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A Special Class of Almost Disjoint Families

Thomas E. Leathrum
Dartmouth College

A SPECIAL CLASS OF ALMOST DISJOINT FAMILIES

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Abstract. The collection of branches (maximal linearly ordered sets of nodes) of the tree ${}^{<\omega}\omega$ (ordered by inclusion) forms an almost disjoint family (of sets of nodes). This family is not maximal – for example, any level of the tree is almost disjoint from all of the branches. How many sets must be added to the family of branches to make it maximal? This question leads to a series of definitions and results: a set of nodes is *off-branch* if it is almost disjoint from every branch in the tree; an *off-branch family* is an almost disjoint family of off-branch sets;

$\mathfrak{o}$ is the minimum cardinality of a maximal off-branch family.

Results concerning $\mathfrak{o}$ include: (in ZFC)

$\mathfrak{a} \leq \mathfrak{o}$, and (consistent with ZFC) $\mathfrak{o}$ is not equal to any of the standard small cardinal invariants $\mathfrak{b}$,

$\mathfrak{a}$, $\mathfrak{d}$, or $\mathfrak{c} = 2^\omega$. Most of these consistency results use standard forcing notions – for example, $\mathfrak{b} = \mathfrak{a} < \mathfrak{o} = \mathfrak{d} = \mathfrak{c}$

in the Cohen model.

Many interesting open questions remain, though – for example, $\mathfrak{d} \leq \mathfrak{o}$.

The results in this paper have arisen from

a study of structural and combinatorial properties of almost disjoint families, in particular the effects of various kinds of forcing on such families. It is known, for example, that if $V \models CH$ and $\mathbb{P}$ is constructed by a finite support product of Cohen forcing, then there is a maximal almost disjoint family $\mathcal{A}$ in V which remains maximal in the extension

$V^{\mathbb{P}}$ [Ku]. Similar results can be shown with different assumptions – e.g. if $V \models MA + \neg CH$, or if $\mathbb{P}$ adds random reals instead. On the other hand, the collection of branches of the tree ${}^{<\omega}\omega$

form an almost disjoint family of size continuum. This family is not maximal, but can be easily extended using Zorn's Lemma. However, any time a forcing extension adds a new real, a new branch through the tree is added – and so, in the extension, the almost disjoint family is no longer maximal. Two general questions arise from such examples: What properties of an almost disjoint family in the ground model can be used to make this distinction more precise? When extending a

particular nonmaximal almost disjoint family to a maximal family, how many new sets must be added? This paper looks closely at the second question, in the

special case given above (the nonmaximal family being the branches through the tree ${}^{<\omega}\omega$).

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1 Basic Invariants.

Definitions of small cardinal invariants have the general form [vD]:

$$\min\{|\mathcal{Q}| : \mathcal{Q} \subseteq [\omega]^\omega \text{ is a family satisfying property } Q\}.$$

This section is devoted to devising a basic list of small cardinal invariants, so it will be necessary to define several different properties Q.

For example:

1.1 **Definition.** Two infinite sets $A, B \subseteq \omega$ are *almost disjoint* if their interesection is finite. An *almost disjoint family* is a collection of infinite subsets of ω which are pairwise almost disjoint.

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Let Q be the property that $\mathcal{Q}$ is an infinite maximal almost disjoint family. Then the resulting small cardinal invariant is called $\mathfrak{a}$.

The columns of $\omega\times\omega$ form a decomposition of $\omega\times\omega$ — so that if Q is the property of being pairwise almost disjoint subsets of $\omega \times \omega$ which are also almost disjoint from every column, and being maximal under this property, then the resulting small cardinal invariant is still equal to $\mathfrak{a}$. So tighten up the property somewhat: require further than every set in the family intersect any given column *at most once* (so that sets in the family can be regarded as infinite partial functions from ω to ω, which are pairwise almost disjoint as sets of pairs). With this stronger Q, the resulting small cardinal invariant is called $\mathfrak{a}_s$. It is clear now that $\mathfrak{a} \leq \mathfrak{a}_s$.

Some cardinal invariants are defined in terms of ${}^\omega\omega$, the space of functions from the natural numbers to the natural numbers, ordered by "eventual domination" $<^*$ — $f <^* g$ if and only if $\{n : g(n) \leq f(n)\}$ is finite. For example, if the property Q over ${}^\omega\omega$ is the property of being unbounded in this ordering, then the resulting small cardinal invariant is called $\mathfrak{b}$.

1.2 **Definition.** A family $\mathcal{D} \subseteq {}^\omega\omega$ is *dominating* if for every $f \in {}^\omega\omega$ there is a $g \in \mathcal{D}$ such that $f <^* g$. //

If the property Q is the property of being dominating, then the resulting small cardinal invariant is called $\mathfrak{d}$. Since any dominating family is unbounded, it is clear that $\mathfrak{b} \leq \mathfrak{d}$.

There are many inequalities provable between these cardinal invariants — for example, the inequalities $\mathfrak{a} \leq \mathfrak{a}_s$ and $\mathfrak{b} \leq \mathfrak{d}$ have already been mentioned.

The only other inequality known in ZFC for these invariants is $\mathfrak{b} \leq \mathfrak{a}$ [vD]. There are many other invariants which have been investigated, and many inequalities between them have been established. Some of these results are quite difficult — the interested reader is referred to [vD] or [BS] for details.

