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The Bernstein Problem for Embedded Surfaces in the Heisenberg Group $\mathbb{H}^1$

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ABSTRACT. In the paper [13] we proved that the only stable C^2 minimal surfaces in the first Heisenberg group $\mathbb{H}^1$ which are graphs over some plane and have empty characteristic locus must be vertical planes. This result represents a sub-Riemannian version of the celebrated theorem of Bernstein.

In this paper we extend the result in [13] to C^2 complete embedded minimal surfaces in $\mathbb{H}^1$ with empty characteristic locus. We prove that every such a surface without boundary must be a vertical plane. This result represents a sub-Riemannian counterpart of the classical theorems of Fischer-Colbrie and Schoen, [16], and do Carmo and Peng, [15], and answers a question posed by Lei Ni.

1. INTRODUCTION

The study of minimal surfaces has been one of the prime drivers of the developments of geometry and calculus of variations in the twentieth century and, in particular, the Bernstein problem has played a central role. In 1915 Bernstein proved his Theorem [4] that a C^2 minimal graph in $\mathbb{R}^3$ must necessarily be an affine plane and, almost fifty years later, a new insight of Fleming [17] generated renewed interest in the problem. The work of De Giorgi, [14], Almgren, [1], Simons, [29], and Bomberi-De Giorgi-Giusti, [5], culminated in the complete solution to the Bernstein problem:

Theorem 1.1. *Let* $S = \{(x, u(x)) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n,\ x_{n+1} = u(x)\}$ *be a* C^2 *minimal graph in* $\mathbb{R}^{n+1}$*, i.e., let* $u \in C^2(\mathbb{R}^n)$ *be a solution of the minimal surface* *equation*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0, \tag{1.1}$$

in the whole space. If $n \le 7$, then there exist $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ such that $u(x) = \langle a, x\rangle + \beta$, i.e., S must be an affine hyperplane. If instead $n \ge 8$, then there exist non affine (*real analytic*) *functions on $\mathbb{R}^n$ which solve* (1.1).

Roughly a decade later, Fischer-Colbrie and Schoen, [16], and do Carmo and Peng, [15], imposing a stability condition, independently proved a far reaching generalization of the Bernstein property:

Theorem 1.2. *Every stable complete minimal surface $S \subset \mathbb{R}^3$ must be a plane.*

Here, stable means that on every compact set S minimizes area up to order two. We note that in Theorem 1.1 stability plays no role since, thanks to the strict convexity of the area functional $\mathcal{A}(u) = \int \sqrt{1+|Du|^2}\,dx$, for Euclidean graphs on $\mathbb{R}^n$ the stability assumption is automatically satisfied, see e.g. [10].

The purpose of this paper is to prove an analogue of Theorem 1.2 in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ (for the relevant definitions we refer the reader to the next section). The study of the Bernstein problem in this setting has received increasing attention over the last decade. The existence of minimal surfaces in sub-Riemannian spaces was established by two of us in [20] by developing in such setting the methods of the geometric measure theory. The study of minimal graphs in the Heisenberg group was first approached by one of us in [26], by Cheng, Hwang, Malchiodi and Yang [8] (who studied the problem in a more general class of pseudohermitian 3-manifolds), by three of us in [12], and by two of us in [21].

Henceforth in this paper, following a perhaps unfortunate but old tradition, by *minimal* we intend a C^2 surface $S \subset \mathbb{H}^1$ whose sub-Riemannian, or horizontal mean curvature $\mathcal{H}$ (see Proposition 2.3 below for its expression) vanishes identically on S. In these initial investigations, a number of nonplanar minimal graphs over the xy-plane are produced ([8, 21, 26]) and indeed are classified (first in [8], with an alternate proof in [21]). A prototypical example is given by the surface $t = xy/2$ which is an entire minimal graph over the xy-plane. However, this example and all other entire minimal graphs over the xy-plane must have non empty characteristic locus (this fact was proved independently in [8] and [21]). We recall that the latter is defined as the set of points of the surface at which the two bracket generating vector fields X_1, X_2 become tangent to the surface itself.

In some of these same papers, new examples were discovered of entire minimal graphs over some plane, but with an empty characteristic locus. In [21], two of us first produced infinitely many examples of such graphs, one of which is given by

$$x = y\tan(\tanh(t)). \tag{1.2}$$

Moreover, as announced in [8] (this and many other examples are shown in more detail in [7]), the surface

$$x = yt \tag{1.3}$$

is also noncharacteristic and minimal. From the point of view of the Bernstein problem, these examples would indicate a failure of the property—there exists a rich reservoir of graphs over the xy-plane which are minimal (although they have characteristic points) and an equally rich reservoir of nonplanar noncharacteristic minimal graphs over the yt-plane (or the xt-plane). In the positive direction, the work [21] shows that graphs over vertical planes must have a specific structure indicating some kind of rigidity (see also [7] for other classification results).

In [11] the first three authors continued the investigation into noncharacteristic graphs by asking a more refined question: are surfaces such as (1.2) or (1.3) local minima? Just as in the classical case, sub-Riemannian minimal surfaces are shown to merely be critical points of the relative area functional (the so-called horizontal perimeter). Since this functional is shown to lack the fundamental convexity property which guarantees in the flat case that critical points are global minimizers, the question of stability becomes central. It could happen in fact that minimal surfaces such as (1.2), (1.3) fail to be locally area minimizing. As the following surprising theorem proved in [11] shows, this is precisely the case.

Theorem 1.3. *Let $\alpha > 0$, $\beta \in \mathbb{R}$; then the surfaces*

$$x = y(\alpha t + \beta), \quad y = x(-\alpha t + \beta),$$

are unstable noncharacteristic entire minimal graphs.

This result first showed that, in the sub-Riemannian setting, the Bernstein property cannot hold unless we assume the surface be noncharacteristic and stable. We emphasize that the surfaces in Theorem 1.3 are also global intrinsic graphs in the sense of [18], [19], see the definition below.

To introduce the next result we recall the interesting fact, proved by Franchi, Serapioni and Serra Cassano [18], that if in $\mathbb{H}^1$ one performs the blow-up à la De Giorgi of sets with locally finite perimeter at a point of its reduced boundary, one obtains a vertical plane

$$\Pi_\gamma = \{(x, y, t) \in \mathbb{H}^1 \mid ax + by = \gamma\}. \tag{1.4}$$

Such planes are also noncharacteristic minimal surfaces in $\mathbb{H}^1$. On the other hand it was proved in Corollary 15.3 in [12] that every plane (1.4) is stable. Therefore, it is natural to assume that these sets are the appropriate self-similar sets in the sub-Riemannian Bernstein problem.

In [13], we continued this line of investigation and provided a positive answer to the following version of the Bernstein problem.

Theorem 1.4 (Bernstein, [13, Theorem 1]). *In $\mathbb{H}^1$ the only stable C^2 minimal entire graphs over some plane, with empty characteristic locus, are the vertical planes* (1.4).

Another approach to the sub-Riemannian Bernstein problem arises when considering the following intrinsic notion of graph introduced in [18] and developed further in [2, 3, 19, 22]. A C^2 surface S is called an *X_1-graph* if there exist a domain $\Omega \subset \mathbb{R}^2_{uv}$ and $\varphi \in C^2(\Omega)$, such that $S = \{(0, u, v) \circ \varphi(u, v)e_1 \mid (u, v) \in \Omega\}$, where $e_1 = (1, 0, 0)$. We note that one basic consequence of this definition is that S has empty characteristic locus. This follows from the fact that the vector field X_1 is always transversal to the surface. Interestingly, if we assume that Ω be bounded, then the horizontal perimeter of S is given by the formula

$$\mathcal{P}_H(S) = \int_\Omega \sqrt{1 + \mathcal{B}_\varphi(\varphi)^2}\, \mathrm{d}u\, \mathrm{d}v, \tag{1.5}$$

where we have denoted by $\mathcal{B}_\varphi(f) = f_u + \varphi f_v$ the linearized Burger's operator. Notice that $\mathcal{B}_\varphi(\varphi) = \varphi_u + \varphi\varphi_v$ is the nonlinear inviscid Burger's operator. In [3], Barone Adesi, Serra Cassano and Vittone prove the following Bernstein theorem for these types of graphs.

Theorem 1.5 (Bernstein, [3, Theorem 2]). *The only C^2 stable minimal entire X_1-graphs are the vertical planes.*

The main result established in this paper is the following theorem.

Theorem A (of Bernstein type). *The vertical planes are the only stable C^2 complete embedded minimal surfaces in $\mathbb{H}^1$ without boundary and with empty characteristic locus.*

We note that Theorem A is not contained in either of the cited Theorems 1.4 or 1.5. For instance the sub-Riemannian *catenoids* in $\mathbb{H}^1$ (the reader should note that from the Euclidean standpoint these surfaces are just the classical hyperboloids of revolution)

$$(t - a)^2 = \frac{4}{b^2}\left(\frac{b}{4}(x^2 + y^2) - 1\right), \quad a, b \in \mathbb{R},\ b > 0, \tag{1.6}$$

are complete embedded minimal surfaces with empty characteristic locus which are not graphs on any plane, nor they are entire intrinsic graphs. The above result shows, in particular, that the minimal surfaces (1.6) must be unstable. Theorem A represents a sub-Riemannian counterpart of the cited classical theorems of Fischer-Colbrie and Schoen, [16], and do Carmo and Peng, [15], and answers a question posed by Lei Ni in 2006.

We now briefly discuss our approach to Theorem A. A central new idea is contained in the notion of *strict intrinsic graphical strip* introduced in Definition

1.6 below. This notion was suggested to us by the analysis of the double Burger equation

$$\mathcal{B}_\varphi(\mathcal{B}_\varphi(\varphi)) = 0. \tag{1.7}$$

Such equation characterizes those X_1-graphs which are minimal, see for instance equation (15.5) in [12]. Suppose we are given some interval $J = (-4\varepsilon, 4\varepsilon) \subset \mathbb{R}$, $\varepsilon > 0$, and functions F, G, $\sigma \in C^2(J)$ satisfying the condition

$$F'(s)^2 < 2\sigma'(s)G'(s), \quad \text{for every } s \in J. \tag{1.8}$$

We note explicitly that (1.8) implies, in particular, that $\sigma'(s)G'(s) > 0$ for every $s \in J$. If we consider the mapping $\Psi : \mathbb{R} \times J \to \mathbb{R}^2$ from the (u,s) to the (u,v) plane defined by $\Psi(u,s) = (u,v)$, where v is defined by the equation

$$v = v(u,s) = G(s)\frac{u^2}{2} + F(s)u + \sigma(s), \tag{1.9}$$

then we see that the Jacobian determinant of Ψ is given by

$$\begin{aligned} \det J_\Psi(u,s) &= \det \begin{pmatrix} 1 & 0 \\ G(s)u + F(s) & G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s) \end{pmatrix} \\ &= G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s). \end{aligned} \tag{1.10}$$

Thanks to (1.8) the Jacobian of Ψ is always different from zero. We emphasize at this moment that the continuity of the first derivatives of the functions F, G and σ, along with the assumption (1.8), guarantee that, possibly restricting the interval $J = (-4\varepsilon, 4\varepsilon)$, we can always force the map Ψ to be globally one-to-one, hence a C^2 diffeomorphism of the infinite strip $\mathbb{R} \times J$ onto its image $\Psi(\mathbb{R} \times J)$. We denote with $\Psi^{-1}(u,v) = (u, s(u,v))$ the inverse C^2 diffeomorphism. When we write $s(u,v)$ we mean the function specified by such inverse diffeomorphism.

