algebra find that student' mathematical deficiencies in general and those pertinent to algebra are often multifaceted where, for instance, difficulties with basic algebra perpetuate into harder tasks like transposition. Most papers studying mathematical deficiencies invariably touch upon the topic of solving equations or transposition of formulae though various authors focus their work on diverse aspects of the problem such as poor algebra skills, reasons behind it, developing diagnostic tools, and intervention techniques. O'Brien and Ní Ríordáin (2017) and Lucariello, Tine, and Ganley (2014) developed tests for diagnosing students' difficulties in algebra. Several publications investigate students' misconceptions in algebra and agree on the fact that the misconceptions form barriers to further learning and need to be addressed and dispelled in class (Lucariello, Tine, & Ganley, 2014; Bush & Karp, 2013; Barbieri & Booth, 2016). The research suggests that, while necessary, focusing an intervention solely on cognition or motivation may not lead to improvement in algebra skills whereas a combined intervention approach that includes error reflection may be beneficial (Barbieri & Booth, 2016).

3. Typical errors

To understand the difficulties associated with the topic of transposition, an error analysis was performed on students' work. The deficiencies identified through the error analysis included: a lack of understanding of the equality sign and equivalent equations, incorrect use of inverse operations, mistakes expanding or simplifying equations, and inability to start a question. The types of errors identified by the error analysis are shown in Table 1, along with examples of students' work.

While all presented classes of deficiencies are major contributors to students' struggles with re-arranging equations, it is the lack of conceptual understanding that plays a major role here as without the deep understanding of what equations are, how they are formed and what the equality sign means other algebra skills become unproductive in the context of transposition. All non-conceptual deficiencies observed are algebraic in nature and constitute the basics necessary, but not sufficient, for manipulating equations and formulae.

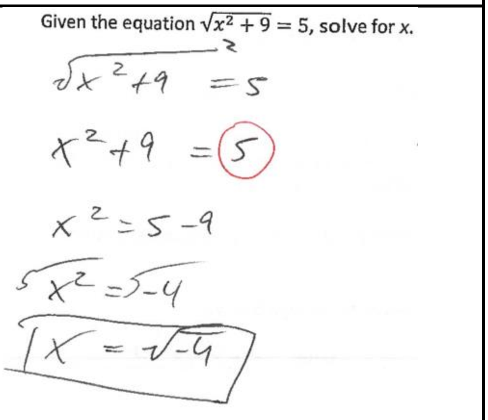
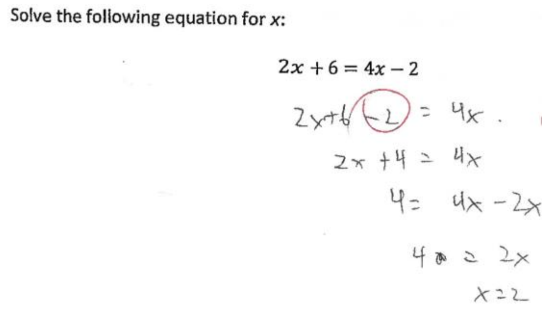
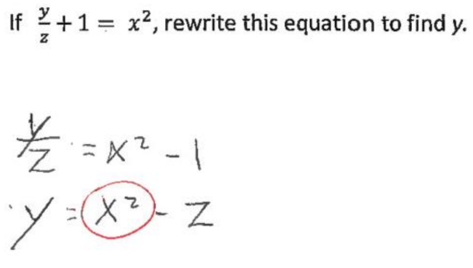
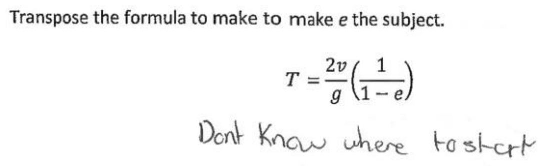
Bush and Karp (2013), Stephens, et al. (2013) and Byrd, McNeil, Chesney, and Matthews (2015) concur the fact that lack of understanding of the equality sign and equivalent equations (i.e. concepts) is at the core of students' problems with solving equations. It is also worth noting that O'Connor and Norton (2016) while examining mathematical difficulties with quadratic equations analyse student error patterns and relationships between them. The findings suggest a strong connection between lack of procedural skills and misconceptions, and lack of conceptual understanding. Just as the students in the study by O'Connor and Norton did not have the tools to factorise quadratics, many of our students did not have the pre-requisite algebra skills to apply to transposition problems. Indeed, conceptual errors and procedural errors are often intertwined, Rittle-Johnson, Siegler, and Alibali, (2001) put forward the idea of an iterative model of the development of conceptual and procedural knowledge which we think may be especially relevant to the process of learning to rearrange equations. This means that while the emphasis on teaching for conceptual understanding is of major importance, procedural deficiencies must be addressed concurrently. These ideas feed into the intervention strategy currently under development.

4. Why is transposition difficult?

At the workshop for staff from the Department of Mathematics and over a series of project meetings, lecturers discussed the following question: *Why is the topic so difficult for students?* We also looked at students that are proficient in transposition.