By forcing techniques, it is possible to construct models in which various *strict* inequalities hold between these cardinal invariants. For example, forcing to add ω_2-many Cohen reals over a model of CH gives a model of $\mathfrak{a}_s < \mathfrak{d}$. Some such consistency questions remain open, though — for example, it is unknown whether $\mathfrak{a} \leq \mathfrak{d}$ is provable in ZFC. In fact, this question is an important motivation for the present research.

2 Definitions.

2.1 **Definition.** The set ${}^{<\omega}\omega$ consists of finite sequences of natural numbers. This set is given an ordering by $\sigma \leq \tau$ if and only if σ is an initial segment of τ — the result is a countably branching, countable height tree ordering. A *node* of the tree is an element of ${}^{<\omega}\omega$. A *branch* through the tree is a maximal linearly ordered set of nodes. The n^{th} *level* of the tree is the set of nodes which, as sequences, all have length n. //

In general, the families this paper deals with will be almost disjoint families of infinite sets of nodes, in particular families extending the (nonmaximal) family of branches of ${}^{<\omega}\omega$.

2.2 **Definition.** An infinite set A of nodes of ${}^{<\omega}\omega$ is *off-branch* if A is almost disjoint from every branch of ${}^{<\omega}\omega$. An *off-branch family* is an almost disjoint family of off-branch sets. //

2.3 **Definition.**

$$\mathfrak{o} = \min\{|\mathcal{O}| : \mathcal{O} \text{ is a maximal off-branch family}\}.$$

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3 Equivalent and Related Invariants.

The first few results concern equivalent definitions of $\mathfrak{o}$. For example, one natural question is whether it makes any difference to define $\mathfrak{o}$ in terms of the binary tree ${}^{<\omega}2$ instead of ${}^{<\omega}\omega$. In order to establish the context for this question, define a set A of nodes of the tree

${}^{<\omega}2$ to be *off-binary* if A is almost disjoint from every branch of

${}^{<\omega}2$, and an *off-binary family* is an almost disjoint family of off-binary sets. By analogy with the definition of $\mathfrak{o}$, define

$$\mathfrak{o}_b = \min\{|\mathcal{O}| : \mathcal{O} \text{ is a maximal off-binary family}\}.$$

3.1 **Lemma.** $\mathfrak{o} = \mathfrak{o}_b$.

Proof. The basic idea of this proof is to construct mappings between ${}^{<\omega}2$ and

${}^{<\omega}\omega$ which carry off-binary families to off-branch families, and vice versa. It turns out that the best thing to do is to simply embed the trees in canonical ways into each other, and look at *pullbacks* of the families.

For one direction, notice first that ${}^{<\omega}2 \subseteq {}^{<\omega}\omega$, so the identity map on ${}^{<\omega}2$ embeds it into ${}^{<\omega}\omega$. Let $\mathcal{O}$ be a maximal off-branch family.

Let

$$\bar{\mathcal{O}} = \{A \cap {}^{<\omega}2 : A \in \mathcal{O} \text{ and } A \cap {}^{<\omega}2 \text{ infinite}\}.$$

(So $\bar{\mathcal{O}}$ is the pullback of $\mathcal{O}$ over the identity map embedding.) Since each $A \in \mathcal{O}$ is off-branch, $\bar{\mathcal{O}}$ is an off-binary family. If $\bar{\mathcal{O}}$ is not maximal, let $B \subseteq {}^{<\omega}2$ be a witness to this fact — i.e. an off-binary set which is almost disjoint from every element of

$\bar{\mathcal{O}}$. Then $B \subseteq {}^{<\omega}\omega$, B is an off-branch set, and B is almost disjoint from every element of $\mathcal{O}$, contradicting maximality of

$\mathcal{O}$. This proves, in particular, that $\mathfrak{o}_b \leq \mathfrak{o}$.

To do the other direction, define an embedding $\pi : {}^{<\omega}\omega \to {}^{<\omega}2$ as follows. Let $\vec{1}_n$ be the length n sequence of 1's. For $\sigma \in {}^{<\omega}\omega$, $\sigma = \langle n_0, n_1, \ldots, n_k \rangle$, let

$$\pi(\sigma) = \vec{1}_{n_0}{}^\frown\langle 0\rangle^\frown\vec{1}_{n_1}{}^\frown\langle 0\rangle^\frown\ldots{}^\frown\vec{1}_{n_k}{}^\frown\langle 0\rangle\,.$$

So the image of π is the collection of binary sequences ending in 0, with the coordinates of σ being coded by the lengths of corresponding blocks of consecutive 1's.

With this π, the argument proceeds much as in the first case. Let $\bar{\mathcal{O}}$ be a maximal off-binary family. For $\bar{A} \in \bar{\mathcal{O}}$, let $A = \{\sigma \in {}^{<\omega}\omega : \pi(\sigma) \in \bar{A}\}$ (so A is the pullback of $\bar{A}$ over π), and let $\mathcal{O} = \{A : \bar{A} \in \bar{\mathcal{O}}, A \text{ infinite}\}$. Then $\mathcal{O}$ is an off-branch family. Furthermore, if $\mathcal{O}$ is not maximal and $B \subseteq {}^{<\omega}\omega$ witnesses the fact that $\mathcal{O}$ is not maximal, then $\pi''B = \{\pi(\sigma) : \sigma \in B\}$ witnesses that $\bar{\mathcal{O}}$ isn't maximal either, which contradicts the assumption. So $\mathfrak{o} \leq \mathfrak{o}_b$. ■

Another equivalent form of $\mathfrak{o}$ adds the requirement that the off-branch family contain a decomposition of the tree. Define by analogy with $\mathfrak{o}$ the cardinal invariant

$$\begin{aligned}\mathfrak{o}_d = \min\{|\mathcal{O}| :\ & \text{there is a decomposition } \mathcal{D} \text{ of } {}^{<\omega}\omega \text{ into infinite sets so that} \\ & \mathcal{O} \cup \mathcal{D} \text{ is a maximal off-branch family}\}.\end{aligned}$$

3.2 **Lemma.** $\mathfrak{o} = \mathfrak{o}_d$.

