Nonlinear filtering and heterogeneous swarms of autonomous agents - exactly solvable model

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Abstract. We consider slightly heterogeneous swarm of agents controlled by one leader. We study the global dynamics by using a newly established connection binding multi-agents dynamics and nonlinear optimal state estimation (nonlinear filtering). For a whole nonlinear class of mutual interactions, we are able to exactly characterize the resulting swarm dynamics. Our leader-follower dynamics is interpretable as a feedback particle filtering problem similar to finite-dimensional, nonlinear filters originally proposed by V. E. Beneš. The state estimation problem can be explicitly solved as it merely a change of probability measure on an Ornstein-Uhlenbeck process. The agents interactions, driven by common observations of the randomly corrupted leaders position, correspond to the innovation kernel that underlies any Bayesian filter. Numerical results fully corroborate our theoretical findings and intuition.

Keywords: Heterogeneous swarm, Multi-agent dynamics, Leader-based model, Nonlinear filtering, Feedback Particle Filter, Exact Results, Numerical Simulations.

1 Introduction

Among the vast and steadily increasing literature devoted to the dynamics of large number of mutually interacting autonomous agents, analytically solvable models stylizing some aspects of reality are definitely welcome (Hongler *et al.*[1], Eftimie[2], Bellomo and Dogbe[3], Bertin *et al.*[4]). Despite specific features inherent to analytical approaches, these contributions enhance our understanding of the emergence of collective phenomena like synchronization, aggregation, pattern formation, behavioral phase transitions, fashion trend formation and many others. Most analytical studies focus on the dynamics of homogeneous swarms (i.e. involving identical agents). Either the agents local rules are given and the ultimate goal is to analytically derive the emerging collective patterns or inversely, given a collective behavior, the goal is to unveil the agents local

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rules and their interactions. Purely homogeneous swarms are however rather scarcely encountered in reality.

In this paper, we focus our attention on slightly heterogeneous populations in which one special agent (we call it the *leader*) is able to drive the whole swarm towards a desired objective (Wang and Han[5]). Several types of leaders can be distinguished depending on the ways they affect their fellows. Either the leader is external and hence is explicitly recognized by ordinary agents of the swarm (Couzin *et al.*[6], Aureli and Porfiri[7]), or it acts as a *shill* who appears ordinary to its fellows while in fact obeying to a hidden master (Dyer *et al.*[8], Gribovskiy *et al.*[9], Wang and Guo[10]). Besides very particular models (Sartoretti and Hongler[11,12]), there is generally little hope for an analytical investigation of the collective behavior of a shill- or leader-infiltrated (and hence heterogeneous) swarm. The objective of this paper is to unveil a class of dynamical models for which this can be achieved.

Our source of inspiration is taken from the realm of estimation problems. In noise filtering, one considers the evolution of a stochastically driven system $\mathcal{S}$, monitored by an observer $\mathcal{O}$, itself delivering noisy information. The filtering goal at time t is to construct the best possible estimation of the state $\mathcal{S}$ by processing information delivered only from $\mathcal{O}$ up to time t. The filtering process is achieved via *sequential Bayesian* steps. Specifically, one starts with a *prediction step* to estimate the relevant conditional probability density function (pdf) based on the $\mathcal{S}$-dynamics, then one updates this pdf based on the $\mathcal{O}$-dynamics. For linear $\mathcal{S}$-evolutions driven by White Gaussian Noise (WGN) and $\mathcal{O}$-measurements also corrupted by WGN, the filtering problem is completely solvable and its explicit solution is known as the *Kalman-Bucy filter*. Indeed, due to linearity and Gaussian driving noise, both the prediction and the updating steps conserve their Gaussian character. Therefore the underlying filtering problem remains finite-dimensional as all operations are expressible via means and covariances only. For nonlinear evolution, the Gaussian character is lost, thus generally leading to infinite-dimensional problems. Analytical treatments are then precluded and only numerical approaches are feasible. One numerical method is given by particle filters, specifically *feedback particles algorithms* (FPA) (Yang *et al.*[13]). These algorithms are directly based dynamics of randomly interacting particles and can therefore be identified with specific agents dynamics. The FPA prediction step is achieved by attributing the $\mathcal{S}$-dynamics to a homogeneous swarm of agents. The updating process, realized by mutual agents interactions will globally minimize, in real time, the Kullback-Leibler distance between the $\mathcal{S}$ pdf and the swarm empirical distribution. In this paper, we view the $\mathcal{S}$-dynamics as playing the role of a leader evolving among a homogeneous swarm of N ordinary agents. When $N \to \infty$, this dynamics is reducible to a *mean-field game* (Pequitoy *et al.*[14], Guéant *et al.*[15]) with here an infinitesimally short time horizon, (as only real time updating – excluding smoothing – is realized). The FPA offers therefore a natural framework to construct leader driven swarms of agents. As a natural consequence, solvable filtering problems, like the Kalman-Bucy case, provide directly solvable heterogeneous swarms dynamics. Here, our intention is to construct a class of multi-agents models which simultaneously keep the associ-