Definition 1.6. Let $\varepsilon > 0$, $J = (-4\varepsilon, 4\varepsilon)$. A C^2 surface $S \subset \mathbb{H}^1$ is an *intrinsic graphical strip* on J if there exist functions F, G, $\sigma \in C^2(J)$ satisfying $(F')^2 \le 2\sigma' G'$ such that, if

$$\Omega = \Psi(\mathbb{R} \times J) = \left\{ (u,v) \mid u \in \mathbb{R},\ v = G(s)\frac{u^2}{2} + F(s)u + \sigma(s) \text{ for } s \in J \right\},$$

then with $\varphi \in C^2(\Omega)$ defined by

$$\varphi(u,v) = F(s(u,v)) + uG(s(u,v)),$$

we have

$$S = \Big\{(0,u,v)\circ(\varphi(u,v),0,0) \mid (u,v)\in\Omega\Big\}$$
$$= \Big\{\Big(\varphi(u,v),u,v-\frac{u}{2}\varphi(u,v)\Big) \mid (u,v)\in\Omega\Big\}.$$

We say that S is a *strict intrinsic graphical strip* on J if F, G, σ satisfy the strict inequality (1.8), and if the map $\Psi:\mathbb{R}\times J\to\Omega$ is globally one-to-one, hence a diffeomorphism of $\mathbb{R}\times J$ onto $\Psi(\mathbb{R}\times J)=\Omega$.

Remark 1.7. A strict intrinsic graphical strip is necessarily a minimal surface. To see this, we observe that the function φ in the above definition satisfies (1.7). The reader will find most of the computations to achieve this in the proof of Corollary 3.6, see formula (3.7) below, and the computations following that formula.

Remark 1.8. In the case of a strict intrinsic graphical strip, without loss of generality we can assume that $G'(s)>0$ on J (this property is needed in the proof of Lemmas 4.1, 4.2 and Theorem B). We can justify this claim as follows. As observed earlier, the condition (1.8) implies $\sigma'(s)G'(s)>0$. Since $\sigma'(s)$ does not change sign, if $\sigma'(s)>0$ this forces $G'(s)>0$. If instead we have $G'(s)<0$ on J, we replace F, G, σ by $\tilde F(s)=F(-s)$, $\tilde G(s)=G(-s)$, $\tilde\sigma(s)=\sigma(-s)$. The newly defined functions satisfy (1.8). We also have $\tilde G'(s)>0$. Now we take $\varphi(u,v)=\tilde F(-s(u,v))+u\tilde G(-s(u,v))$. We see that the surface parameterized by this new φ has the same trace as the one with the original φ.

Remark 1.9. We emphasize here that any vertical plane such as (1.4) is an intrinsic graphical strip, but not a strict intrinsic graphical strip. One has in fact if $a\neq 0$, that $\varphi(u,v)=\gamma/a-(b/a)u$, so that $F(s)\equiv\gamma/a$, $G(s)=-b/a$, $\sigma(s)\equiv 0$. Therefore, $2\sigma'G'-(F')^2\equiv 0$.

Notice that, as a consequence of the smoothness hypothesis on F, G, an intrinsic graphical strip is a surface of class C^2. Definition 1.6 takes advantage of a change of coordinates introduced in [3] which is one of the essential tools in the proof of Theorem 1.5. The motivation behind Definition 1.6 will be explained in Section 5. With this definition in place, in Section 4 we can adapt some of the ideas from [11], [13] to construct a variation on an intrinsic graphical strip which decreases the horizontal perimeter, proving the following basic result.

Theorem B. *Let S be a strict intrinsic graphical strip as in Definition 1.6. There exists a $\psi\in C_0^2(S)$ such that*

$$\mathcal{V}_{II}^H(S,\psi X_1)<0,$$

where $\mathcal{V}_{II}^H(S;X)$ denotes the second variation of the horizontal perimeter along the vector field X. As a consequence, S is unstable.

The relevance of Theorem B is in the following theorem, which we prove in Section 5.

Theorem C. *Every C^2 complete noncompact embedded minimal surface without boundary with empty characteristic locus and which is not itself a vertical plane contains a strict intrinsic graphical strip.*

Our proof of Theorem C hinges on a close analysis of the representation results of [21]. Theorems B and C are the main novel technical points of the present paper. From them, the proof of the Bernstein type Theorem A immediately follows.

Finally, we would like to mention that after completion of our paper we have received the preprint [25] in which the authors use a somewhat different approach to establish results which are closely related to those in this paper. For instance, Theorem 4.7 in [25] is precisely our Theorem A. However, in their Theorem 6.1 they are also able to classify C^2 stable complete immersed minimal surface with non-empty characteristic locus.

2. Definitions

In this section we recall some definitions and known results which will be needed in the paper. We recall that the Heisenberg group $\mathbb{H}^n$ is the graded, nilpotent Lie group of step 2 with underlying manifold $\mathbb{C}^n \times \mathbb{R} \cong \mathbb{R}^{2n+1}$, whose points we indicate $g = (x, y, t)$, $g' = (x', y', t')$, etc. The non-Abelian group law in $\mathbb{H}^n$ is prescribed by the left-translations

$$L_g(g') = g \circ g' = \left(x+x',\ y+y',\ t+t'+\frac{1}{2}(x\cdot y' - x'\cdot y)\right). \tag{2.1}$$

Here, and throughout the paper, we will use $v \cdot w$ to denote the standard Euclidean inner product of two vectors v and w in $\mathbb{R}^n$. The grading of the Heisenberg algebra is given by $\mathfrak{h}_n = V_1 \oplus V_2$, where $V_1 = \mathbb{R}^{2n} \times \{0\}$, $V_2 = \{0\} \times \mathbb{R}$. Accordingly, elements of the horizontal layer V_1 have degree one, whereas elements of the vertical layer V_2 are assigned the degree two. We recall that, identifying $\mathfrak{h}_n$ with $\mathbb{R}^{2n+1}$, we have for the bracket

$$[g, g'] = (0,\ 0,\ x\cdot y' - x'\cdot y).$$

It is then clear that $[V_1, V_1] = V_2$, and that V_2 is the group center. Associated with the grading, one has in $\mathbb{H}^n$ the following non-isotropic dilations

$$\delta_\lambda(g) = (\lambda x, \lambda y, \lambda^2 t), \quad \lambda > 0, \tag{2.2}$$

whose homogeneous dimension is given by $Q = 2n + 2$.

Henceforth, we will focus on the first Heisenberg group $\mathbb{H}^1$. Applying the differential $(L_g)_*$ of (2.1) to the standard basis $\{\partial_x, \partial_y, \partial_t\}$ of $\mathbb{R}^3$, we obtain the three distinguished vector fields

$$\begin{aligned} X_1 &= (L_g)_*(\partial_x) = \partial_x - \frac{y}{2}\,\partial_t, \\ X_2 &= (L_g)_*(\partial_y) = \partial_y + \frac{x}{2}\,\partial_t, \\ T &= (L_g)_*(\partial_t) = \partial_t. \end{aligned}$$

The horizontal bundle $H\mathbb{H}^1$ is the subbundle of $T\mathbb{H}^1$ whose fiber at a point $g \in \mathbb{H}^1$ is given by

$$H_g = \operatorname{span}\{X_1(g), X_2(g)\}.$$

We endow $\mathbb{H}^1$ with a left-invariant Riemannian metric $\{g_{ij}\}$, whose inner product we will denote by $\langle \cdot, \cdot \rangle$, with respect to which $\{X_1, X_2, T\}$ constitute an orthonormal basis. If $S \subset \mathbb{H}^1$ is a C^2 oriented surface, we will indicate with $\mathbf{N}$ a (non-unit) Riemannian normal with respect to $\langle \cdot, \cdot \rangle$, and with $\boldsymbol{\nu} = \mathbf{N}/|\mathbf{N}|$ the corresponding Gauss map. We will let

$$p = \langle \mathbf{N}, X_1 \rangle, \quad q = \langle \mathbf{N}, X_2 \rangle, \quad W = \sqrt{p^2 + q^2}, \quad \omega = \langle \mathbf{N}, T \rangle. \tag{2.3}$$

The characteristic locus of S is the closed subset of S defined by

$$\Sigma(S) = \{g \in S \mid W(g) = 0\}.$$

We notice explicitly that $\Sigma(S) = \{g \in S \mid T_g S = H_g\}$. We also set on $S \setminus \Sigma(S)$

$$\bar{p} = \frac{p}{W}, \quad \bar{q} = \frac{q}{W}, \quad \bar{\omega} = \frac{\omega}{W}. \tag{2.4}$$

Definition 2.1. Let $S \subset \mathbb{H}^1$ be a C^2 oriented surface. A horizontal normal of S is defined as

$$\mathbf{N}^H = pX_1 + qX_2,$$

whereas on $S \setminus \Sigma(S)$ the horizontal Gauss map is defined as

$$\boldsymbol{\nu}^H = \frac{1}{W}\mathbf{N}^H = \bar{p}X_1 + \bar{q}X_2.$$

The horizontal perimeter measure of S has the following form.

Proposition 2.2. *Let $S \subset \mathbb{H}^1$ be a C^2 oriented surface, then the horizontal perimeter of S is*

$$\mathcal{P}_H(S) = \int_S \sqrt{\langle \boldsymbol{\nu}, X_1 \rangle^2 + \langle \boldsymbol{\nu}, X_2 \rangle^2}\, d\sigma = \int_S \frac{W}{|\mathbf{N}|}\, d\sigma,$$

where $d\sigma$ *is the Riemannian surface area element associated to* $\langle \cdot, \cdot \rangle$.

To investigate minimal surfaces, we recall the notion of horizontal mean curvature $\mathcal{H}$ introduced in [12], [26], [21]. Such notion is obtained by projecting the horizontal Levi-Civita connection onto the so-called horizontal tangent bundle $HTS = TS \cap H\mathbb{H}^1$. If we assume, as we may, that the Riemannian normal field on S, $\mathbf{N}^H$, can be extended to a neighborhood of S, and continuing to denote by $\bar{p}$, $\bar{q}$ the quantities introduced in (2.4) relative to such extension, then it has been shown in the above cited references that $\mathcal{H}$ can be computed by the following proposition.

Proposition 2.3. *For $g \in S \setminus \Sigma(S)$, the horizontal mean curvature of S at g is given by*

$$\mathcal{H}(g) = X_1\bar{p}(g) + X_2\bar{q}(g).$$

For $g \in \Sigma(S)$, we define $\mathcal{H}(g) = \lim_{g' \to g, g' \in S \setminus \Sigma(S)} \mathcal{H}(g')$, whenever the limit exists. A surface S is said to be *minimal* if its horizontal mean curvature vanishes identically.