Proof. Clearly $\mathfrak{o} \leq \mathfrak{o}_d + \omega = \mathfrak{o}_d$. So it only remains to show that $\mathfrak{o}_d \leq \mathfrak{o}$. Let $f : \omega \to \mathcal{O}$ be an injection, and let $g : \omega \to {}^{<\omega}\omega$ be a bijection. Define a function h by

$$h(n) = (\{g(n)\} \cup f(n)) \smallsetminus \bigcup_{k<n} f(k).$$

Then the range of h is a decomposition of ${}^{<\omega}\omega$ into infinite off-branch sets, and for any off-branch set A, $h(n) \cap A$ is infinite if and only if $f(n) \cap A$ is infinite. Letting $\mathcal{O}' = \mathcal{O} \smallsetminus ran(f)$ provides a witness to $\mathfrak{o}_d \leq \mathfrak{o}$. ■

The situation changes somewhat when considering a *particular* decomposition of the tree. Given a decomposition $\mathcal{D}$ of of ${}^{<\omega}\omega$ into off-branch sets, define, by analogy with $\mathfrak{o}$, the cardinal invariant

$$\tilde{\mathfrak{o}}(\mathcal{D}) = \min\{|\mathcal{O}| : \mathcal{O} \cup \mathcal{D} \text{ is a maximal off-branch family}\}.$$

So $\mathfrak{o}_d = \min_{\mathcal{D}} \mathfrak{o}(\mathcal{D})$. Clearly $\mathfrak{o}_d \leq \mathfrak{o}(\mathcal{D})$ for any $\mathcal{D}$ — but what about $\mathfrak{o}(\mathcal{D}) \leq \mathfrak{o}_d$?

It turns out that it is easier to approach this problem directly, in terms of $\mathfrak{o}$ rather than $\mathfrak{o}_d$. However, this result requires the use of the fact that $\mathfrak{a} \leq \mathfrak{o}$, which will be proved later.

3.3 **Lemma.**
For any decomposition $\mathcal{D}$ of of
${}^{<\omega}\omega$ into off-branch sets, $\mathfrak{o}(\mathcal{D}) = \mathfrak{o}$.

Proof. Deferred to next section.

Antichains in the tree are clearly off-branch — they intersect any given branch at most once. So what happens to the cardinal invariant if you strengthen the off-branch condition to talking about antichains? (Notice that every infinite off-branch set contains an infinite antichain.) Define, by analogy with $\mathfrak{o}$, yet another cardinal invariant:

$$\bar{\mathfrak{o}} = \min\{|\mathcal{O}| : \mathcal{O} \text{ is a maximal almost disjoint family of antichains of } {}^{<\omega}\omega\}.$$

Again, clearly $\mathfrak{o} \leq \bar{\mathfrak{o}}$. While it seems counterintuitive that $\bar{\mathfrak{o}} \leq \mathfrak{o}$, constructing a model in which these two invariants are different seems quite difficult.

4 Results in ZFC — $\mathfrak{a} \leq \mathfrak{o}$.

Recall that $\mathfrak{a}$ is the minimum size of a maximal almost disjoint family of subsets of a countable set. It is not obvious at first that $\mathfrak{o}$ is related to $\mathfrak{a}$, since the off-branch family does *not* contain the branches.

4.1 **Theorem.** *$\mathfrak{a} \leq \mathfrak{o}$.*

(The following version of this proof was suggested by the referee to show more clearly the connection with $\mathfrak{a}_s$ later.)

Proof. Suppose, for contradiction, that $\mathfrak{o} < \mathfrak{a}$. Let $\mathcal{O}$ be a maximal off-branch family of size $\mathfrak{o}$. For each n, let b_n be any branch containing the node $\langle n \rangle$.

Then the b_n are all distinct, and in fact disagree at the very first level. For each $O \in \mathcal{O}$, define:

$$\bar{O} = \{\langle n, i\rangle : \exists \sigma \in O \cap b_n \ ht(\sigma) = i\}.$$

Let Col be the collection of columns of $\omega \times \omega$ — then $\{\bar{O} : O \in \mathcal{O},\ \bar{O} \text{ infinite}\} \cup Col$ is an almost disjoint family of subsets of $\omega \times \omega$. However, this family cannot be maximal, since it has size $< \mathfrak{a}$. Let $\bar{B} \subseteq \omega \times \omega$ be almost disjoint from every column and from every $\bar{O}$. Define $B = \{\sigma : \langle n, i\rangle \in \bar{B},\ \sigma \in b_n,\ ht(\sigma) = i\}$. Then B is off-branch and almost disjoint from every $O \in \mathcal{O}$, contradicting maximality of $\mathcal{O}$. ■

It is now possible to prove Lemma 3.3, that $\mathfrak{o} = \mathfrak{o}(\mathcal{D})$.