ated FPA finitely dimensional and yet escapes from the pure Gaussian world. The idea on which we base our construction, is to consider a class of "Girsanov-changes" of probability measures applied on Ornstein-Uhlenbeck dynamics (i.e. linear dynamics with Gaussian noise sources). Here one considers the class of Girsanov changes of measures studied in Taylor[16], Hongler[17], Dai Pra[18], Benes[19] and Daum[20].
We organize the paper as follows: in section 2, the explicit connection between the filtering problem and the driving of a swarm of agents infiltrated by a leader is exposed. In section 3, we introduce our specific example of non-Gaussian interacting agents, controlled by a leader and for which the associated FPA is analytically solvable. Section 4, we report numerical experiments to illustrate our analytical findings.

2 Multi-agent dynamics and Feedback Particle Filtering

Consider a swarm of N Brownian agents $\mathcal{A}_i$, $i = 1, 2, \cdots, N$, and one additional leader agent $\mathcal{A}$ with dynamics:

$$\begin{cases} dX_i(t) = f\left(X_i(t)\right) dt + \mathcal{K}\left(X_i(t), \boldsymbol{X}(t), dZ(t)\right) + \sigma dW_i(t), \\ \text{leaders dynamics} \begin{cases} dY(t) = f\left(Y(t)\right) dt + \sigma dW(t), \\ dZ(t) = h\, Y(t) dt + \sigma_o dW_y(t), \end{cases} \end{cases} \tag{1}$$

where $h > 0$ is a constant, $f : \mathbb{R} \to \mathbb{R}$, $dW_i(t)$, $dW(t)$ and $dW_y(t)$ are mutually independent WGN processes and the vector $\boldsymbol{X}(t) = (X_1(t), X_2(t), \cdots, X_N(t))$ describes the dynamic state of the N homogeneous agents. The *leader* agent $Y(t)$ affects the dynamics of the $X_i(t)$ via the interaction kernel $\mathcal{K}\left(X_i(t), \boldsymbol{X}(t), dZ(t)\right)$. We emphasize that the leaders dynamics $Y(t)$ itself is independent from the swarms state $\boldsymbol{X}(t)$. Agents are only able to observe the corrupted leaders position $Z(t)$, (the leader effectively hides its real position $Y(t)$ from the other fellows).
In Eq.(1), we focus our attention on the class of interactions kernels:

$$\mathcal{K}\left[X_i(t), \boldsymbol{X}(t), dZ(t)\right] = \nu\left(X_i(t), t\right) \otimes \left\{ dZ(t) - \underbrace{\frac{h}{2}\left[X_i(t) + \frac{1}{N}\sum_{k=1}^{N} X_k(t)\right]}_{\mathcal{G}[X_i(t), \boldsymbol{X}(t)]} dt \right\}, \tag{2}$$

where the coupling strength $\nu = \nu(X_i(t), t)$ is a positive convex function in $X_i(t)$ and where, due to the presence of multiplicative WGN processes, we define $\otimes$ to denote the Stratonovich interpretation of the underlying stochastic integrals (Jazwinski[21]). In Eq.(2), $\mathcal{G}\left[X_i(t), \boldsymbol{X}(t)\right]$ is a consensual position given by the average between agent $\mathcal{A}_i$'s position and the whole swarm barycenter. The interaction kernel relates the position increment $\mathcal{G}dt$ with the leader's unveiled position increment $dZ(t)$ and weights this stimulus with the coupling strength ν. The assumptions on ν imply that $\mathcal{K}\left[X_i(t), \boldsymbol{X}(t), dZ(t)\right]$ tends, in real time, to steadily reduce the distance between $\mathcal{G}\left[X_i(t), \boldsymbol{X}(t)\right] dt$ and $dZ(t)$. While the multiplicative factor $\nu(X_i(t), t)$ in Eq.(2) remains yet undetermined, its complete specification can be fixed by introducing a cost structure. In general, one requires that for some running cost functional $\mathcal{J}\left[\mathcal{K}, X_i(t), \boldsymbol{X}(t), \mathcal{Z}(t), t\right]$ and final cost $\Psi\left(X_i(T), \boldsymbol{X}(T), dZ(T)\right)$ at time horizon T, the interaction $\mathcal{K}$ is a minimizer of the associated optimization problem. Formally, the interaction kernel (and hence ν) would be the unique minimizer over a set $\mathbb{K}$ of admissible controls, namely:

$$\begin{aligned} &\mathcal{K}\left[X_i(t), \boldsymbol{X}(t), dZ(t)\right] \\ &\quad = \min_{K \in \mathbb{K}} \left\{ \left(\int_t^T \mathcal{J}\left[K, X_i(s), \boldsymbol{X}(s), dZ(s), s\right] ds \right) + \Psi\left(X_i(T), \boldsymbol{X}(T), dZ(T)\right) \right\}. \end{aligned} \tag{3}$$