A number of reasons were identified which are summarised in Figure 2. Crucially on the technical side, transposition of formulae is a demanding task as it is a culmination of many algebra skills and concepts coming into play at once and lacking one component of this 'portfolio' often results in a failure. On the non-technical side, there are several factors contributing to the problem. Staff observed that many students like to have a procedure involving a defined set of steps to follow also noted by Marjoram, et al. (2012); this is not the case for transposition. Though there are some 'rules', they can be applied in multiple ways generating different paths to the correct solution. A question on transposition often involves many steps, students do not like long problems and perceive them as "hard"! There is also the fact that the students have been exposed to the topic before therefore some already know it and are bored in class distracting others whereas some think they know it (when they do not) and are not paying attention. This does not create an optimum learning environment.

Table 1: Error classification

Common errors	Class of maths deficiency	Some examples
Lack of understanding of the equality sign.	Conceptual understanding	
Incorrect use of inverse operations, including mistakes with signs.	Insufficient knowledge of the 'rules' of transposition	
Incorrect use of distributive law, factorising, fraction arithmetic and simplifying algebraic expressions. Including mistakes with signs.	Prerequisite algebra skills	
Not knowing where to start.	Difficult to identify: possibly all of the deficiencies listed	

Our discussions with the lecturers from CIT and a survey of second and third level textbooks revealed an inconsistency in the technical language used which is not helpful and may be confusing to the students. Common terminology used to describe the manipulation of an equation includes: transpose entities, reverse operations, 'move' terms, 'cross-multiply' and 'bring across equals sign'. This poses natural questions. Does the terminology matter? Does diversity in terminology cause confusion with the students? Are some terms more correct or appropriate than the others? Should using the terms 'move' and 'bring across' be avoided as it gives students an idea of moving entities around in a random fashion?

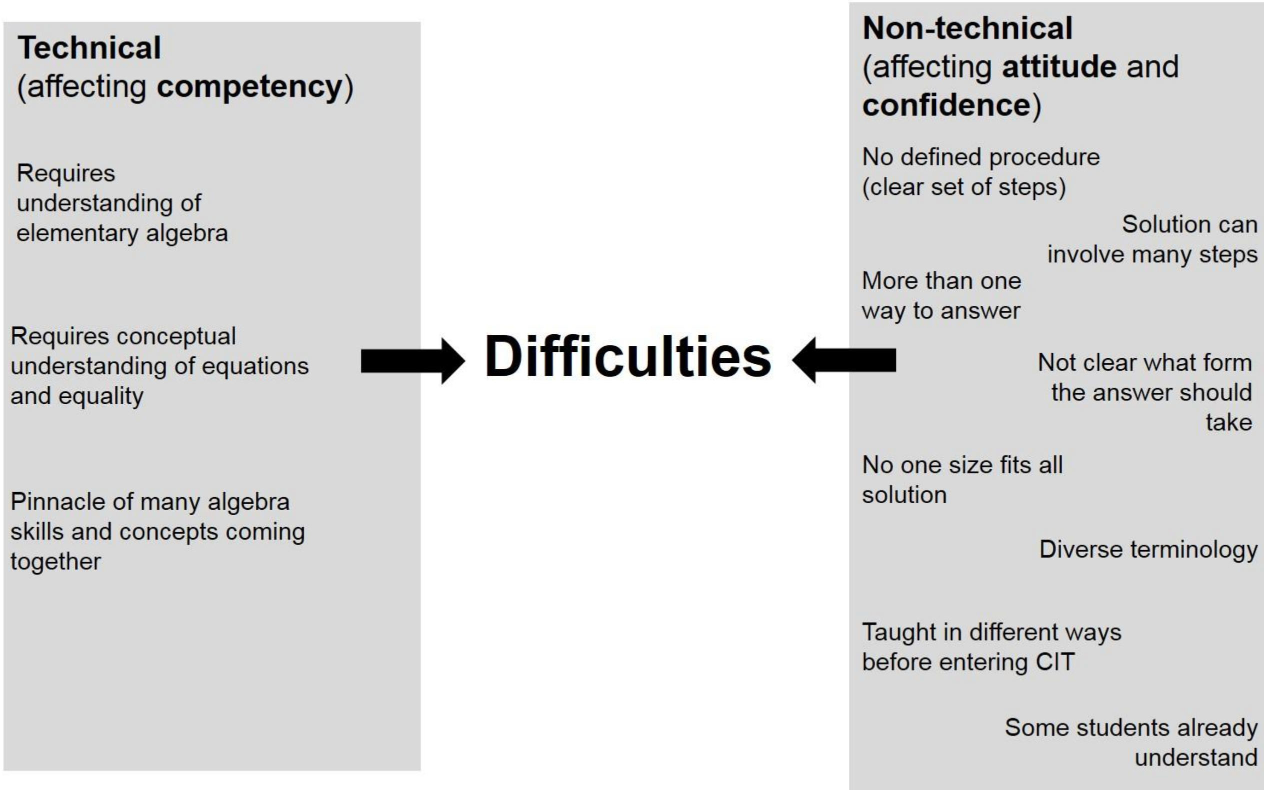


Figure 2: Summary of students' difficulties with transposition.