4.2 *Proof of Lemma 3.3.* Since $\mathfrak{o} = \mathfrak{o}_d \leq \mathfrak{o}(\mathcal{D})$, it only remains to show $\mathfrak{o}(\mathcal{D}) \leq \mathfrak{o}$. Let $\mathcal{O}$ be a maximal off-branch family of size $\mathfrak{o}$,

let $\mathcal{D} = \{D_n : n < \omega\}$ be a decomposition of ${}^{<\omega}\omega$ into off-branch sets, and for each $A \in \mathcal{O}$ define $\mathcal{D} \restriction A = \{D_n \cap A : n < \omega, D_n \cap A \text{ infinite}\}$.

Then $\mathcal{D} \restriction A$ is a countable pairwise disjoint collection of subsets of A. So let $\mathcal{B}_A$ be such that $(\mathcal{D} \restriction A) \cup \mathcal{B}_A$ is a maximal almost disjoint family *of subsets of* A, and $|\mathcal{B}_A| = \mathfrak{a}$.

Since each A is off-branch, each element of $\mathcal{B}_A$ is also off-branch, so $\mathcal{B} = \bigcup_{A \in \mathcal{O}} \mathcal{B}_A$ is an off-branch family. Also, for each $B \in \mathcal{B}$ and each $n < \omega$, $B \cap D_n$ is finite, so $\mathcal{B} \cup \mathcal{D}$ is an off-branch family. Furthermore, $|\mathcal{B}| = \mathfrak{o} \cdot \mathfrak{a} = \mathfrak{o}$ (since $\mathfrak{a} \leq \mathfrak{o}$), so it remains only to show that $\mathcal{B} \cup \mathcal{D}$ is a *maximal* off-branch family.

To show this, let C

be an off-branch set.

Since $\mathcal{O}$ is a maximal off-branch family, there is some $A \in \mathcal{O}$ such that $C \cap A$ is infinite. But $(\mathcal{D} \restriction A) \cup \mathcal{B}_A$ is also maximal, so C has infinite intersection with some element of

$(\mathcal{D} \restriction A) \cup \mathcal{B}_A$, and thus with some element of $\mathcal{B} \cup \mathcal{D}$. ■

Consider now the relationship between $\mathfrak{o}$ and $\mathfrak{a}_s$. The important distinction is not in the nature of the decomposition, but in how often a set intersects one of the specified sets — $\mathfrak{a}_s$ (where the sets intersect columns at most once) bears a much closer relationship to $\bar{\mathfrak{o}}$ (where the sets intersect branches at most once) than to $\mathfrak{o}(\mathcal{D})$.

4.3 **Theorem.** $\mathfrak{a}_s \leq \bar{\mathfrak{o}}$.

Proof. This proof is exactly like the proof that $\mathfrak{a} \leq \mathfrak{o}$, with the following exception: Notice that (using the same notation as above) $\bar{O}$ now intersects each column of $\omega \times \omega$ *at most*

once. Therefore, the family

$\{\bar{O} : O \in \mathcal{O},\ \bar{O} \text{ infinite}\}$ is a family as in $\mathfrak{a}_s$. The remaining details are left to the reader. ■

5 Cohen Reals — $Con(\mathfrak{a} < \mathfrak{o} = \mathfrak{d})$.

The Cohen forcing notion used here will be $\mathbb{P} = {}^{<\omega}\omega$ (finite sequences of natural numbers), ordered by end-extension — in other words, the tree turned upside-down. This is equivalent, as a forcing notion, to the usual $Fn(\omega, 2)$ Cohen forcing.

5.1 **Theorem.**

If $\mathcal{O}$ is an off-branch family (in the ground model V), $\mathbb{P}$ is the forcing notion given by the set ${}^{<\omega}\omega$ ordered by $p \leq q$ if and only if $p \supseteq q$,

and G is $\mathbb{P}$-generic over V, then in the extension $V[G]$, $\mathcal{O}$ is not maximal.

Proof. For a sequence $\langle n_0, n_1, \ldots, n_k \rangle \in {}^{<\omega}\omega$, define

$$rs(\langle n_0, n_1, \ldots, n_k \rangle) = \langle n_0, n_1, \ldots, n_k + 1 \rangle .$$

This is a "right shift" function in the tree — if the successors of $\langle n_0, n_1, \ldots, n_{k-1} \rangle$ are ordered according to n_k, left to right, then rs shifts $\langle n_0, n_1, \ldots, n_k \rangle$ one

place to the right. The Cohen-generic set G will simply (and exactly) be a branch through the tree ${}^{<\omega}\omega$. The idea of this proof is to show that the "hair" of the branch G,

$\bar{G} = \{rs(\sigma) : \sigma \in G\}$, is off-branch (which is obvious, since it is an antichain) and almost disjoint from all members of the off-branch family $\mathcal{O}$.

This is accomplished by a density argument. The idea is to show that, for every $A \in \mathcal{O}$, the set of sequences whose right shifts are *not* in A contains a dense *open* set in the forcing notion (which is just the upside down tree). Formally, define

$$D_A = \{\sigma \in {}^{<\omega}\omega : \forall \tau \in {}^{<\omega}\omega\ rs(\sigma^\frown\tau) \notin A\}.$$

It remains to show that D_A is dense for each $A \in \mathcal{O}$. This is accomplished by an appeal to the fact that A is an off-branch set.