The coupled set of Eqs.(1), (2) and (3) can be interpreted as a multi-players differential game (Bensoussan[22]). For large populations, one can use the empirical density $P^{(N)}(x,t)$ to construct the mean field posterior density $P(x, t \mid \mathcal{Z}(t), x_0)$:

$$P^{(N)}(x,t)dx = \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}\left\{X_n(t) \in [x, x+dx]\right\} \approx P(x, t \mid \mathcal{Z}(t), x_0)dx, \tag{4}$$

where the condition $\mathcal{Z}(t)$ stands for the information history of the process Z until time t and x_0 for the common initial location of the whole swarm. In the $N \to \infty$ limit, we have:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} X_k(t) = \int_{\mathbb{R}} x' P(x', t \mid \mathcal{Z}(t), x_0)dx' = \mathbb{E}\left\{X(t) \mid \mathcal{Z}(t)\right\}. \tag{5}$$

The Fokker-Planck equation which governs this mean field posterior density reads (with a self explaining abuse of notation for $\mathcal{K}$):

$$\begin{aligned} \tfrac{\partial}{\partial t} P(x, t \mid \mathcal{Z}(t), x_0) = &-\tfrac{\partial}{\partial x}\left\{\left[a(x) + \mathcal{K}\left(x, \mathbb{E}\left\{X(t) \mid \mathcal{Z}(t)\right\}\right)\right] P(x, t \mid \mathcal{Z}(t), x_0)\right\} \\ &+ \tfrac{\sigma^2}{2}\tfrac{\partial^2}{\partial x^2} P(x, t \mid \mathcal{Z}(t), x_0). \end{aligned} \tag{6}$$

Note that Eqs.(6) and (3) define in a forward/backward coupling a so called differential *mean-field game problem.*

Feedback particles filters. For vanishing forward time horizon (i.e. $T = t$) in Eq.(3), a simpler situation arises (the backward in time coupling becomes trivial) and the minimization is reduced to solving an Euler-Lagrange variational problem (ELP) for $\Psi\left(x, \mathbb{E}\left\{X(t) \mid \mathcal{Z}(t)\right\}\right)$. Choosing the objective criterion Ψ to be the Kullback-Leibler distance d_K:

$$\begin{cases} \Psi\left(x, \mathbb{E}\left\{X(t)\right\}, dZ(t)\right) := d_K\left\{P(x', t \mid \mathcal{Z}(t), x_0); Q(x, t \mid x_0)\right\}, \\ d_K\left\{P(x', t \mid \mathcal{Z}(t), x_0); Q(x, t \mid x_0)\right\} := \int_{\mathbb{R}} P(x', t \mid \mathcal{Z}(t), x_0)\left\{\ln\left[\frac{P(x', t \mid \mathcal{Z}(t), x_0)}{Q(x', t \mid x_0)}\right]\right\} dx', \end{cases} \tag{7}$$

with $Q(x, t \mid x_0)$ being the transition probability density of the diffusion process $Y(t)$ defined in Eq.(1) we find the ELP:

$$\begin{cases} -\frac{\partial}{\partial x}\left\{\frac{1}{P(x,t \mid \mathcal{Z}(t), x_0)}\frac{\partial}{\partial x}\left[P(x, t \mid \mathcal{Z}(t), x_0)\nu(x,t)\right]\right\} = \frac{h}{\sigma^2}, \\ \lim_{|x|\to\infty} P(x, t \mid \mathcal{Z}(t), x_0)\nu(x,t) = 0, \end{cases} \tag{8}$$

which leads to:

$$\begin{cases} \nu(x,t) = \frac{h}{\sigma^2 P(x,t \mid \mathcal{Z}(t), x_0)}\left\{\int_{-\infty}^{x}\left[\mathbb{E}\left\{X(t) \mid \mathcal{Z}(t)\right\} - x'\right] P(x', t \mid \mathcal{Z}(t), x_0) dx'\right\}, \\ \mathbb{E}\left\{X(t) \mid \mathcal{Z}(t)\right\} = \int_{-\infty}^{+\infty} x' P(x', t \mid \mathcal{Z}(t), x_0) dx'. \end{cases} \tag{9}$$

Eqs.(1), (2) together with $\nu(x,t)$ given in Eq.(9) produce a nonlinear continuous time feedback particle filter. This allows for a direct reinterpretation of the leader-based dynamics in terms of a stochastic filtering problem. A class of examples is detailed in the next section.
It is worthwhile noting that the leader influences the swarm through the variance σ (and the parameter h), and not only through its position. As σ grows, the agents uncertainties on the actual leader position increase. Consequently the coupling strength $\nu(x,t)$ decreases, the agents variances increase and the swarm tends to form a widespread group of agents around the leader. Alternatively, small values for σ will allow for very compact swarm formation.