It is now well known ([8, 12, 21, 26, 27]) that critical points of the perimeter are characterized by having $\mathcal{H} \equiv 0$ away from the characteristic locus. We mention that recent work of Cheng, Hwang and Yang ([9]) and Ritoré and Rosales ([28]) have clarified the behavior of such critical points at the characteristic locus. However, since we will be restricting to the category of noncharacteristic surfaces, we will not discuss these results here.

3. The Second Variation of the Horizontal Perimeter and the Stability of Minimal Surfaces

In this section, we recall the first and second variation of the horizontal perimeter for intrinsic graphs. We mention that formulas for the first and second variation of the horizontal perimeter have been derived a number of times in various contexts ([2, 3, 6, 8, 12, 22–24, 27, 28]).

Let $S \subset \mathbb{H}^1$ be an oriented C^2 surface with empty characteristic locus, and consider vector fields $\mathcal{X} = aX_1 + bX_2 + kT$, with $a, b, k \in C_0^2(S)$. We define the *first variation* of the horizontal perimeter with respect to the deformation of S,

$$S^\lambda = S + \lambda\mathcal{X},$$

as

$$\mathcal{V}_I^H(S;\mathcal{X}) = \frac{d}{d\lambda}P_H(S^\lambda)\Big|_{\lambda=0}.$$

We say that S is *stationary* if $\mathcal{V}_I^H(S;\mathcal{X}) = 0$, for every $\mathcal{X}$. We define the *second variation* of the horizontal perimeter as

$$\mathcal{V}_{II}^H(S;\mathcal{X}) = \frac{d^2}{d\lambda^2}P_H(S^\lambda)\Big|_{\lambda=0}.$$

We say that S is *stable* is $\mathcal{V}_{II}^H(S;\mathcal{X}) \geq 0$ for every $\mathcal{X}$.

Henceforth, to simplify the formulas we introduce the following notation

$$F_{\mathcal{X}} \stackrel{\text{def}}{=} \bar{p}a + \bar{q}b + \bar{\omega}k = \frac{\langle \mathcal{X}, \mathbf{N} \rangle}{\langle \boldsymbol{\nu}^H, \mathbf{N} \rangle}. \tag{3.1}$$

Henceforth, we will follow the notation in [12]. In particular, we will indicate with σ_H the horizontal perimeter measure on a surface S which is given by Proposition 2.2. For instance, if S is an X_1-graph, then one can easily see that

$$d\sigma_H = \sqrt{1 + \mathcal{B}_\varphi(\varphi)^2}\, du\, dv.$$

Furthermore, $\nabla^{H,S}$ denotes the tangential horizontal gradient on S which on a function $f : S \to \mathbb{R}$ is defined by

$$\nabla^{H,S} \bar{f} = \nabla^H \bar{f} - \langle \nabla^H \bar{f}, \boldsymbol{\nu}^H \rangle \boldsymbol{\nu}^H,$$

where $\bar{f}$ is any extension of f to the whole $\mathbb{H}^1$. The following result was proved independently by several people in various contexts, see [2, 3, 6, 8, 12, 22–24, 27, 28].

Theorem 3.1. *Let $S \subset \mathbb{H}^1$ be an oriented C^2 surface with empty characteristic locus, then*

$$\mathcal{V}_I^H(S;\mathcal{X}) = \int_S \mathcal{H} F_{\mathcal{X}}\, d\sigma_H. \tag{3.2}$$

In particular, S is stationary if and only if it is minimal.

To state the next result we introduce a notation. Given the quantity $\bar{\omega}$ we let

$$\mathcal{A} = -\nabla^{H,S}\bar{\omega}.$$

The following second variation formula was proved in [12].

Theorem 3.2. *Let $S \subset \mathbb{H}^1$ be a minimal surface with empty characteristic locus, then*

$$\mathcal{V}_{II}^H(S;\mathcal{X}) = \int_S \left\{ |\nabla^{H,S} F_{\mathcal{X}}|^2 + (2\mathcal{A} - \bar{\omega}^2) F_{\mathcal{X}}^2 \right\} d\sigma_H.$$

As a consequence, S is stable if and only if for any $\mathcal{X}$ one has

$$\int_S (\bar{\omega}^2 - 2\mathcal{A}) F_{\mathcal{X}}^2\, d\sigma_H \leq \int_S |\nabla^{H,S} F_{\mathcal{X}}|^2\, d\sigma_H.$$

The following result is Corollary 15.4 in [12]. Let $\varphi : \Omega \subset \mathbb{R}^2_{(u,v)} \to \mathbb{R}$ give an intrinsic X_1-graph S, we recall the formula (1.5) for the horizontal perimeter of S.

Corollary 3.3. *Let S be a C^2 minimal, intrinsic X_1-graph; then for any $\mathcal{X}$ one has*

$$\mathcal{V}_{II}^H(S;\mathcal{X}) = \int_\Omega \frac{\mathcal{B}_\varphi(F_\mathcal{X})^2}{\sqrt{1+\mathcal{B}_\varphi(\varphi)^2}}\, du\, dv - \int_\Omega \frac{\varphi_v^2 + 2\mathcal{B}_\varphi(\varphi_v)}{\sqrt{1+\mathcal{B}_\varphi(\varphi)^2}} F_\mathcal{X}^2\, du\, dv,$$

where $F_\mathcal{X}$ is as in (3.1).

We next derive the second variation formula for special deformations of the intrinsic graph S. We consider compactly supported vector fields on S of the type $\mathcal{X} = \psi X_1$, where $\psi \in C_0^2(S)$. For this family of deformations we obtain, from Corollary 3.3, the following result.

Theorem 3.4. *Let S be a C^2 minimal, intrinsic X_1-graph, given by a function $\varphi : \Omega \subset \mathbb{R}^2_{(u,v)} \to \mathbb{R}$, then for any $\psi \in C_0^2(S)$ one has*

$$\mathcal{V}_{II}^H(S,\psi X_1) = \int_\Omega \frac{\mathcal{B}_\varphi(\psi)^2}{(1+\mathcal{B}_\varphi(\varphi)^2)^{3/2}}\, du\, dv - \int_\Omega \frac{\psi^2}{(1+\mathcal{B}_\varphi(\varphi)^2)^{3/2}} (2(\mathcal{B}_\varphi(\varphi))_v - \varphi_v^2)\, du\, dv. \tag{3.3}$$

Remark 3.5. In the statement of the above result the function $\psi \in C_0^2(S)$. Slightly abusing the notation in the integral in the right-hand side of (3.3) we have continued to indicate with ψ the function in $C_0^2(\Omega)$ obtained by composing the original ψ with the parametrization of the surface S

$$\Omega \ni (u,v) \longmapsto \left(\varphi(u,v), u, v - \frac{u}{2}\varphi(u,v)\right).$$

Proof. We note that with $\mathcal{X} = \psi X_1$, we have $a = \psi$, $b = k = 0$. We also recall, see formulas (15.1) in [12], that for an intrinsic X_1-graph one has

$$\bar{p} = \frac{1}{\sqrt{1+\mathcal{B}_\varphi(\varphi)^2}} \qquad \bar{q} = -\frac{\mathcal{B}_\varphi(\varphi)}{\sqrt{1+\mathcal{B}_\varphi(\varphi)^2}},$$

and therefore from (3.1) one has

$$F_\mathcal{X} = \frac{\psi}{\sqrt{1+\mathcal{B}_\varphi(\varphi)^2}}. \tag{3.4}$$

From this formula a simple computation gives

$$\mathcal{B}_\varphi(F_\mathcal{X}) = \frac{\mathcal{B}_\varphi(\psi)}{\sqrt{1+\mathcal{B}_\varphi(\varphi)^2}} - \frac{\mathcal{B}_\varphi(\varphi)\mathcal{B}_\varphi(\mathcal{B}_\varphi(\varphi))}{(1+\mathcal{B}_\varphi(\varphi)^2)^{3/2}}.$$

We now recall that the minimality of S is equivalent to φ being a solution of the double Burger equation

$$\mathcal{B}_\varphi(\mathcal{B}_\varphi(\varphi)) = 0.$$

We thus conclude that

$$\mathcal{B}_\varphi(F_X) = \frac{\mathcal{B}_\varphi(\psi)}{\sqrt{1+\mathcal{B}_\varphi(\varphi)^2}}. \tag{3.5}$$

Using (3.4) and the identity

$$(\mathcal{B}_\varphi(\varphi))_v - \mathcal{B}_\varphi(\varphi_v) = \varphi_v^2,$$

we thus obtain

$$\begin{aligned} -\int_\Omega \frac{\varphi_v^2 + 2\mathcal{B}_\varphi(\varphi_v)}{\sqrt{1+\mathcal{B}_\varphi(\varphi)^2}} F_X^2 \, \mathrm{d}u\, \mathrm{d}v \\ = -\int_\Omega \frac{\psi^2}{(1+\mathcal{B}_\varphi(\varphi)^2)^{3/2}} (2(\mathcal{B}_\varphi(\varphi))_v - \varphi_v^2)\, \mathrm{d}u\, \mathrm{d}v. \end{aligned}$$

On the other hand, (3.5) gives

$$\int_\Omega \frac{\mathcal{B}_\varphi(F_X)^2}{\sqrt{1+\mathcal{B}_\varphi(\varphi)^2}} \, \mathrm{d}u\, \mathrm{d}v = \int_\Omega \frac{\mathcal{B}_\varphi(\psi)^2}{(1+\mathcal{B}_\varphi(\varphi)^2)^{3/2}} \, \mathrm{d}u\, \mathrm{d}v.$$

Combining the last two equations we reach the desired conclusion. □

Next, we apply Theorem 3.4 to the case of a strict intrinsic graphical strip as in Definition 1.6. We recall the diffeomorphism $\Psi : \mathbb{R} \times J \to \Omega = \Psi(\mathbb{R} \times J) \subset \mathbb{R}^2_{u,v}$ given by $\Psi(u,s) = (u,v) = (u, (u^2/2)G(s) + F(s)u + \sigma(s))$, see (1.9). As before, in the statement of the next result given a function $\psi \in C_0^2(S)$ slightly abusing the notation we will write $\psi \in C_0^2(\Omega)$. What we mean by this is the composition of the original ψ with the parametrization of the surface S

$$\Omega \ni (u,v) \longmapsto \left(\varphi(u,v), u, v - \frac{u}{2}\varphi(u,v)\right)$$

provided in Definition 1.6.

Corollary 3.6. *Let S be a strict intrinsic graphical strip defined by functions F, G, $\sigma \in C^2(J)$ and $\varphi(u,v) = F(s(u,v)) + uG(s(u,v))$, as in Definition 1.6.*

One has for any $\psi \in C_0^2(S)$,

(3.6) $$\mathcal{V}_{II}^H(S,\psi X_1) =$$
$$= \int_{\mathbb{R}\times J} \left(\left(\frac{\partial}{\partial u}(\psi\circ\Psi)(u,s) \right)^2 \times \frac{G'(s)u^2/2 + F'(s)u + \sigma'(s)}{(1+G(s)^2)^{3/2}} + \frac{(\psi\circ\Psi)(u,s)^2}{(1+G(s)^2)^{3/2}} \frac{F'(s)^2 - 2\sigma'(s)G'(s)}{G'(s)u^2/2 + F'(s)u + \sigma'(s)} \right) \mathrm{d}u\,\mathrm{d}s,$$

where we have indicated with $\Psi : \mathbb{R}\times J \to \Omega$ *the diffeomorphism defined by* (1.9).