To see how this works, let G be a generic set for the Cohen forcing. So for each $A \in \mathcal{O}$ there is a $\sigma \in G \cap D_A$. Now if $\tau \in G$ is such that $rs(\tau) \in A$, then $\tau \subseteq \sigma$, by the definition of D_A. In particular, then, $\bar{G} \cap A$ is finite. ■

5.2 **Theorem.**

If $\mathbb{P}$ is a forcing notion constructed as a finite support product of κ-many Cohen forcing notions (where κ is an uncountable cardinal) and G is $\mathbb{P}$-generic over V, then in the extension $V[G]$, there are no maximal off-branch families of size less than κ.

The technique in this proof is a completely standard way of dealing with products of Cohen reals [Ku] — nothing more will be said here.

5.3 **Corollary.**

$Con(\omega_1 = \mathfrak{b} = \mathfrak{a} < \mathfrak{o} = \mathfrak{d} = 2^\omega = \kappa)$.

Proof. Start with a model $V \vDash ZFC + CH$. Force to add κ-many Cohen reals.

By the above Theorem, together with previous results concerning the effects of Cohen forcing on the other cardinal invariants [vD, BS], this will give the desired model. ■

6 Random Reals — $Con(\mathfrak{d} < \mathfrak{o})$.

The random forcing notion used here will be the Boolean algebra given by the measure algebra on the product space 2^κ — this algebra, used as a forcing notion, is said to *add κ-many random reals.* The following standard fact for random forcing (see [So] or [Je]) will not be proven here.

6.1 **Fact.** *Let $\mathbb{B}$ be the measure algebra on the product space 2^κ, and let G be $\mathbb{B}$-generic over V. If $r \in V[G]$ is an countable subset of V, then there is a countable set $X \subseteq \kappa$ such that $r \in V[G \restriction X]$.*

6.2 *Note.* The forcing notion $\mathbb{B}$ which adds κ many random reals has *ccc*. Since κ is assumed to be infinite, the measure algebra on 2^κ is equivalent (as a forcing notion) to the measure algebra on $2^{\kappa\times\omega}$. Then a generic set G gives a function $f_G : \kappa \times \omega \to 2$, and thus κ many reals (subsets of ω), r_α for $\alpha < \kappa$, such that $n \in r_\alpha$ if and only if $f_G(\alpha, n) = 1$. Now the measure algebra on $2^{\omega\times\omega}$ adds countably many random reals, and the measure algebra on $2^{1\times\omega}$ is said to *add a single random real* – but these two measure algebras are equivalent (as forcing notions).

It is now possible to apply random forcing in the context of off-branch families. The author owes much of the credit for the mode of presentation for this proof to the guidance and insistence of Prof. James E. Baumgartner.

6.3 **Theorem.**

If $\mathcal{O}$ is an off-branch family (in the ground model V), $\mathbb{B}$ is the measure algebra on 2^ω (so that forcing with this Boolean algebra adds a single random real), and G is $\mathbb{B}$-generic over V, then in the extension $V[G]$, $\mathcal{O}$ is not maximal.

This proof is essentially the same as the Cohen reals proof, but the new context of random reals makes some of the calculations more delicate.

It will be carried out using off-binary families instead of off-branch families, but these are entirely equivalent since $\mathfrak{o} = \mathfrak{o}_b$.

Proof. The set of branches through the tree ${}^{<\omega}2$ can be identified with the product space 2^ω, so the sets of branches inherit a measure algebra structure from the interval. As Boolean algebras, and thus as a forcing notions, these measure algebras are equivalent. So rather than forcing with the measure algebra on 2^ω, use the

equivalent measure algebra on the branches through ${}^{<\omega}2$. Given a sequence $\sigma \in {}^{<\omega}2$, associate with σ the Baire interval, denoted $[\sigma]$, of branches which contain σ (all the branches with σ as a common stem), with measures for these

Baire intervals given by

$$m([\sigma]) = 2^{-len(\sigma)}.$$

It is easy to see that this is consistent with the measure on 2^ω. Finally, rather than using the Boolean algebra, this proof deals explicitly with the pre-order on Baire sets (elements of the σ-algebra generated by the Baire intervals), where the ordering is by almost containment: $p \leq q$ if and only if $m(p \setminus q) = 0$, where p and q are Baire sets.

Thus, countable unions and intersections of Baire sets correspond to

countable infima and suprema (respectively) in the Boolean algebra.

As with the single Cohen real proof, the idea of this proof is to look at the "hair"

of the generic branch added by forcing with this measure algebra.

If $G \subseteq \mathbb{B}$ is a generic ultrafilter,

define $g = \bigcup\{\sigma : [\sigma] \in G\}$ — so g is in ${}^{\omega}2$, and is essentially the generic branch added by forcing with $\mathbb{B}$. Define the "hair" on g to be the set

$H = \{\tau^\frown\langle 1-i\rangle : \exists n\ g\restriction n = \tau^\frown\langle i\rangle\}$.

Then H is an antichain in ${}^{<\omega}2$, so it remains only to show that H is almost disjoint from every element of $\mathcal{O}$.