3 Finite dimensional filtering with Weber parabolic functions

Let us now introduce a specific filtering problem which will be related to the control of the multi-agents dynamics. The nonlinear filtering problem is to estimate the value of the one-dimensional state $Y(t)$, at time t, given a set of measurements prior to t: $\mathcal{Z}(t) = \{Z(s) \mid 0 \leq s \leq t\}$. We will treat hereafter time-continuous measurements and assume that the leader state $Y(t)$ – starting at position y_0 – evolves according to the stochastic differential equation:

$$\begin{aligned} dY(t) &= \overbrace{\left\{\frac{d}{dy}\left[\log \mathcal{Y}_B(y)\right]\Big|_{y=Y(t)}\right\}}^{:= f_B[Y(t)]} dt + dW(t), \\ Y(0) &= y_0 \end{aligned} \tag{10}$$

in which $W(t)$ is the standard Brownian motion and where $\mathcal{Y}_B(y)$ is the Weber parabolic function, solution to the ordinary differential equation:

$$\frac{d^2}{dy^2}\mathcal{Y}_B(y) = \left[\frac{y^2}{4} + \left(B - \frac{1}{2}\right)\right]\mathcal{Y}_B(y) \tag{11}$$

with B a control parameter. From the definition of $f_B[Y(t)]$, we easily see that

$$\frac{d}{dy}f_B(y) + f_B^2(y) = \frac{\frac{d^2}{dy^2}\mathcal{Y}_B(y)}{\mathcal{Y}_B(y)} = \frac{y^2}{4} + \left(B - \frac{1}{2}\right). \tag{12}$$

This leads to a Beneš type finite-dimensional filtering problem (a fully analytical treatment of filtering problems in the Beneš class can be found in Daum[20]). In the sequel, we impose the parameter range $B \in \mathbb{R}^+$ which ensures the positivity of $\mathcal{Y}_B(y)$ ($\forall\, y \in \mathbb{R}$). For $B \in [0, 1/2]$, we further observe that the generalized potential $-\log[\mathcal{Y}_B(y)]$ is locally attractive near the origin and asymptotically repulsive for $|y| \to \infty$. In the parameter range $B > 1/2$, the potential is systematically repulsive $\forall y \in \mathbb{R}$ (Hongler[17]). Figure 1 shows the shape of $\mathcal{Y}_B(y)$ and $f_B(y)$ for different values of the control parameter B.

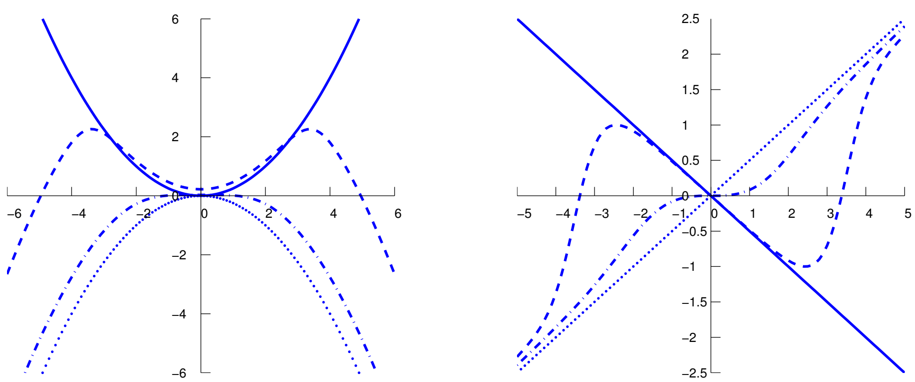

Fig. 1. Shape of $\mathcal{Y}_B(y)$ (left) and $f_B(y)$ (right) for $B = 0$ (plain line), $B = 0.01$ (stripped line), $B = 0.5$ (stripped-dotted line) and $B = 1$ (dotted line). For $B = 0$ the filtering problem is linear and the dynamics stable. For $B = 1$ the filtering problem is again linear but with unstable dynamics. In between, we have a nonlinear filtering problem and the conditional probability density changes – with increasing B – from unimodal to bimodal and back to unimodal.

For $B = 0$ and $B = 1$, we respectively obtain linear dynamics:

$$\begin{cases} \mathcal{Y}_0(y) = e^{-\frac{1}{4}y^2} \quad \Rightarrow \quad f_0(y) = -\frac{1}{2}y \\ \mathcal{Y}_1(y) = e^{+\frac{1}{4}y^2} \quad \Rightarrow \quad f_1(y) = +\frac{1}{2}y. \end{cases} \tag{13}$$

Using the framework introduced in Daum[20], the continuous time filter is given by the normalized probability density $P(y, t|\mathcal{Z}_t)$ of observing $Y(t) := y$

conditioned on the set of measurements up to time t, $\mathcal{Z}(t)$, and can be written as (computational details are given in the Appendix):

$$P(y,t) := P(y,t \mid \mathcal{Z}(t)) = \frac{\mathcal{Y}_B(y) \cdot e^{-\frac{(y-m)^2}{2s}}}{\mathcal{J}_0(m,s,B)} \tag{14}$$

with $\mathcal{J}_0(m,s,B)$ the normalization function (see Appendix):

$$\mathcal{J}_0(m,s,B) = 2\sqrt{\frac{\pi s}{2+s}} \left[\sqrt{\frac{2+s}{2-s}}\right]^B e^{\frac{m^2 s}{2(4-s^2)}} \mathcal{Y}_B\left(\frac{2m}{\sqrt{4-s^2}}\right). \tag{15}$$