Identifying the processes used by good learners is a powerful resource for designing educational interventions (Rittle-Johnson, et al., 2001). Therefore, in our search of ways to improve teaching the topic of transposition we also wanted to determine what it is that a student proficient in transposition understands that sets him or her apart from others. A cohort of second year level 8 engineering students, who are found to be competent with transposition, were asked to write down key ideas that they would use to explain transposition to their peers. Though the language of replies varied, the most common and relevant feedback included (a) "apply opposite/inverse operations" and (b) "do the same to both sides of the equation". This is telling as the students showed understanding of the concept of transposition and stated exactly the two ideas that we will examine later as ideas *instrumental* to successful transposition.

5. Potential intervention strategy

Our early findings indicate that there are a number of technical and non-technical issues surrounding transposition that need to be considered in the development of an intervention strategy. On the technical side, the intervention must build up an understanding of the concepts of equations and equality, as well as address key algebraic deficiencies and misconceptions.

As educators, we always wonder if our students have developed a conceptual understanding. Conceptual understanding is knowing more than just isolated facts and methods. A successful student not only understands the ideas, but also has the ability to apply this knowledge to new contexts (Fosnot, 2018). In other words, conceptual knowledge is flexible and not tied to specific problems and is therefore generalizable (Rittle-Johnson, et al., 2001). Eric Mazur wondered about conceptual understanding too and proposed a peer instruction model to be used in conjunction with lectures (Mazur, 1997). Peer instruction is designed to engage students during class through activities that require each student to apply the core concepts being presented, and then to explain and discuss those concepts with their fellow students. Mazur's results show a significant increase in the percentage of students answering the concept question correctly. Mazur's successful findings have been replicated across disciplines at second and third level (Cummings and Roberts, 2008; Smith, Wood, Krauter, & Knight, 2011). Additionally, Smith, et al. (2009) found that combining peer discussion with instructor explanation increases student learning from in-class concept questions. The logic underlying the success of peer discussion based on concept questions is that it continuously engages students' minds and provides feedback to both students, and the lecturer, about the level of understanding. It also allows students to construct their own knowledge of the topic. There is a transformative effect of the shift from a lecturer-centred "transmissionist" environment to a more learner-centred constructivist classroom (Smith, et al., 2011). Engaging students in a learning activity in one class predisposes them to learn from a subsequent lecture.

If a teaching model based on peer-discussion is to be used in classes on transposition of formulae, what are the main ideas underlying transposition and what concept questions do the students need to be confronted with in order to foster a greater understanding? Physics, Biology, Engineering and other applied disciplines lend themselves well to concept based teaching and testing. It is not as straightforward to test conceptual understanding in Mathematics. There are several aspects to consider. What are the big mathematical ideas underlying a particular topic? What is the indication of one's understanding of such ideas? There are three aspects to such understanding: knowing which mathematical ideas are key and why they are important, knowing which ideas are useful in a particular context of problem solving and the ability to justify the approach to solving a problem. An ability to find an error in someone else's work and explain it is another hallmark of conceptual understanding. This prompts one to think about the right questions to ask when testing or discussing a concept: "why this works", "why can you do this", "how do you explain this", "justify your solution" and so on. When students have acquired conceptual understanding, they can give arguments to explain why some steps are consequences of the others.

What are the key ideas underlying transposition? The big key idea is that a*ny equation can be represented in an infinite number of different but* <u>*equivalent*</u> *ways.* Additionally, the following ideas are instrumental to successful transposition:

1. respect the equality i.e. do the same to both sides of the equation to obtain an equivalent equation, and;
2. simplify the equation/formula step by step by applying the inverse operations.

Note that the two ideas instrumental to transposition were exactly the ones that proficient students highlighted.

On par with fostering the correct conceptual understanding, it is vital to also dispel common misconceptions surrounding transposition of formula that persist in the student community. Therefore, the intervention strategy currently being developed within this project will focus on both building the right concepts while also highlighting and resolving misconceptions.

6. Conclusions and future work

The origins of students' deficiencies with transposition of formulae are complex, multifaceted and perpetuate far beyond a mathematics class. These difficulties have to be addressed in an efficient way to ensure graduates and future professionals are proficient at the technique. A peer discussion based teaching model may be a suitable format for cultivating better conceptual understanding in class.

Future work will mainly focus in two directions:

- development of an intervention strategy for teaching transposition, and;
- development of a diagnostic test that assesses students' conceptual understanding on the topic and measures the effect of the intervention.

We intend to develop and implement an intervention strategy which will involve peer-discussion based teaching with an increased emphasis on conceptual understanding while also dispelling the students' misconceptions relevant to transposition of formulae. The diagnostic test and post-test analysis will aim to quantify the impact of peer instruction on student attainment in the topic of transposition in a first-year service mathematics course.

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