For each $A \in \mathcal{O}$, let

$f : \omega \to A$ enumerate A. First, notice that

$$\bigcup_n \sum_{k\geq n} [f(k)] = \emptyset$$

To see this, suppose it is not true and let $b \in \bigcup_n \sum_{k\geq n}[f(k)]$. Now $b \in 2^\omega$ — interpret b as an element of ${}^{\omega}2$, so that $\{b\restriction i : i < \omega\}$ is a branch through the tree. Since there are infinitely many k for which $b \in [f(k)]$, there are infinitely many k for which $b\restriction i = f(k)$ for some i. But this contradicts A being off-branch.

The sets $\bigcup_{k\geq n}[f(k)]$ are nested (i.e.

$\bigcup_{k\geq n}[f(k)] \supseteq \bigcup_{k\geq n+1}[f(k)]$ for all n), so

$$m\left(\bigcap_n \bigcup_{k\geq n} [f(k)]\right) = \lim_{n\to\infty} m\left(\bigcup_{k\geq n}[f(k)]\right) = 0.$$

For each n, let $B_n \subseteq \{f(k) : k \geq n\}$ be the antichain of minimal elements (with respect to the tree ordering) of the set $\{f(k) : k \geq n\}$. Then

$\bigcup_{k\geq n}[f(k)] = \bigcup_{\sigma\in B_n}[\sigma]$. Since the intervals $[\sigma]$ are *disjoint* for $\sigma \in B_n$,

$$m\left(\bigcup_{\sigma\in B_n}[\sigma]\right) = \sum_{\sigma\in B_n} m([\sigma]).$$

So

$$\lim_{n\to\infty} \sum_{\sigma\in B_n} m([\sigma]) = 0.$$

For all $\sigma \in {}^{<\omega}2$, define

$$pred(\sigma) = \begin{cases} \tau & \sigma = \tau^\frown\langle 0\rangle \text{ or } \sigma = \tau^\frown\langle 1\rangle\,; \\ \langle\rangle & \text{otherwise.} \end{cases}$$

Now for all $k \geq n$, there is a $\sigma \in B_n$ such that $[pred(f(k))] \subseteq [pred(\sigma)]$, and in this case $m([pred(f(k))]) \leq 2m([\sigma])$. Therefore,

$$m\Bigg(\bigcup_{k\geq n}[pred(f(k))]\Bigg) \leq 2 \cdot \sum_{\sigma\in B_n} m([\sigma])$$

Since the sets $\bigcup_{k\geq n}[pred(f(k))]$ are also nested,

$$m\Bigg(\bigcap_n \bigcup_{k\geq n}[pred(f(k))]\Bigg) = \lim_{n\to\infty} m\Bigg(\bigcup_{k\geq n}[pred(f(k))]\Bigg) \leq \lim_{n\to\infty} 2 \cdot \sum_{\sigma\in B_n} m([\sigma]) = 0$$

The Baire set $p = \bigcap_n \bigcup_{k\geq n}[pred(f(k))]$ is in the equivalence class represented by the Boolean value $[\![\forall n\ \exists k \geq n\ \exists i\ pred(f(k)) = g\restriction i]\!]$, so the statement that $m(p) = 0$ means

that this value is $= \mathbf{0}$ in the Boolean algebra.

To see why this suffices to finish the proof, notice that

$$\forall q\ \exists n\ m\Bigg(\bigcup_{k\geq n}\ [pred(f(k))]\Bigg) < m(q).$$

Let $q' = q \smallsetminus \bigcup_{k\geq n}\ [pred(f(k))]$. Then for all m and for

all $k \geq n$,

$q' \not\equiv \Vdash g\restriction m \neq pred(f(k))$. But notice that if

$\sigma \in H \cap A$, then there is an m such that

$pred(\sigma) = g\restriction m$ — thus, since there are only finitely many m such that

$g\restriction m = pred(f(k))$ for some k, it follows that $H \cap A$ is also finite. ■

6.4 **Theorem.**

If $\mathbb{B}$ is a Boolean algebra forcing notion which adds κ-many random reals where κ is an uncountable cardinal,

and G is $\mathbb{B}$-generic over V,

then in the extension $V[G]$, there are no maximal off-branch families of size less than κ.

As with Cohen reals, the technique of this proof is standard — see [So] or [Je].

6.5 **Corollary.**

$Con(\omega_1 = \mathfrak{b} = \mathfrak{a} = \mathfrak{d} < \mathfrak{o} = 2^\omega = \kappa)$.

Proof. Start with a model $V \models ZFC + CH$. Force to add κ-many random reals.

By the above Theorem, together with previous results concerning the effects of random forcing on the other cardinal invariants [vD, BS],

this will give the desired model. ■

7 Sacks Reals — $Con(\mathfrak{o} < 2^\omega)$.

7.1 **Definition.** *(Sacks forcing)* The partial ordering $\mathbb{P}$ is the set of perfect binary trees — i.e.

$$\begin{aligned}\mathbb{P} = \{p : & p \subseteq {}^{<\omega}2, \forall \sigma \in p\ \forall \tau \subseteq \sigma\ \tau \in p, \text{and} \\ & \forall \sigma \in p\ \exists \tau \in {}^{<\omega}2 \text{ both } \sigma^\frown\tau^\frown\langle 0\rangle \in p \text{ and } \sigma^\frown\tau^\frown\langle 1\rangle \in p\}; \\ p \leq q & \text{ iff } p \subseteq q.\end{aligned}$$

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Let p be a Sacks condition (i.e. a perfect binary tree).