The measurement *dependent* quantities $m := m(Z(t);t)$ are given by

$$m = m(Z(t);t) = \frac{\tanh(pt)}{p}\left[h\int_0^t \frac{\sinh(ps)}{\sinh(pt)} dZ(s) + \frac{py_0}{\sinh(pt)}\right] \tag{16}$$

and similarly, the measurement *independent* quantities $s := s(t)$ read as:

$$s = s(t) = \frac{1}{p}\tanh(pt) \tag{17}$$

with the definition $p = \sqrt{h^2 + \frac{1}{4}}$. With this expression for $P(y,t)$, we have for the conditional mean:

$$\langle Y_t \rangle := \mathbb{E}(Y_t|\mathcal{Z}(t)) = \frac{4m}{4-s^2} + \frac{2s}{\sqrt{4-s^2}} f_B\left[\frac{2m}{\sqrt{4-s^2}}\right] \tag{18}$$

and after tedious elementary manipulations the conditional variance:

$$var(Y_t) := \mathbb{E}((Y_t - \langle Y_t \rangle)^2|\mathcal{Z}(t)) = \frac{2s}{2+s} + \frac{4s^2}{4-s^2}\left\{\frac{m^2}{4-s^2} + B - f_B^2\left(\frac{2m}{\sqrt{4-s^2}}\right)\right\}. \tag{19}$$

Remark: For the linear cases $B = 0$ and $B = 1$ from Eq.(13), we consistently find the following classical results:

$$P(y,t) = \frac{\exp\left\{-\frac{((2+s)y-2m)^2}{4s(2+s)}\right\}}{\sqrt{2\pi\frac{2s}{2+s}}}, \qquad \langle Y_t \rangle = \frac{2}{2+s}m, \qquad var(Y_t) = \frac{2}{2+s}s \tag{20}$$

for $B = 0$ and

$$P(y,t) = \frac{\exp\left\{-\frac{((2-s)y-2m)^2}{4s(2-s)}\right\}}{\sqrt{2\pi\frac{2s}{2-s}}}, \qquad \langle Y_t \rangle = \frac{2}{2-s}m, \qquad var(Y_t) = \frac{2}{2-s}s \tag{21}$$

for $B = 1$. As predicted by the linear version of the feedback filter, when $B = 0$ and $B = 1$ the coupling strength $\nu(x,t)$ reduces to the standard **state independent** Kalman gain:

$$\nu(x,t) = \nu(t) = \frac{h}{\sigma^2} var(Y_t). \tag{22}$$

4 Numerical results

Numerical results are obtained by simulating (1) for a finite swarm of agents and one leader. Thanks to the consistency of the estimator (see Yang *et al.*[13]), one can still use the results from the mean field analysis for large enough N. In this case $P(y,t)$ must be fitted to the empirical histogram of agents' position at time t, to find the values for m and s. The control $\nu(x,t)$ can then be computed from its integral expression Eq.(9), while $\langle Y \rangle_t$ can be computed from Eq.(18). The derivative $\frac{d}{dx}\nu(x,t)$ is computed by using the relation $\frac{\frac{d}{dx}P(x,t)}{P(x,t)} = f_B(x) + \frac{x-m}{s}$, which can be written as:

$$\frac{d}{dx}\nu(x,t) = \frac{h}{\sigma^2}\left(\langle Y \rangle_t - x\right) - \nu(x,t) \cdot \left(f_B(x) - \frac{x-m}{s}\right). \tag{23}$$

Note that this natural fitting strategy to estimate $P(y,t)$ is – computationally – very costly. Extensive numerical computations have shown that $\nu(x,t)$ can safely be computed from Eq.(9) when using directly the empirical histogram of the agents' position instead of the fitted function in Eq.(14). The derivative $\frac{d}{dx}\nu(x,t) \simeq \frac{\nu(x+h,t)-\nu(x,t)}{h}$ is computed by selecting a sufficiently small value h.

Numerical results in linear cases. Figures 2 and 3 show in red the time evolution of the noisy leader's unveiled position. The mean value from the swarm of agents (likewise the output of the feedback particle filter) produces the smooth blue curve. As the agents' control is updated based on the unveiled position of the leader, a small delay can be observed between the leaders movements and the swarms reactions. The filter performs well: as expected, the swarms barycentric position is nearly always closer to the actual position of the leader than the unveiled position. This means that the control on the agents leads to a better approximation of the actual leaders position.