Proof. Since every strict intrinsic graphical strip is an intrinsic X_1-graph, we can apply the second variation formula (3.3) in Theorem 3.4. In this formula we want to use the global diffeomorphism $\Psi : \mathbb{R}\times J \to \Omega$ to convert the integral on Ω to an integral on $\mathbb{R}\times J$. By (1.10)

$$\begin{aligned}
\det J_\Psi(u,s) &= \det\begin{pmatrix} 1 & 0 \\ v_u & v_s \end{pmatrix} \\
&= \det\begin{pmatrix} 1 & 0 \\ G(s)u + F(s) & G'(s)\dfrac{u^2}{2} + F'(s)u + \sigma'(s) \end{pmatrix} \\
&= G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s).
\end{aligned}$$

We emphasize that since we are assuming that S is a strict graphical strip, then (1.8) is in force, and therefore the Jacobian of Ψ is always different from zero. Recall that we are also assuming that Ψ is globally one-to-one. The Inverse Function Theorem gives at every point $(u,v) = \Psi(u,s)$

$$J_{\Psi^{-1}}(u,v) = \begin{pmatrix} 1 & 0 \\ -\dfrac{G(s)u+F(s)}{G'(s)u^2/2 + F'(s)u + \sigma'(s)} & \dfrac{1}{G'(s)u^2/2 + F'(s)u + \sigma'(s)} \end{pmatrix}.$$

We thus have

(3.7) $$\begin{aligned}
s_u &= -\frac{G(s)u + F(s)}{G'(s)u^2/2 + F'(s)u + \sigma'(s)}, \\
s_v &= \frac{1}{G'(s)u^2/2 + F'(s)u + \sigma'(s)}.
\end{aligned}$$

Using (3.7) and the assumption that $\varphi(u,v) = F(s) + uG(s)$, we thus find

$$\begin{aligned}\mathcal{B}_\varphi(\varphi) = \varphi_u + \varphi\varphi_v &= G(s) + (G'(s)u + F'(s))s_u + \varphi(G'(s)u + F'(s))s_v \\ &= G(s) - \frac{(F'(s)+uG'(s))(F(s)+uG(s))}{G'(s)u^2/2 + F'(s)u + \sigma'(s)} \\ &\qquad + \frac{(F'(s)+uG'(s))(F(s)+uG(s))}{G'(s)u^2/2 + F'(s)u + \sigma'(s)} = G(s).\end{aligned}$$

This gives,

$$(\mathcal{B}_\varphi(\varphi))_v = G'(s)s_v = \frac{G'(s)}{G'(s)u^2/2 + F'(s)u + \sigma'(s)},$$

$$(\varphi_v)^2 = (F'(s) + uG'(s))^2 s_v^2 = \left(\frac{F'(s)+uG'(s)}{G'(s)u^2/2 + F'(s)u + \sigma'(s)}\right)^2.$$

Combining these formulas yields

$$2(\mathcal{B}_\varphi(\varphi))_v - \varphi_v^2 = \frac{2\sigma'(s)G'(s) - F'(s)^2}{G'(s)u^2/2 + F'(s)u + \sigma'(s)}.$$

Substituting this into the second integral in the right-hand side of (3.3) gives

$$\begin{aligned}\mathcal{V}_{II}^H(S,\psi X_1) = \int_\Omega &\frac{1}{(1+G(s)^2)^{3/2}} \\ &\times\left(\mathcal{B}_\varphi(\psi)^2 + \psi^2\left(\frac{F'(s)^2 - 2\sigma'(s)G'(s)}{G'(s)u^2/2 + F'(s)u + \sigma'(s)}\right)\right)\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

Now, to complete the proof, we make the change of variable $(u,v) = \Psi(u,s)$, with $(u,s) \in \mathbb{R}\times J$. The Jacobian of such diffeomorphism is given by (1.10) which gives

$$\mathrm{d}u\,\mathrm{d}v = \left(G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)\right)\mathrm{d}u\,\mathrm{d}s.$$

Observe furthermore that

$$\begin{aligned}\mathcal{B}_\varphi(\psi) = \psi_u + \varphi\psi_v &= \psi_u + (F+Gu)\psi_v = \psi_u + v_u\psi_v \\ &= \frac{\partial}{\partial u}\psi(u,v(u,s)) = \frac{\partial}{\partial u}(\psi\circ\Psi)(u,s).\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}\mathcal{V}_{II}^H(S,\psi X_1) = \int_{\mathbb{R}\times J}&\left(\left(\frac{\partial}{\partial u}(\psi\circ\Psi)(u,s)\right)^2\frac{G'(s)u^2/2 + F'(s)u + \sigma'(s)}{(1+G(\sigma)^2)^{3/2}}\right. \\ &\left.+\frac{(\psi\circ\Psi)(u,s)^2}{(1+G(s)^2)^{3/2}}\frac{F'(s)^2 - 2\sigma'(s)G'(s)}{G'(s)u^2/2 + F'(s)u + \sigma'(s)}\right)\mathrm{d}u\,\mathrm{d}s,\end{aligned}$$

which proves (3.6). ◻

4. Proof of Theorem B: Strict Intrinsic Graphical Strips Are Unstable

In this section we suitably adapt some of the ideas in [11] and the modifications of [13] to prove Theorem B. We do this by constructing a variation which strictly decreases the horizontal area of a strict intrinsic graphical strip, that is, we find a test function ψ for which $\mathcal{V}_{II}^{H}(S, \psi X_1) < 0$. This proves that such surfaces are unstable, thus establishing Theorem B.

To construct such a ψ we start by constructing a sequence ψ_k. We will show that for large enough k, we have $\mathcal{V}_{II}^{H}(S, \psi_k X_1) < 0$. For any given $\delta > 0$, we fix a function $\chi \in C_0^\infty(\mathbb{R})$ so that $0 \le \chi(s) \le 1$, $\chi(s) = 1$ for $|s| \le \delta$, $\chi(s) = 0$ for $|s| \ge 2\delta$, and $|\chi'| \le C = C(\delta)$. For each $k \in \mathbb{N}$, we let $\chi_k(s) = \chi(s/k)$ and hence

- $\chi_k(s) = 0$ for $|s| \ge 2\delta k$,
- $\chi_k(s) = 1$ for $|s| \le \delta k$,
- $|\chi_k'(s)| \le C/k$.

Next, fix a function $\zeta \in C_0^\infty(\mathbb{R})$ with $\zeta \ge 0$, $\operatorname{supp}(\zeta) = [-1, 1]$ and $\int_{\mathbb{R}} \zeta \, \mathrm{d}s = 1$. Letting $\zeta_k(s) = k\zeta(ks)$, we have that

$$\operatorname{supp}(\zeta_k) = [-1/k, 1/k] \quad \text{and} \quad \int_{\mathbb{R}} \zeta_k(s) \, \mathrm{d}s = 1.$$

Let F, G and σ be the functions in Definition 1.6 with

$$F'(s)^2 - 2\sigma'(s)G'(s) < 0 \quad s \in J. \tag{4.1}$$

As we have mentioned in the introduction, without loss of generality we assume that $G', \sigma' > 0$ in J. We define $F_k = F \star \zeta_k$, $G_k = G \star \zeta_k$, $\sigma_k = \sigma \star \zeta_k$. Since F, G and σ are continuous on J, shrinking J slightly if necessary, we may assume that they are uniformly continuous on $\bar{J}$. Therefore $F_k \to F$, $F_k' \to F'$, $G_k \to G$, $G_k' \to G'$, $\sigma_k \to \sigma$ and $\sigma_k' \to \sigma'$ uniformly on $\bar{J}$. The condition (4.1) now carries over to F_k, G_k, σ_k, that is, there is a positive integer k_0 such that if $k > k_0$ (relabeling the sequence if necessary, we take $k_0 = 1$) then for every $s \in J$, $F_k'(s)^2 - 2\sigma_k'(s)G_k'(s) < 0$. The left hand side of this inequality is precisely the discriminant of the quadratic expression in the variable u:

$$G_k'(s)\frac{u^2}{2} + F_k'(s)u + \sigma_k'(s).$$

Since the discriminant is strictly negative, $G_k'(s)u^2/2 + F_k'(s)u + \sigma_k'(s)$ never vanishes for $u \in \mathbb{R}$ and $s \in J$. Next, we construct a sequence of test functions ψ_k

to be used in the formula (3.6). We let

$$\psi_k(u,s) \stackrel{\text{def}}{=} \frac{\chi(s)\chi_k(u)}{(G_k'(s)u^2/2 + F_k'(s)u + \sigma_k'(s))^{1/2}}. \tag{4.2}$$

We note that $\psi_k \in C_0^\infty(\mathbb{R}\times J)$ due to the above considerations. With ψ_k in hand, we analyze $\mathcal{V}_{II}^H(S, \psi_k X_1)$. Before proceeding to the computations, we remark that the function ψ in (3.6) is defined on $\Omega = \Psi(\mathbb{R}\times J)$. Our ψ_k's have been already defined on the (u,s) space, that is on $\mathbb{R}\times J$. Therefore, occurrences of $\psi\circ\Psi$ in (3.6) will be replaced by ψ_k in the proof of the subsequent two lemmas. We start with the second term in the right hand side of (3.6).

Lemma 4.1. *We have*

$$\begin{aligned}\lim_{k\to\infty}\int_{\mathbb{R}\times J} &\frac{\psi_k(u,s)^2}{(1+G(s)^2)^{3/2}}\frac{F'(s)^2 - 2\sigma'(s)G'(s)}{G'(s)u^2/2 + F'(s)u + \sigma'(s)}\,\mathrm{d}u\,\mathrm{d}s \\ &= -2\pi\int_J \frac{\chi(s)^2}{(1+G(s)^2)^{3/2}}\frac{G'(s)}{(2\sigma'(s)G'(s) - F'(s)^2)^{1/2}}\,\mathrm{d}s.\end{aligned}$$

Proof. Substituting the quantity $\psi\circ\Psi$ with ψ_k in the second term of the right hand side of (3.6) and recalling the definition of ψ_k we have

$$\lim_{k\to\infty}\int_{\mathbb{R}\times J}\frac{\psi_k(u,s)^2}{(1+G(s)^2)^{3/2}}\frac{F'(s)^2 - 2\sigma'(s)G'(s)}{G'(s)u^2/2 + F'(s)u + \sigma'(s)}\,\mathrm{d}u\,\mathrm{d}s \tag{4.3}$$

$$\begin{aligned}&= \lim_{k\to\infty}\int_J \chi(s)^2\frac{F'(s)^2 - 2\sigma'(s)G'(s)}{(1+G(s)^2)^{3/2}} \\ &\quad\times\left(\int_{\mathbb{R}}\frac{\chi_k(u)^2}{(G_k'(s)u^2/2 + F_k'(s)u + \sigma_k'(s))(G'(s)u^2/2 + F'(s)u + \sigma'(s))}\,\mathrm{d}u\right)\mathrm{d}s \\ &= \int_J \chi(s)^2\frac{F'(s)^2 - 2\sigma'(s)G'(s)}{(1+G(s)^2)^{3/2}}\left(\int_{\mathbb{R}}\frac{1}{(G'(s)u^2/2 + F'(s)u + \sigma'(s))^2}\,\mathrm{d}u\right)\mathrm{d}s.\end{aligned}$$