The set $split(p) = \{\sigma \in p : \sigma^\frown\langle 0\rangle \in p \text{ and } \sigma^\frown\langle 1\rangle \in p\}$ contains the "branching points" or "splitting points" of p. The set

$split_n(p) = \{\sigma \in split(p) : |\{\sigma' \subseteq \sigma : \sigma' \in split(p)\}| = n\}$ contains the "n^{th} branching (or splitting) points" of p. For two Sacks conditions p and q, define $p \leq_n q$ if and only if $p \leq q$ and $split_n(q) = split_n(p)$

— that is, p agrees with q up to and including the n^{th} branching points. These orderings can be used to show that Sacks forcing satisfies

Axiom A (see [Je]) — in particular, if $p_{n+1} \leq_n p_n$ for each n and $q = \cap_n p_n$, then q is a perfect binary tree and $q \leq_n p_n$ for each n.

The following result deals with antichain families instead of off-branch families, in order to get a stronger result — that forcing to add many Sacks reals over a model of CH leaves $\bar{o} = \omega_1$ in the extension. Since this proof deals with *preserving* ground model families rather than destroying them as in the Cohen and random reals proofs, the case of adding a single Sacks real will not be handled separately — it would simply repeat much of the same argument as below.

7.2 **Theorem.** *(CH)*

Suppose $\mathbb{P}$ is the notion of forcing which adds κ many Sacks reals with a countable support product, where κ is an uncountable cardinal with $cf(\kappa) > \omega$. There is a maximal antichain family $\mathcal{O}$ in V such that $\mathcal{O}$ remains maximal in $V^{\mathbb{P}}$.

The following technical lemmas for Sacks forcing will be useful here (see [Je]):

7.2.1 **Lemma.** *If $\mathbb{P}_\kappa$ is the forcing to add κ-many Sacks reals with a countable support product, $\mathbb{P}_\omega$ is the forcing to add ω-many Sacks reals (with full support product), G is $\mathbb{P}_\kappa$-generic over V, and*

$\rho \in V[G]$ is an countable subset of V, then there is a $G' \in V[G]$

which is $\mathbb{P}_\omega$-generic over V such that $\rho \in V[G']$.

For the purposes of the next lemma, it will be convenient to define a notion of "splitting point" for $\mathbb{P}_\omega$.

For $p \in \mathbb{P}_\omega$, an n^{th} *splitting point* of p is a sequence $\langle \sigma_0, \ldots, \sigma_n\rangle$ of elements of ${}^{<\omega}2$ such that for each $i \leq n$, $\sigma_i \in split_{n-i}(p(i))$. Then $split_n(p) = \{\vec{\sigma} : \vec{\sigma}$ an n^{th} splitting point of $p\}$ and $split(p) = \bigcup_n split_n(p)$. Finally $p \restriction \vec{\sigma}$ is defined by

$$(p \restriction \vec{\sigma})(i) = \begin{cases} p(i) \restriction \sigma_i & i \leq n; \\ p(i) & \text{otherwise.} \end{cases}$$

7.2.2 **Lemma.** *If $p \in \mathbb{P}_\omega$ and τ is a $\mathbb{P}_\omega$-name for an infinite subset of ω, then there is a $q \leq p$ such that for every n and every*
$\vec{\sigma} \in split_n(q)$, $q \restriction \vec{\sigma} \parallel n \in \tau$.

Now q gives a function $\pi : split(q) \to 2$ by $\pi(\vec{\sigma}) = 1$ if and only if $q \restriction \vec{\sigma} \Vdash n \in \tau$. Such a q is said to *code τ as π*.

Proof of Theorem. Enumerate ${}^{<\omega}\omega$ as $\{\eta_n : n < \omega\}$. Then the two Lemmas apply also to (names for) infinite off-branch sets. By the first Lemma, it suffices to consider only $\mathbb{P}_\omega$ — if $\mathcal{O}$ is a maximal antichain family in V which is indestructible under forcing with $\mathbb{P}_\omega$, then $\mathcal{O}$ will also be indestructible under forcing with $\mathbb{P}_\kappa$. In $\mathbb{P}_\omega$, given names τ_1 and τ_2 (for infinite antichains) suppose q codes both τ_1 and τ_2 as π. Then $q \Vdash \tau_1 = \tau_2$. (This determines an "isomorphism" between
τ_1 and τ_2.) By the second Lemma, the set $D = \{q : q \text{ codes some name } \tau \text{ for an infinite antichain as some } \pi\}$
is dense in $\mathbb{P}_\omega$. By CH, $|\mathbb{P}_\omega| = \omega_1$, so it is possible to enumerate all pairs $\langle (q_\alpha, \pi_\alpha) : \alpha < \omega_1 \rangle$ where $q_\alpha \in D$ and $\pi : split(q_\alpha) \to 2$.

For each $\alpha < \omega_1$, choose a name τ_α such that q_α codes τ_α as π_α. (So the τ_α are representative members of their "isomorphism classes.") For any $p \in \mathbb{P}_\omega$ and any $\mathbb{P}_\omega$-name τ
for an infinite antichain, there is an α such that $q_\alpha \leq p$ and $q_\alpha \Vdash \tau = \tau_\alpha$.