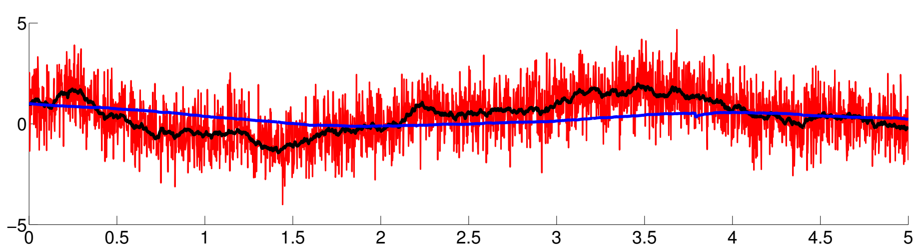

Fig. 2. Leader's position $Y(t)$ from Eq.(10) (black), along with its unveiled position $\mathcal{Z}(t)$ (red), for $B = 0$, $\sigma = h = 1$ and $t \in [0; 5]$. In blue the mean value $\langle Y \rangle_t$ measured from a swarm of $N = 1000$ agents. The particles start with $Y_i(0) = x_0 = 1\ \forall i$, while $Z(0) = Y(0) = x_0$.

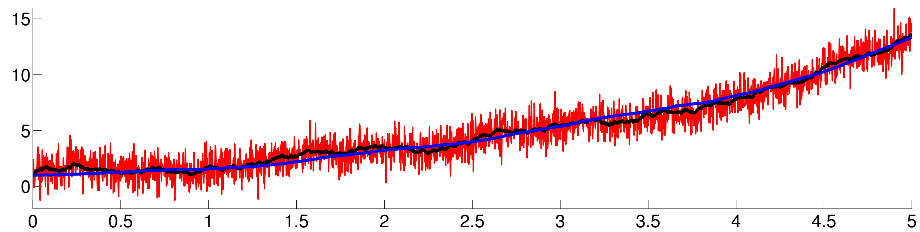

Fig. 3. Leader's position $Y(t)$ from Eq.(10) (black), along with its unveiled position $\mathcal{Z}(t)$ (red), for $B = 1$, $\sigma = h = 1$ and $t \in [0; 5]$. In blue the value $\langle Y \rangle_t$ measured from a swarm of $N = 1000$ agents. The particles start with $Y_i(0) = x_0 = 0.1\ \forall i$, while $Z(0) = Y(0) = x_0$.

Numerical results in nonlinear cases. We now consider the parameter range $0 < B < 0.5$ where the dynamics of the leader is nonlinear and exhibits an attractive potential in the central region (i.e., around the origin), and a repulsive potential for $|x| \gg 0$. Between these two regions, the potential changes from attractive to repulsive and the agents experience strong nonlinear dynamics. Note that the dynamics in the attractive region is meta-stable and a leader starting within this region ultimately escapes to infinity.
During the sojourn time of the leader in the attractive region, the close-by agents undergo quasi linear dynamics. They stay in this attractive region, self-arrange in the vicinity of the leaders position to empirically build the a posteriori distribution $P(y, t)$. As soon as the leader escapes from the attractive region, the other agents start feeling their barycentric control and ultimately follow the leader outside the attractive region. For an infinite swarm, its barycenter follows the leaders position with nearly no delay; but in our case, as $N < \infty$ agents, a delay can possibly be observed between the exit times of the leader and the agents. Figure 4 and 5 show the results of a representative numerical simulation for $N = 1000$ agents, with a very narrow and shallow attractive region ($B = 0.49$). Observe the explicit delay between the exit times of the leader and the swarm for $\sigma = 5$ in Figure 5.

5 Summary and Conclusion

Heterogeneous multi-agent systems are notoriously difficult to describe analytically and most especially if the underlying dynamics is intrinsically nonlinear. In this note, we present a class of dynamics where explicit and fully analytical results can be derived. The core of our construction relies on recent approaches that have been obtained in the realm of nonlinear estimation problems. The so-called particle filter method can be reinterpreted as a general leader-follower problem in which a swarm of interacting agents try to follow a leader whose unveiled position is corrupted by noise. In stochastic filtering, only finite-dimensional problems can possibly be solved analytically. Hence, when the

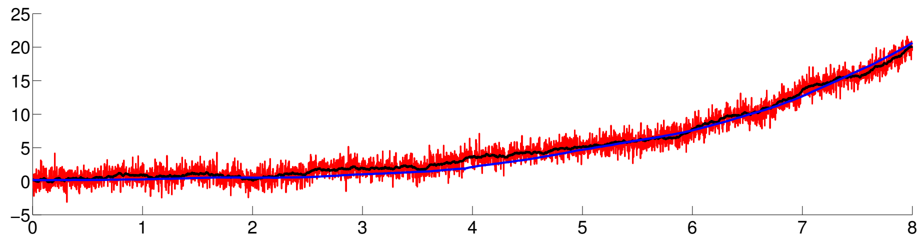

Fig. 4. Leader's position $Y(t)$ from Eq.(10) (black), along with its unveiled position $\mathcal{Z}(t)$ (red), for $B = 0.49$, $\sigma = h = 1$ and $t \in [0; 8]$. In blue the value $\langle Y \rangle_t$ measured from a swarm of $N = 1000$ agents. The particles start with $Y_i(0) = x_0 = 0.2\ \forall i$, while $Z(0) = Y(0) = x_0$.