In the above, we have used the fact that since for each $u\in\mathbb{R}$,

$$G_k'(s)\frac{u^2}{2} + F_k'(s)u + \sigma_k'(s) \longrightarrow G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s) \quad \text{as } k\to\infty$$

uniformly for $s\in\bar{J}$, and the latter quantity never vanishes, we have

$$\begin{aligned}\frac{1}{2}\left|G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)\right| &< \left|G_k'(s)\frac{u^2}{2} + F_k'(s)u + \sigma_k'(s)\right| \\ &< 2\left|G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)\right|.\end{aligned}$$

Hence, Lebesgue dominated convergence theorem allows taking the limit inside the integral. Next, we want to compute the integral

$$\int_{\mathbb{R}} \frac{1}{(G'(s)u^2/2 + F'(s)u + \sigma'(s))^2}\, du.$$

Using standard integration techniques we obtain

$$\int \frac{1}{(Au^2 + Bu + C)^2}\, du$$
$$= \frac{2Au + B}{(4AC - B^2)(Au^2 + Bu + C)} + \frac{4A}{(4AC - B^2)^{3/2}} \arctan\left(\frac{2Au + B}{\sqrt{4AC - B^2}}\right).$$

This implies if $A > 0$

$$\int_{\mathbb{R}} \frac{1}{(Au^2 + Bu + C)^2}\, du = \frac{4\pi A}{(4AC - B^2)^{3/2}}.$$

Since we have that $G'(s) > 0$, letting $A = G'(s)/2$, $B = F'(s)$ and $C = \sigma'(s)$ we have

$$\int_{\mathbb{R}} \frac{1}{(G'(s)u^2/2 + F'(s)u + \sigma'(s))^2}\, du = 2\pi \frac{G'(s)}{(2\sigma'(s)G'(s) - F'(s)^2)^{3/2}}. \tag{4.4}$$

Substituting (4.4) in (4.3) we reach the desired conclusion. ◻

Now we turn to the first term in the right hand side of (3.6).

Lemma 4.2. *We have*

$$\lim_{k\to\infty} \int_{\mathbb{R}\times J} \left(\left(\frac{\partial \psi(u,s)}{\partial u}\right)^2 \frac{G'(s)u^2/2 + F'(s)u + \sigma'(s)}{(1 + G(s)^2)^{3/2}} \right) du\, ds$$
$$= \frac{\pi}{2} \int_J \frac{\chi(s)^2}{(1 + G(s)^2)^{3/2}} \frac{G'(s)}{(2\sigma'(s)G'(s) - F'(s)^2)^{1/2}}\, ds.$$

Proof. Again, we closely follow the development in [13]. By recalling (4.2) we first obtain

$$\frac{\partial \psi_k}{\partial u}(u,s) = \frac{\chi(s)}{2} \left(\frac{2\chi_k'(u) Q_k(u,s) - \chi_k(u) D_k(u,s)}{Q_k(u,s)^{3/2}} \right),$$

where we have let

$$Q_k(u,s) = G_k'(s)\frac{u^2}{2} + F_k'(s)u + \sigma_k'(s)$$

and

$$D_k(u,s) = uG_k'(s) + F_k'(s).$$

For the computations that follow, it is convenient to also let

$$\begin{aligned}
Q(u,s) &= G'(s)\frac{u^2}{2} + F'(s)u + \sigma'(s)\,, \\
D(u,s) &= \frac{\partial}{\partial u}Q(u,s) = uG'(s) + F'(s).
\end{aligned}$$

It follows that

$$\begin{aligned}
\left(\frac{\partial\psi_k}{\partial u}(u,s)\right)^2 &= \\
= \chi(s)^2 &\left(\frac{\chi_k'(u)^2}{Q_k(u,s)} - \frac{1}{2}(\chi_k(u)^2)'\frac{D_k(u,s)}{Q_k(u,s)^2} + \frac{1}{4}\chi_k(u)^2\frac{D_k(u,s)^2}{Q_k(u,s)^3}\right).
\end{aligned}$$

Substituting the quantity $\psi \circ \Psi$ in the first term of the right hand side of (3.6), and using the above expression for $\psi_{k,u}$, we have

$$\begin{aligned}
\int_{\mathbb{R}\times J} &\left(\frac{\partial\psi_k(u,s)}{\partial u}\right)^2 \frac{G'(s)u^2/2 + F'(s)u + \sigma'(s)}{(1+G(s)^2)^{3/2}}\,\mathrm{d}u\,\mathrm{d}s \\
&= \int_J \frac{\chi(s)^2}{(1+G'(s)^2)^{3/2}}\left(\boxed{1} + \boxed{2} + \boxed{3}\right)\mathrm{d}s,
\end{aligned}$$

where

$$\begin{aligned}
\boxed{1} &= \int_{\mathbb{R}} \chi_k'(u)^2\frac{Q(u,s)}{Q_k(u,s)}\,\mathrm{d}u, \\
\boxed{2} &= -\frac{1}{2}\int_{\mathbb{R}} (\chi_k^2(u))'Q(u,s)\frac{D_k(u,s)}{Q_k(u,s)^2}\,\mathrm{d}u, \\
\boxed{3} &= \frac{1}{4}\int_{\mathbb{R}} \chi_k(u)^2Q(u,s)\frac{D_k(u,s)^2}{Q_k(u,s)^3}\,\mathrm{d}u.
\end{aligned}$$

Since $|\chi_k'(u)| \le C/k$, by Lebesgue dominated convergence theorem we have

$$\lim_{k\to\infty}\boxed{1} = 0. \tag{4.5}$$

In addition, since $D_k(u,s) \to D(u,s)$, $Q_k(u,s) \to Q(u,s)$, and $\chi_k(s) \to 1$ when $k \to \infty$, we obtain

$$\lim_{k\to\infty} \boxed{3} = \frac{1}{4}\int_{\mathbb{R}} \frac{D(u,s)^2}{Q(u,s)^2}\,\mathrm{d}u \tag{4.6}$$
$$= -\frac{1}{4}\int_{\mathbb{R}} \frac{\partial}{\partial u}Q(u,s)\frac{\partial}{\partial u}\left(\frac{1}{Q(u,s)}\right)\mathrm{d}u$$
$$= \frac{1}{4}\int_{\mathbb{R}} \frac{\partial^2 Q(u,s)}{\partial u^2}\frac{1}{Q(u,s)}\,\mathrm{d}u$$
$$= \frac{1}{4}\int_{\mathbb{R}} \frac{G'(s)}{G'(s)u^2/2 + F'(s)u + \sigma'(s)}\,\mathrm{d}u$$
$$= \frac{\pi G'(s)}{(2\sigma'(s)G'(s) - F'(s)^2)^{1/2}}.$$

The third equality above is obtained by integration by parts whereas in the last equality, we have used the fact that $G'(s) > 0$ and standard calculus techniques. Now we turn to the quantity $\boxed{2}$.

$$\lim_{k\to\infty} \boxed{2} = -\lim_{k\to\infty} \frac{1}{2}\int_{\mathbb{R}} (\chi_k(u)^2))' Q(u,s)\frac{D_k(u,s)}{Q_k(u,s)}\,\mathrm{d}u \tag{4.7}$$
$$= -\lim_{k\to\infty} \frac{1}{2}\int_{\mathbb{R}} \chi_k(u)^2 \frac{\partial}{\partial u}\left(\frac{Q(u,s)D_k(u,s)}{Q_k(u,s)^2}\right)\mathrm{d}u$$
$$= -\lim_{k\to\infty} \frac{1}{2}\int_{\mathbb{R}} \chi_k(u)^2 \left(\frac{Q_u(u,s)\,\mathrm{d}_k(u,s)}{Q_k(u,s)^2} + \frac{Q(u,s)D_{k,u}(u,s)}{Q_k(u,s)^2} - 2\frac{Q(u,s)\,\mathrm{d}_k(u,s)Q_{k,u}(u,s)}{Q_k(u,s)^3}\right)\mathrm{d}u$$
$$= -\frac{1}{2}\int_{\mathbb{R}} \frac{Q_u(u,s)D(u,s)}{Q(u,s)^2} + \frac{D_u(u,s)}{Q(u,s)} - 2\frac{D(u,s)Q_u(u,s)}{Q(u,s)^2}\,\mathrm{d}u$$
$$= -\frac{1}{2}\int_{\mathbb{R}} \frac{G'(s)}{Q(u,s)}\,\mathrm{d}u - \frac{1}{2}\int_{\mathbb{R}} \frac{\partial}{\partial u}Q(u,s)\frac{\partial}{\partial u}\left(\frac{1}{Q(u,s)}\right)\mathrm{d}u$$
$$= -\frac{1}{2}\int_{\mathbb{R}} \frac{G'(s)}{Q(u,s)}\,\mathrm{d}u + \frac{1}{2}\int_{\mathbb{R}} \frac{Q_{uu}(u,s)}{Q(u,s)}\,\mathrm{d}u = 0,$$

since $Q_{uu}(u,s) = G'(s)$. Combining (4.5), (4.6) and (4.7), we obtain the desired conclusion. □

Combining (3.6) with Lemmas 4.1 and 4.2 we can now prove Theorem B in the introduction.

Proof of Theorem B. Let ψ_k be the function constructed in (4.2) and consider $\psi_k \circ \Psi^{-1} \in C_0^2(\Omega)$, where Ψ is the diffeomorphism in (1.9). If we lift this function to the surface, and by abuse of notation we continue to indicate with ψ_k such lifted function, we obtain a function in $C_0^2(S)$. From Corollary 3.6, Lemmas 4.1, 4.2 and the fact that $G'(s) > 0$ on J we deduce that

$$\lim_{k\to\infty} \mathcal{V}_{II}^H(S, (\psi_k X_1)) =$$

$$= \left(\frac{\pi}{2} - 2\pi\right) \int_J \frac{\chi(s)^2}{(1+G(s)^2)^{3/2}} \frac{G'(s)}{(2\sigma'(s)G'(s) - F'(s)^2)^{1/2}}\, ds < 0.$$

Therefore, for large enough k we have $\mathcal{V}_{II}^H(S, \psi_k X_1) < 0$. This completes the proof. ◻

5. PROOF OF THEOREM C: EXISTENCE OF STRICT INTRINSIC GRAPHICAL STRIPS

The main objective of this section is establishing that every complete minimal surface without boundary and with empty characteristic locus contains a strict intrinsic graphical strip, unless the surface is a vertical plane. This will prove Theorem C in the introduction. Our approach hinges on the following basic representation theorem for minimal surfaces which is a consequence of the results in [21], and which, in a different way, has already proved crucial in [13].