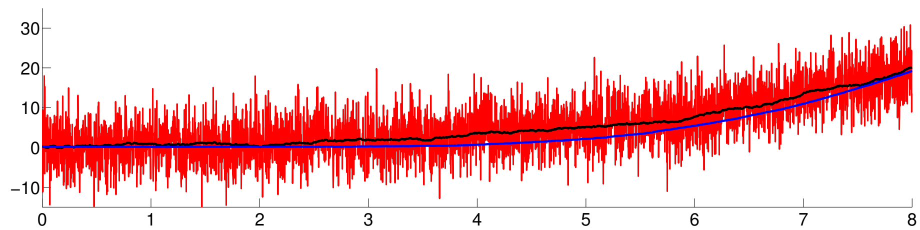

Fig. 5. Leader's position $Y(t)$ from Eq.(10) (black), along with its unveiled position $\mathcal{Z}(t)$ (red), for $B = 0.49$, $\sigma = 5$, $h = 1$ and $t \in [0; 8]$. In blue the value $\langle Y \rangle_t$ measured from a swarm of $N = 1000$ agents. The particles start with $Y_i(0) = x_0 = 0.2\ \forall i$, while $Z(0) = Y(0) = x_0$.

dynamics is driven by Gaussian noise, all relevant probability distributions remain always remain Gaussian and hence calculations are limited to the first two moments (Kalman-Bucy filter). The intimate connection existing between multi-agent systems and estimation problems show that for nonlinear dynamics, analytical results are in general hopeless. For one class of non-Gaussian filtering finite-dimensional problems however, explicit analytical models are available and were pioneered by Benes. It is therefore fully natural to study how the Benes' class enables to construct nonlinear solvable multi-agents systems, as it is done here. The core analytical tools leading to solvable finite-dimensional filtering problems rely on an underlying Riccati equation that we explicitly solved (in the scalar situation) via Weber parabolic cylinder functions. Using these special functions, we are able to answer two open questions originally formulated in Daum[20]. Our multi-agent class of dynamics enables to explicitly observe how a leader can control the spreading factor of the agents around its position, by tuning the strength of the observation noise.

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Appendix - Details of calculations

For the readers ease we introduce notations and collect formulas useful for the computation of the conditional probability density.

5.1 Collection of useful formulas

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y).$$

$$\int_{\mathbb{R}} e^{-ax^2-2bx-c}dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2-ac}{a}}, \qquad a>0$$

$$\int_{\mathbb{R}} \cosh[x\alpha]e^{-\frac{(x-\mu)^2}{\gamma}}dx = \sqrt{\pi\gamma}\cosh[\mu\alpha]e^{\frac{1}{4}\gamma\alpha^2} \tag{24}$$

$$\int_{\mathbb{R}} x\cosh[x\alpha]e^{-\frac{(x-\mu)^2}{\gamma}}dx = \sqrt{\pi\gamma}\left[\frac{\alpha\gamma}{2}\sinh(\mu\alpha) + \mu\cosh(\mu\alpha)\right]e^{\frac{1}{4}\gamma\alpha^2} \tag{25}$$

$$\int_{\mathbb{R}} x\sinh[x\alpha]e^{-\frac{(x-\mu)^2}{\gamma}}dx = \sqrt{\pi\gamma}\left[\frac{\alpha\gamma}{2}\cosh(\mu\alpha) + \mu\sinh(\mu\alpha)\right]e^{\frac{1}{4}\gamma\alpha^2} \tag{26}$$

From sections 9.24 and 9.25 of Gradshteyrn and Ryzhik[23], we extract:

$$\mathcal{D}_{-B}(x) = \frac{e^{-\frac{x^2}{4}}}{\Gamma(B)}\int_{\mathbb{R}^+} e^{-x\,\zeta-\frac{\zeta^2}{2}}\zeta^{B-1}d\zeta, \quad (\mathcal{R}(B)>0) \quad \text{(see Gradshteyrn and Ryzhik[23], 9.241/2)} \tag{27}$$

$$\begin{cases} \mathcal{Y}_B(x) := \frac{1}{2}\left[\mathcal{D}_{-B}(x) + \mathcal{D}_{-B}(-x)\right] = \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{x^2}{4}}}{\Gamma(B)}\int_{\mathbb{R}^+}\cosh(x\,\zeta)e^{-\frac{\zeta^2}{2}}\zeta^{B-1}\,d\zeta, \\ \frac{d^2}{dx^2}\left\{\mathcal{Y}_B(x)\right\} = \left[\frac{x^2}{4} + \left(B-\frac{1}{2}\right)\right]\mathcal{Y}_B(x), \quad (B\geq 0), \quad \text{(see Gradshteyrn and Ryzhik[23], 9.255/1)} \end{cases} \tag{28}$$

5.2 Quadratures

Let us define the couple of quadratures:

$$\mathcal{J}_i(m,s,B) = \int_{\mathbb{R}} x^i\mathcal{Y}_B(x)e^{-\frac{(x-m)^2}{2s}}dx, \quad i=0,1. \tag{29}$$

0^{th}-order moment - $\mathcal{J}_0(m,B)$ Using the integral representation given in Eq.(28), we can write:

$$\mathcal{J}_0(m,s,B) = \int_{\mathbb{R}} \left\{ \mathcal{Y}_B(x) e^{-\frac{(x-m)^2}{2s}} \right\} dx$$

$$= \sqrt{\tfrac{2}{\pi}} \tfrac{1}{\Gamma(B)} \int_{\mathbb{R}^+} \zeta^{[B-1]} e^{-\frac{\zeta^2}{2}} \left\{ \int_{\mathbb{R}} \cosh(\zeta x) e^{-\frac{(x-m)^2}{2s} - \frac{x^2}{4}} dx \right\} d\zeta$$

$$\mathcal{J}_0(m,s,B) = \sqrt{\tfrac{2}{\pi}} \frac{e^{-\frac{m^2}{2(2+s)}}}{\Gamma(B)} \int_{\mathbb{R}^+} \zeta^{[B-1]} e^{-\frac{\zeta^2}{2}} \left\{ \int_{\mathbb{R}} \cosh(\zeta x) e^{-\frac{(2+s)}{4s}\left(x - \frac{2m}{2+s}\right)^2} dx \right\} d\zeta$$

Now we use Eq.(24) with $\gamma = 4s/(2+s)$ and $\mu = 2m/(2+s)$ to get

$$\mathcal{J}_0(m,s,B) = 2\sqrt{\frac{2}{\pi}} \sqrt{\frac{\pi s}{(2+s)}} \frac{e^{-\frac{m^2}{2(2+s)}}}{\Gamma(B)} \int_{\mathbb{R}^+} \zeta^{[B-1]} e^{-\frac{\zeta^2}{2}\left[\frac{2-s}{2+s}\right]} \cosh\left[\frac{2m\zeta}{2+s}\right] d\zeta$$

Let us introduce the renormalization $\eta := \zeta \sqrt{\frac{2-s}{2+s}}$, which implies

$$\mathcal{J}_0(m,s,B) = \sqrt{\frac{2}{\pi}} 2 \sqrt{\frac{\pi s}{2+s}} \frac{e^{-\frac{m^2}{2(2+s)}}}{\Gamma(B)} \left[\sqrt{\frac{2+s}{2-s}}\right]^B \underbrace{e^{+\frac{m^2}{(4-s^2)}} e^{-\frac{m^2}{(4-s^2)}}}_{=1} \int_{\mathbb{R}^+} \eta^{[B-1]} \cosh\left[\frac{2m\,\eta}{\sqrt{4-s^2}}\right] e^{-\frac{\eta^2}{2}} d\eta. \tag{30}$$

Finally, using the definition Eq.(28), we end up with:

$$\mathcal{J}_0(m,s,B) = 2\sqrt{\frac{\pi s}{2+s}} \left[\sqrt{\frac{2+s}{2-s}}\right]^B e^{\frac{m^2 s}{2(4-s^2)}} \mathcal{Y}_B\left(\frac{2m}{\sqrt{4-s^2}}\right). \tag{31}$$

First order moment - $\mathcal{J}_1(m,s,B)$ From the definitions Eqs.(27) and (29), we observe that one can write:

$$\mathcal{J}_1(m,s,B) := \int_{\mathbb{R}} \left\{ x\, \mathcal{Y}_B(x) e^{-\frac{(x-m)^2}{2s}} \right\} dx$$

From the previous equation and the definition of $\mathcal{J}_0(m,s,B)$ given in Eq.(29), let us observe that we can write:

$$\tfrac{d}{dm} \mathcal{J}_0(m,s,B) = \int_{\mathbb{R}} \left\{ \left[\tfrac{(x-m)}{s}\right] \mathcal{Y}_B(x) e^{-\frac{(x-m)^2}{2s}} \right\} dx = \tfrac{1}{s} \mathcal{J}_1(m,s,B) - \tfrac{m}{s} \mathcal{J}_0(m,s,B).$$

This is equivalent to the relation:

$$\mathcal{J}_1(m,s,B) = m\mathcal{J}_0(m,s,B) + s\left[\frac{d}{dm}\mathcal{J}_0(m,s,B)\right]. \tag{32}$$

Using Eqs.(31) and (32), the conditioned expectation reads:

$$\mathbb{E}(x|\mathcal{Z}_t) = \left.\tfrac{\mathcal{J}_1(m.b,B)}{\mathcal{J}_0(m,s,B)}\right|_t = m + s\left[\tfrac{d}{dm}\left(\log\left\{\mathcal{J}_0(m,s,B)\right\}\right)\right]\Big|_t = \tfrac{4}{4-s^2} m(z) + \tfrac{2s}{\sqrt{4-s^2}} f_B\left(\tfrac{2\,m(z)}{\sqrt{4-s^2}}\right) \tag{33}$$