Theorem 5.1. *Let S be a C^2 complete embedded non-characteristic minimal surface without boundary and assume that it is not a vertical plane. Let $g_0 \in S$ be a point admitting a neighborhood (in S) that may be written as a graph over the plane $t = 0$. There exist a neighborhood U of g_0, an interval J, and functions $h_0 \in C^2(J)$, $\gamma \in C^3(J, \mathbb{R}^2)$, with $|\gamma'(s)| = 1$ for $s \in J$, such that U is parameterized by $\mathcal{L} : \mathbb{R} \times J \to \mathbb{H}$*

$$\mathcal{L}(r,s) = \left(\gamma(s) + r(\gamma')^{\perp}(s), h_0(s) - \frac{r}{2}\gamma(s)\cdot\gamma'(s)\right) \tag{5.1}$$

for $s \in J$, $r \in \mathbb{R}$. Moreover, with $W_0(s) = h_0'(s) + \frac{1}{2}\gamma' \cdot \gamma^{\perp}(s)$ and $\kappa(s) = \gamma'' \cdot (\gamma')^{\perp}$, we have that

$$1 - 2W_0(s)\kappa(s) < 0, \quad s \in J. \tag{5.2}$$

The proof of Theorem 5.1 will be presented after Corollary 5.5 below. We first develop some preparatory results. Our first lemma was proved in [21], but it is also contained in Proposition 4.1 of [8].

Lemma 5.2. *Let $D \subset \mathbb{R}^2$ be an open set, $g \in C^2(D)$, and consider the C^2 map $G : D \to \mathbb{H}^1$ given by $G(x,y) = (x, y, g(x,y))$. Suppose that $S = G(D)$ is a non-characteristic minimal surface. Then S is foliated by horizontal straight lines which are the integral curves of $\mathbf{v}_H^{\perp} = \bar{q}X_1 - \bar{p}X_2$.*

The next lemma was also proved in [21], but the reader should also see Proposition 6.16 in the subsequent work [27].

Lemma 5.3. *Suppose S be a C^2 non-characteristic minimal surface such that no open subset of S may be written as a graph over the xy-plane. Then, S is a piece of a vertical plane and, hence, is foliated by horizontal straight lines which are the integral curves of $\mathbf{v}_H^\perp$.*

In the next lemma we combine into a single result the two different situations considered in Lemmas 5.2 and 5.3.

Lemma 5.4. *Let S be a C^2 minimal surface in $\mathbb{H}^1$ with empty characteristic locus, and let p be a point in the interior of S (in the relative topology). Then, there exists a neighborhood Δ of p in S which is foliated by horizontal straight line segments which are integral curves of $\mathbf{v}_H^\perp$.*

Proof. For every $p \in \mathring{S}$, there exists an open set $U \subset \mathbb{H}^1$ and a $\varphi \in C^2(U)$ such that $\nabla\varphi \neq 0$ in U and $\Sigma = S \cap U = \{(x, y, t) \in U \mid \varphi(x, y, t) = 0\}$. Let $S_1 = \{(x, y, t) \in \Sigma \mid \varphi_t(x, y, t) \neq 0\}$, $S_2 = \{(x, y, t) \in \Sigma \mid \varphi_t(x, y, t) = 0\}$. Notice that, either $\varphi_t \equiv 0$ on Σ and in such case $S_2 = \Sigma$ is a vertical cylinder over a curve in the xy plane, or there exists an open set $V \subset \mathbb{H}^1$ such that $S_2 \cap V$ is a C^1 curve in $\mathbb{H}^1$. In the former case we can invoke Lemma 5.3 to conclude that $\Delta = \Sigma$ is foliated by horizontal straight line segments which are integral curves of $\mathbf{v}_H^\perp$. We are thus left with the case in which $S_1 \neq \emptyset$. By shrinking Σ if necessary we can assume that $\Sigma = S_1 \cup S_2$, where S_2 is a C^1 curve.

In our arguments, we consider integral curves of $\mathbf{v}_H^\perp$ passing through points on the surface S. To make this notion precise, we recall that as S is a C^2 submanifold of $\mathbb{H}^1 = \mathbb{R}^3$, every point $p \in S$ is contained in a coordinate chart $i : D \subset \mathbb{R}^2 \to S$ with $i \in C^2(D)$. For any C^1 vector field, U_0, defined on $i(D)$, the integral curve of U_0 passing through $q \in i(D)$ is simply $i(\gamma)$ where $\gamma \subset D$ is a solution to the initial value problem:

$$\begin{aligned} \gamma'(t) &= i_*^{-1}(U_0)(\gamma(t)), \\ \gamma(0) &= i^{-1}(q). \end{aligned}$$

Direct calculation then shows that

$$\frac{d}{dt} i(\gamma) = i_* i_*^{-1} U_0(\gamma(t)) = U_0(i(\gamma(t))),$$

and $i(\gamma(0)) = i(i^{-1}(q)) = q$. As U_0 (and hence $i^* U_0$) is C^1, the standard theorems concerning solutions to ODE apply to the integral curves of U_0 on S. In particular, we may conclude that given $q \in S$, there exists (at least for a short time) a unique integral curve of U_0. Similarly, we conclude that integral curves of U_0 on S have continuous dependence on parameters.

By Lemma 5.2, each point in S_1 is contained in a neighborhood which is foliated by straight line segments which are integral curves of $\nu_H^\perp$. Thus, those portions of integral curves of $\mathbf{v}_H^\perp$ contained in S_1 are at least piecewise linear. By the fact that $\mathbf{v}_H^\perp$ is C^1 and the uniqueness of solutions to ode's, we must have that these portions of integral curves are straight lines. We may extend each such line segment maximally within S_1. If a limit point of a maximally extended line segment were in S_1, we could apply Lemma 5.2 to extend it further, violating the assumption that we had extended maximally. Thus we conclude that the limit points of the line segment are in $\partial S_1 \cup S_2$.

Consider $p \in S_2$ and let c be the integral curve of $\mathbf{v}_H^\perp$ with $c(0) = p$. Let B_ε be the metric ball of radius ε centered at p and $c_\varepsilon = c \cap B_\varepsilon$. Then, there exists an $\varepsilon > 0$ sufficiently small so that one of the following possibilities occurs:

(1) $c_\varepsilon \cap S_2$ is closed and has no interior;
(2) $c_\varepsilon \cap S_2$ is closed with nonempty interior and p is in the interior;
(3) $c_\varepsilon \cap S_2$ is closed with nonempty interior and p is contained in the boundary of the interior of $c_\varepsilon \cap S_2$.

In the first case, $c_\varepsilon \cap S_1$ is open and dense in c_ε. By Lemma 5.2, every point in $c_\varepsilon \cap S_1$ is contained in an open line segment which is a subset of c_ε. As $c_\varepsilon \cap S_2$ is closed and is contained in the boundary of $c_\varepsilon \cap S_1$, we conclude that c_ε is piecewise linear. By the smoothness of $\mathbf{v}_h^\perp$ and the uniqueness of solutions to ODE, we conclude c_ε is a single straight line segment.

In the second case, we may shrink ε so that $c_\varepsilon \cap S_2 = c_\varepsilon$ and S_2 divides $B_\varepsilon \cap S$ into exactly two pieces N_1, N_2. We next show that if $q \in N_1$ is contained in a line segment, $L \subset N_1$, which reaches the boundary of N_1, then the length of L is at least $2(\varepsilon - \delta)$ where δ is the Euclidean distance from p to q. Observe that the endpoints of L cannot be in S_2. If one were in S_2, then by the uniqueness of solutions of ODE, we conclude that L and S_2 coincide. This contradicts our assumption that $q \notin S_2$. Thus, L must be a line segment in B_ε which has both its boundary points in ∂B_ε. By construction, the Euclidean distance from p to the endpoints of L is ε. Denoting the Euclidean distance from p to q by δ, the triangle inequality implies that the length of L is at least $2(\varepsilon - \delta)$.

Let $q_i \in N_1$ be a sequence of points converging to p and let L_i be the maximal line segment which is the integral curve of $\mathbf{v}_H^\perp$ through q_i which is contained in N_1. By the continuous dependence on parameters of the solutions to an ODE and the fact the $\mathbf{v}_H^\perp$ is C^1, we know $L = \lim_{i\to\infty} L_i$ exists and is an integral curve of $\mathbf{v}_H^\perp$ passing through p. Moreover, since L is the limit of lines segments each of whose lengths are bounded below by $2(\varepsilon - \delta_i)$ (where δ_i is the Euclidean distance from p to q_i), we conclude L is a line segment of length at least 2ε. Note that so far, we have shown that every point in S_1 and every point in S_2 that fall in cases one and two are contained in an open line segment which is an integral curve of $\mathbf{v}_H^\perp$.

We are left with points of S_2 which fall into the third category. The collection of such points in S_2 is, by construction, closed and has empty interior. Thus, c_ε contains an open dense set of points that are either in S_1 or fall in one of the first two cases above. For each such points, Lemma 5.2 or the discussion of the first two cases yields an open line segment containing the point which is a subset of c_ε. Thus, as in the argument for case one, c_ε is piecewise linear and, by the smoothness of $\mathbf{v}_H^\perp$, must be a single straight line segment.

Using the arguments above for points in S_2 and Lemma 5.2 for points in S_1, we see that integral curve of $\mathbf{v}_H^\perp$ through any point contains a line segment through that point. Thus, all such integral curves are piecewise linear and, by the smoothness of $\mathbf{v}_H^\perp$, must be straight lines. Combining all of these arguments shows that Σ is foliated by straight line segments which are integral curves of $\mathbf{v}_H^\perp$. □

Corollary 5.5. *Let S be a C^2 connected complete non-characteristic minimal surface without boundary in $\mathbb{H}^1$. Then, S is foliated by horizontal straight lines which are integral curves of $\mathbf{v}_H^\perp$.*

Proof. Since S is assumed to have no boundary, for any $p \in S$ Lemma 5.4 implies that there exists an open neighborhood of p which is foliated by such straight line segments. By the smoothness of $\mathbf{v}_H^\perp$, we have that S itself is foliated by such straight line segments. It remains to show that the entirety of each line is contained in S.

Let $L : (-\varepsilon, \varepsilon) \to S$ be a line segment with $L(0) = p \in S$ and $L'(t) = \mathbf{v}_H^\perp(L(t))$ and let $\tilde{L} : \mathbb{R} \to \mathbb{H}^1$ be the full line containing L so that $\tilde{L}(t) = L(t)$ for $t \in (-\varepsilon, \varepsilon)$. Let

$$I = \{t \in \mathbb{R} \mid \tilde{L}(t) \in S\}.$$

By construction, I is not empty since $0 \in I$. Let $t_i \in I$ be a sequence of parameters so that $t_i \to t_\infty$ where t_∞ is a limit point of I. By completeness of S, we must have that $\lim_{i\to\infty} \tilde{L}(t_i) = \tilde{L}(t_\infty)$ is an element of S. Thus, I is closed as it must contain all of its limit points. But, I is open as well. To see this, consider $p = \tilde{L}(t)$ for a fixed $t \in I$. As $\partial S = \emptyset$, p is in the interior of S and so, by Lemma 5.4, p is contained in a neighborhood which is foliated by straight lines which are integral curves of $\mathbf{v}_H^\perp$. Thus, I must contain an open neighborhood of t. Since I is both open and closed, we conclude that $I = \mathbb{R}$ and that $\tilde{L}(\mathbb{R}) \subset S$. □

Proof of Theorem 5.1. By Corollary 5.5, we have that S is foliated by horizontal straight lines which are integral curves of $\mathbf{v}_H^\perp$. Let O be an open neighborhood of g_0 which may be written as a graph $(x, y, h(x, y))$ with $h \in C^2$. Consider a unit tangential vector field, $\mathcal{W}$, defined on O which is perpendicular (with respect to the fixed Riemannian metric) to $\mathbf{v}_H^\perp$. Let $(\gamma_1(s), \gamma_2(s), h_0(s))$ be an integral curve of $\mathcal{W}$ so that $\gamma(0) = g_0$ with domain J. Note that, as $\mathbf{v}_H^\perp$ is C^1, we have γ_1, γ_2, $h_0 \in C^2(J)$. Let N be the collection of lines in the foliation which pass through point of the curve $(\gamma_1(J), \gamma_2(J), h_0(J))$. From the formula

$$aX_1 + bX_2 + cT = \left(a, b, c + \frac{bx - ay}{2}\right), \tag{5.3}$$

which allows to pass from the standard representation in terms of the Cartesian coordinates in $\mathbb{H}^1$ to that with respect to the orthonormal basis $\{X_1, X_2, T\}$, for a fixed $s_0 \in J$ we find

$$\begin{aligned}\mathcal{L}'_{s_0}(r) &= \Big(\gamma_2'(s_0), -\gamma_1'(s_0), -\frac{1}{2}(\gamma_1(s_0), \gamma_2(s_0)) \cdot (\gamma_1'(s_0), \gamma_2'(s_0))\Big)\\ &= \gamma_2'(s_0)X_1 - \gamma_1'(s_0)X_2 = \boldsymbol{\nu}_H^{\perp}.\end{aligned}$$

Then, the line of the foliation passing through $(\gamma_1(s_0), \gamma_2(s_0), h_0(s_0))$ is given by

$$\begin{aligned}\mathcal{L}_{s_0}(r) = \Big(&\gamma_1(s_0) + r\gamma_2'(s_0), \gamma_2(s_0) - r\gamma_1'(s_0), h_0(s_0)\\ &- \frac{r}{2}(\gamma_1(s_0), \gamma_2(s_0)) \cdot (\gamma_1'(s_0), \gamma_2'(s_0))\Big).\end{aligned}$$

Thus, N may be parametrized by $\mathcal{L} : \mathbb{R} \times J \to \mathbb{H}^1$ given by

$$\mathcal{L}(r, s) = \Big(\gamma_1(s) + r\gamma_2'(s), \gamma_2(s) - r\gamma_1'(s), h_0(s) - \frac{r}{2}\gamma(s) \cdot \gamma'(s)\Big). \tag{5.4}$$

It remains to show that $\gamma = (\gamma_1, \gamma_2) \in C^3(J)$. As O is a graph over a region $\bar{O}$ of the xy-plane, $\bar{\mathcal{L}}(r_0, s) = (\gamma_1(s) + r\gamma_2'(s), \gamma_2(s) - r\gamma_1'(s))$ parametrizes a subset of $\bar{O}$ with $s \in J$, $r \in (-\varepsilon, \varepsilon)$ for ε sufficiently small. Under this parametrization, $V = \bar{p}\,\partial_x + \bar{q}\,\partial_y = \gamma_1'(s)\,\partial_x + \gamma_2'(s)\,\partial_y$. We first observe that, for a fixed $r = r_0$, the curve $s \to \bar{\mathcal{L}}(r_0, s)$ coincides with the integral curve of V through the point $\bar{\mathcal{L}}(r_0, 0)$ on their mutual domain of definition (we may assume, by shrinking J if necessary, that J is the mutual domain of definition). To see this, note that the definition of $\bar{\mathcal{L}}$ gives

$$\bar{\mathcal{L}}_s(r, s) = (\gamma_1'(s) + r\gamma_2''(s), \gamma_2'(s) - r\gamma_1''(s)).$$

This implies

$$\langle \bar{\mathcal{L}}_s(r_0, s), V^{\perp}\rangle = \gamma_2'\gamma_1' + r\gamma_2'\gamma_2'' - \gamma_1'\gamma_2' + r\gamma_1''\gamma_1' = 0.$$

The last equality follows from the fact that $|\gamma'| \equiv 1$ on J. Let $\bar{c} \subset \mathbb{R}^2$ be the integral curve of V passing through $\bar{\mathcal{L}}(r_0, 0)$. We note that $\bar{c}$ is parameterized by arc-length and, to avoid confusion, we will denote its parameter by ξ. Since V is C^1, we have that $\bar{c} \in C^2(\xi)$. Moreover, since O is given by $(x, y, h(x, y))$ with $h \in C^2$, we see that $c(\xi) = h(\bar{c}(\xi))$ is $C^2(\xi)$ as well.

To facilitate our computations, we note that

$$|\bar{\mathcal{L}}_s(r_0, s)| = |1 - r_0\kappa(s)|.$$

This can be verified as follows. Recalling that $|\gamma'| = 1$ and that $\kappa = \gamma_1''\gamma_2' - \gamma_2''\gamma_1'$, one easily obtains

$$|\bar{\mathcal{L}}_s(r_0, s)|^2 = 1 - 2r\kappa(s) + r^2(\gamma_1''(s)^2 + \gamma_2''(s)^2).$$

Now, some elementary considerations give

$$\kappa(s)^2 = ((\gamma_1''(s)^2 + \gamma_2''(s)^2)|\gamma'(s)|^2 - 2(\gamma'(s) \cdot \gamma''(s))^2 = (\gamma_1''(s)^2 + \gamma_2''(s)^2),$$

and this implies the desired conclusion. Let now $\kappa_0 = \sup_{s \in J} |\kappa(s)|$. If $\kappa_0 = 0$, then γ is a line segment and hence γ is certainly C^3. Assuming $\kappa_0 > 0$, we pick $r_0 < \min\{\kappa_0^{-1}, \varepsilon\}$ which implies that $|\bar{\mathcal{L}}_s(r_0, s)| = |1 - r_0\kappa(s)| = 1 - r_0\kappa(s)$. We note that ξ is differentiable in s as $\bar{c}(\xi)$ is the reparameterization by arclength of $\bar{\mathcal{L}}(r_0, s)$ and that $d\xi/ds = 1 - r_0\kappa(s)$. Similarly,

$$\frac{ds}{d\xi} = \frac{1}{1 - r_0\kappa(s)}$$

which, by our choice of r_0, is equal to $\sum_{n=0}^{\infty}(r_0\kappa(s))^n$. Next, we compute

$$\begin{aligned}
c'(\xi) &= \frac{d}{d\xi}h(\bar{c}(\xi)) = \frac{\partial}{\partial s}(h(\gamma_1(s) + r\gamma_2'(s), \gamma_2(s) - r\gamma_1'(s)))\frac{ds}{d\xi} \\
&= \frac{\partial}{\partial s}\left(h_0(s) - \frac{r_0}{2}\gamma(s) \cdot \gamma'(s)\right)\frac{1}{1 - r_0\kappa(s)} \\
&= \left(h_0'(s) - \frac{r_0}{2} - \frac{r_0}{2}\gamma(s) \cdot \gamma''(s)\right)\frac{1}{1 - r_0\kappa(s)} \\
&= \left(h_0'(s) - \frac{r_0}{2} - \frac{r_0}{2}\gamma(s) \cdot \gamma''(s)\right)\left(\sum_{n=0}^{\infty}(r_0\kappa(s))^n\right) \\
&= h_0'(s) + r_0\alpha(s) + r_0^2\kappa(s)\alpha(s) + r_0^3\kappa(s)^2\alpha(s) + \cdots
\end{aligned}$$

where

$$\alpha(s) = -\frac{1}{2} - \frac{1}{2}\gamma(s) \cdot \gamma''(s) + \kappa(s)h_0'(s).$$

At this point we can make some simplifications. First, we note that as $\kappa(s) = \gamma'' \cdot (\gamma')^{\perp}$, and $\gamma' \cdot \gamma'' = 0$ (as $|\gamma'(s)| = 1$), we have

$$\gamma''(s) = \kappa(s)(\gamma'(s))^{\perp}.$$

So, letting $\beta(s) = -\frac{1}{2}\gamma \cdot (\gamma'(s))^{\perp} + h_0'(s)$, we rewrite $\alpha(s) = -\frac{1}{2} + \kappa(s)\beta(s)$. Moreover,

$$\begin{aligned}
r_0\alpha(s) + r_0^2\kappa(s)\alpha(s) + r_0^3\kappa(s)^2\alpha(s) + \cdots \\
&= r_0\alpha(s)\Big(\sum_{n=0}^{\infty}(r_0\kappa(s))^n\Big) \\
&= \frac{r_0\alpha(s)}{1 - r_0\kappa(s)} \\
&= -\left(\frac{r_0}{2}\frac{1}{1 - r_0\kappa(s)} - \beta(s)\frac{r_0\kappa(s)}{1 - r_0\kappa(s)}\right) \\
&= -\left(\frac{r_0}{2}\frac{1}{1 - r_0\kappa(s)} + \beta(s) - \frac{\beta}{1 - r_0\kappa(s)}\right) \\
&= -\left(\beta(s) + \frac{r_0 - 2\beta(s)}{1 - r_0\kappa(s)}\right).
\end{aligned}$$

We conclude that

$$c'(\xi) = h_0'(s) - \beta(s) - \frac{1}{2}\frac{r_0 - 2\beta(s)}{1 - r_0\kappa(s)}.$$

Since $c'(\xi)$ is again differentiable in ξ, and ξ is differentiable in s, we conclude, by the chain rule, that $c'(\xi)$ is also differentiable in s. Noting that $h_0'(s)$ and $\beta(s)$ are once differentiable in s, we conclude that $(1 - r_0\kappa(s))^{-1}$, and hence $\kappa(s)$, is differentiable in s. But, since $\gamma''(s) = \kappa(s)(\gamma'(s))^{\perp}$, $\gamma''(s)$ is differentiable and hence $\gamma \in C^3(s)$.

Lastly, we examine the impact of the assumption that S contains no characteristic points on the neighborhood N. Using the parametrization derived above, we see that the tangent space is spanned by $\nu_H^{\perp}$ and

$$\hat{W} = (\gamma_1'(s) + r\gamma_2''(s))X_1 + (\gamma_2'(s) - r\gamma_1''(s))X_2 + \left(W_0(s) - r + \frac{r^2}{2}\kappa(s)\right)T$$

where, as in the statement of the theorem, we let $W_0(s) = h_0'(s) + \frac{1}{2}\gamma' \cdot \gamma^{\perp}$ and $\kappa(s) = \gamma'' \cdot (\gamma')^{\perp}$. S will have a characteristic point when $\langle \hat{W}, T\rangle = 0$, i.e., when $r = (1 \pm \sqrt{1 - 2W_0(s)\kappa(s)}\,)/(2W_0(s))$. Thus, S is noncharacteristic if and only if $1 - 2W_0(s)\kappa(s) < 0$. $\square$

Note that, without loss of generality (by simply reparametrizing γ), we may assume that any fixed $s \in J$ may be treated as $s = 0$. We will use such a normalization and assume that J is a neighborhood of 0.

We wish to examine the behavior of this patch with respect to the notion of an X_1-graph. Consider the following definitions.

Definition 5.6. Let $C_1(x_0, y_0, t_0)$ denote the integral curve of the vector field X_1 passing through the point (x_0, y_0, t_0). In other words,

$$C_1(x_0, y_0, t_0) = \left\{\left(x_0 + r, y_0, t_0 - \frac{y_0}{2}r\right) \mid r \in \mathbb{R}\right\}.$$

Using Definition 5.6 we next introduce the notion of intrinsic projection of a point to the plane $x = 0$.

Definition 5.7. We define the *intrinsic projection map*

$$\Pi(x_0, y_0, t_0) = \{(0, y, t)\} \cap C_1(x_0, y_0, t_0) = (0, y_0, t_0 + y_0 x_0/2).$$

The following equation follows directly from the definition.

$$\Pi \circ \mathfrak{L}(r, s) = \tag{5.5}$$
$$= \left(0, \gamma_2(s) - r\gamma_1'(s), h_0(s) + \frac{1}{2}\gamma_1(s)\gamma_2(s) - r\gamma_1(s)\gamma_1'(s) - \frac{r^2}{2}\gamma_1'(s)\gamma_2'(s)\right).$$

Lemma 5.8. *Let S be a portion of a minimal surface parameterized by a seed curve/height function pair $(\gamma(s), h_0(s))$ via (5.1) with $r \in \mathbb{R}$, $s \in I$. Let $P(s, r) = \Pi \circ \mathfrak{L}(r, s)$ be given as in (5.5). There exists an interval $J \subset I$ containing XXXXX so that $P : \mathbb{R} \times J \subset \mathbb{R}^2_{(r,s)} \to \mathbb{R}^2_{(y,t)}$ is a one-to-one C^2 diffeomorphism onto its image.*

Proof. The following properties of the seed curve $\gamma : I \to \mathbb{R}^2$ are essential to our proof. We gather them here for the sake of convenience.

(i) $|\gamma'(s)| = 1$.
(ii) $1 - 2W_0(s)\kappa(s) < 0$.
(iii) There exists an interval $J \subset I$ such that for all $s \in J$, $\gamma_1'(s) \neq 0$.

Properties (i), (ii) and the definitions of W_0 and κ were establish in Theorem 5.1. Suppose (iii) is not true, then together with (i) we would have $\gamma'(s) = (0, 1)1$ for all $s \in I$. This would imply $\kappa(s) = \gamma''(s) \cdot \gamma'(s)^{\perp}$ vanishes identically on I and hence (ii) would not be possible. Therefore, by the continuity of γ_1', we can extract a sub-interval J of I on which $\gamma_1'(s) \neq 0$. To continue we define two auxiliary functions ζ and Ψ by means of γ as follows.

$$\zeta : \mathbb{R} \times J \to \mathbb{R}^2, \qquad \zeta(r, s) = (\gamma_2(s) - r\gamma_1'(s), s),$$
$$\Psi : \zeta(\mathbb{R} \times J) \to \mathbb{R}^2, \quad (u, v) = \Psi(u, s) = \left(u, \sigma(s) + F(s)u + \frac{G(s)}{2}u^2\right).$$

where F, G, $\sigma : J \to \mathbb{R}$ are given by

$$\begin{aligned} F(s) &= \gamma_1(s) + \frac{\gamma_2(s)\gamma_2'(s)}{\gamma_1'(s)} = \frac{\gamma \cdot \gamma'}{\gamma_1'}, \\ G(s) &= -\frac{\gamma_2'(s)}{\gamma_1'(s)}, \\ \sigma(s) &= h_0(s) - \frac{1}{2}\gamma_2(s)F(s). \end{aligned} \tag{5.6}$$

Due to property (iii) above and to the fact that $\gamma \in C^3(I)$, the functions ζ, Ψ, F, G, σ are well defined and are $C^2(J)$. One can verify by a straightforward computation that

$$\Pi \circ \mathcal{L}(r,s) = \Psi \circ \zeta(r,s).$$

Therefore, if we show that $\Psi \circ \zeta : \mathbb{R} \times J \to \mathbb{R}^2$ is one one, then $\Pi \circ \mathcal{L}$ is also one one. To this end, we will show separately that both ζ and Ψ are one to one. The fact that ζ is one to one is easy to verify and follows from the fact that $\gamma_1'(s) \neq 0$ on J. We also note that

$$\zeta(\mathbb{R} \times J) = \mathbb{R} \times J.$$

To show that Ψ is one to one, we first consider its second component: $v(u,s) = \sigma(s) + F(s)u + (G(s)/2)u^2$. We have

$$\frac{\partial}{\partial s} v(u,s) = \sigma'(s) + F'(s)u + \frac{G'(s)}{2}u^2.$$

Although it is tedious, nevertheless one can verify by straightforward computations that the following identity holds for any $s \in J$ and any $u \in \mathbb{R}$:

$$F'(s)^2 - 2\sigma'(s)G'(s) = 1 - 2W_0(s)\kappa(s) + (|\gamma'(s)|^2 + 1)(|\gamma'(s)|^2 - 1) < 0.$$

The strict inequality above is due to properties (i) and (ii) of γ. This in turn implies that the quadratic expression in u

$$\frac{\partial}{\partial s} v(u,s) = \sigma'(s) + F'(s)u + \frac{G'(s)}{2}u^2$$

does not vanish for any fixed $u \in \mathbb{R}$ and any $s \in J$. Hence we have

$$\left| \frac{\partial}{\partial s} v(u,s) \right| > 0, \quad s \in J$$

that is, $v(u,s)$ is monotone in s for any fixed $u \in \mathbb{R}$. We infer from this fact and the definition of Ψ that Ψ is one to one. This completes the proof. ◻

Several important facts about the functions F, G, σ, Ψ were established in the proof of Lemma 5.8; we single them out here for references.

Proposition 5.9. *The functions F, G, σ satisfy*

$$F'(s)^2 - 2\sigma'(s)G'(s) < 0. \tag{5.7}$$

The function $\Psi : \mathbb{R} \times J \to \mathbb{R}^2$ is invertible on its image. We let $(u,s) = \Psi^{-1}(u,v)$. In particular, $s = s(u,v)$ is the second component of Ψ^{-1}.

These two lemmas show that every C^2 noncharacteristic complete noncompact embedded minimal surface which is not itself a vertical plane contains a subsurface which can be written as an intrinsic graph. To make the presentation as clean as possible, we prove an intermediate lemma.

Lemma 5.10. *Let S be a C^2 noncharacteristic complete noncompact embedded minimal surface which is not itself a vertical plane and let J and the functions F, G, σ, Ψ be the ones from the proof of Lemma 5.8, and s as in Proposition 5.9. If $\varphi : \Psi(\mathbb{R} \times J) \to \mathbb{R}^2$ is given by*

$$\varphi(u,v) = F(s(u,v)) + uG(s(u,v)) \quad \text{for } (u,v) \in \Omega = \Psi(\mathbb{R} \times J).$$

Then

$$S_0 = \left\{(0,u,v) \circ (\varphi(u,v),0,0) \mid (u,v) \in \Omega\right\}$$

is a sub-surface of S.

Proof. With the functions Ψ, φ, s, F, G, σ, and Ω as in the statement of the lemma, we define $\Phi : \Omega \to \mathbb{H}^1$ as follows

$$\Phi(u,v) = \left(\varphi(u,v), u, v - \frac{1}{2}u\varphi(u,v)\right).$$

Our intention is to show that $\Phi(\Omega) = \mathcal{L}(\mathbb{R} \times J)$. We begin by comparing the second components of Φ and $\mathcal{L}$. Note that if

$$u = \gamma_2(s) - r\gamma_1'(s), \tag{5.8}$$

then

$$\begin{aligned}
\varphi(u,v) &= F(s(u,v)) + uG(s(u,v)) \\
&= F(s) + (\gamma_2(s) - r\gamma_1'(s))G(s) \\
&= \gamma_1(s) + \frac{\gamma_2(s)\gamma_2'(s)}{\gamma_1'(s)} - (\gamma_2(s) - r\gamma_1'(s))\frac{\gamma_2'(s)}{\gamma_1'(s)} \quad \text{(by (5.6))} \\
&= \frac{\gamma_1(s)\gamma_1'(s) + \gamma_2(s)\gamma_2'(s) - \gamma_2(s)\gamma_2'(s) + r\gamma_1'(s)\gamma_2'(s)}{\gamma_1'(s)} \\
&= \gamma_1(s) + r\gamma_2'(s),
\end{aligned} \tag{5.9}$$

which is the first component of $\mathcal{L}$. We now turn to the third component of Φ. Keeping in mind that for $(u,v) \in \Omega = \Psi(\mathbb{R} \times J)$ we have

$$v = \sigma(s) + F(s)u + \frac{G(s)}{2}u^2$$

hence

$$
\begin{aligned}
&v - \frac{1}{2}u\varphi(u,v) \\
&= \sigma(s) + F(s)u + \frac{G(s)}{2}u^2 - \frac{1}{2}u\varphi(u,v) \\
&= h_0(s) - \frac{1}{2}\gamma_2(s)\left(\gamma_1(s) + \frac{\gamma_2(s)\gamma_2'(s)}{\gamma_1'(s)}\right) \\
&\quad + \left(\gamma_1(s) + \frac{\gamma_2(s)\gamma_2'(s)}{\gamma_1'(s)}\right)(\gamma_2(s) - r\gamma_1'(s)) - \frac{1}{2}\frac{\gamma_2'(s)}{\gamma_1'(s)}(\gamma_2(s) - r\gamma_1'(s))^2 \\
&\quad - \frac{1}{2}(\gamma_2(s) - r\gamma_1'(s))(\gamma_1(s) + r\gamma_2'(s)) \quad \text{(by (5.8), (5.6) and (5.9))} \\
&= h_0(s) - \frac{r}{2}\gamma(s)\cdot\gamma'(s)
\end{aligned}
$$

which is the third component of $\mathcal{L}$. ◻

Finally, we turn to the proof of Theorem C.

Proof of Theorem C. Since S is not itself a vertical plane, Lemma 5.3 guarantees the existence of a point $g_0 \in S$ and a neighborhood N of g_0 such that N can be written as a graph over the plane $t = 0$. Theorem 5.1 then provides the necessary parameterization of such a neighborhood by the map $\mathcal{L}$ whose domain is $\mathbb{R} \times J$. Lemmas 5.8, 5.10 and Proposition 5.9 then show that the portion $\mathcal{L}(\mathbb{R} \times J) \subset S$ can be reparameterized to conform to Definition 1.6 hence, establishing the required δ-graphical strip. ◻

Combining this with Theorem C, we can now easily prove our main result of Bernstein type, i.e., Theorem A.

Proof of Theorem A. Suppose S is a C^2 complete embedded noncharacteristic minimal surface without boundary which is not a vertical plane. Then, Theorem C shows that S contains an intrinsic graphical strip, S_0, and thus, by Theorem B, S_0, and hence S, is not stable. ◻

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