Inductively construct antichains A_α for $\alpha < \omega_1$ to satisfy

(1) $\forall \beta < \alpha \; |A_\beta \cap A_\alpha| < \omega$;
(2) if

$$(*) \qquad q_\alpha \Vdash \text{"}\tau_\alpha \text{ an infinite antichain" and } \forall \beta < \alpha \; q_\alpha \Vdash |\tau_\alpha \cap A_\beta| < \omega,$$

then there is some $q \leq q_\alpha$ such that

$$q \Vdash |\tau_\alpha \cap A_\alpha| = \omega.$$

This construction establishes the context for a density argument: Suppose the construction works, so that (1) and (2) hold.

Given a name τ for an infinite antichain, consider the set $D_\tau = \{q : \exists \xi \; q \Vdash \tau \cap A_\xi \text{ infinite}\}$. For any $p \in \mathbb{P}_\omega$ let α be such that $q_\alpha \leq p$ and $q_\alpha \Vdash \tau = \tau_\alpha$. Now for this pair (q_α, τ_α), if $(*)$ fails then there is an
$\eta < \alpha$ and a $q \leq q_\alpha$ such that $q \Vdash \tau \cap A_\eta$ infinite (i.e. $q \in D_\tau$), while if $(*)$ holds then there is a $q \leq q_\alpha$ such that
$q \Vdash \tau \cap A_\alpha$ infinite (i.e. $q \in D_\tau$). So D_τ is dense in $\mathbb{P}_\omega$. The only difficulty then is in constructing q when $(*)$ holds.

Enumerate $\{A_\beta : \beta < \alpha\}$ as $\{B_i : i < \omega\}$. By $(*)$, for each $i < \omega$,

$$q_\alpha \Vdash |\tau_\alpha \setminus (B_0 \cup \cdots \cup B_i)| = \omega.$$

The idea of this proof is to construct another "fusion sequence" $\langle p_n : n < \omega \rangle$ with $p_0 = q_\alpha$ and $p_{n+1} \leq p_n$, constructing along
with p_n a finite "block" $S_n \subseteq {}^{<\omega}\omega$ of A_α.

At stage 0 of this construction, let $p_0 = q_\alpha$ and $S_0 = \emptyset$. It will be necessary to carry several hypotheses through the induction —
in particular, after completing stage $n+1$, $\bigcup_{m \leq n+1} S_m$ must be an antichain and p_{n+1} must force that infinitely many nodes $\eta \in \tau_\alpha \setminus (B_0 \cup \cdots \cup B_n)$

are incomparable with everything in
$\bigcup_{m\leq n+1} S_m$. More formally, let I_n be (a name for) the set

$$I_n = \Big\{ \eta \in {}^{<\omega}\omega : \eta \in \tau_\alpha \setminus (B_0 \cup \cdots \cup B_n) \text{ and } \eta \text{ incomparable with each } \eta' \in \bigcup_{m\leq n} S_m \Big\}.$$

The inductive construction will work with the assumption that
$p_n \Vdash I_n$ infinite and will build p_{n+1} and S_{n+1} so that $p_{n+1} \Vdash I_{n+1}$ infinite. Notice that both S_{n+1} and B_{n+1} are involved in the definition of I_{n+1} — so all intersections of $A_\alpha = \bigcup_n S_n$ with B_n will happen by stage n, and A_α must then be almost disjoint from all of the B_n's. It is easy to check that
$p_0 \Vdash I_0$ infinite, since $S_0 = \emptyset$.

At stage $n+1$, construct p_{n+1} and S_{n+1} as follows: Enumerate $split_{n+1}(p_n)$ as $\{\vec{\sigma}_i : i \leq k\}$. Let $r'_0 = p_n \restriction \vec{\sigma}_0$. Extend r'_0 to $r''_0 \leq r'_0$ so that $r''_0 \Vdash a_0 \subseteq I_n$ for some
finite antichain a_0 satisfying $|a_0| = 2^k$ and
$a_0 \cap B_{n+1} = \emptyset$
(this is possible because of $(*)$). Form $r_{0,0} \leq p_n$ as follows:

$$r_{0,0}(i) = \begin{cases} r''_0(i) \cup \bigcup\{p_n(i)\restriction\sigma : \sigma \in split_{n-i+1}(p_n(i)), \sigma \neq \vec{\sigma}_0(i)\}, & i \leq n+1; \\ r''_0(i) & \text{otherwise.} \end{cases}$$

So $r_{0,0}$ amalgamates $r''_0(i)$ with the "other branches"
of $p_n(i)$ for each i. Let $r'_1 = r_{0,0} \restriction \vec{\sigma}_1$. The problem now is that r'_1 may not force the same information about a_0 that r''_0 does — in particular, r'_1 may have an extension which forces all but finitely much of
I_n to be comparable with a single node from a_0, ruining efforts to get
$p_{n+1} \Vdash I_{n+1}$ infinite.

So group the elements of a_0 into pairs — since the elements t_1 and t_2 in any one such pair are incomparable,
r'_1 forces that one of t_1 and t_2
must have infinitely many nodes in I_n incomparable to it (if infinitely many nodes are comparable to one, then
infinitely many of those same nodes are
incomparable with the other). So extend r'_1 to decide which of t_1 and t_2 keeps infinitely much of I_n incomparable to it. Do this for each of the 2^{k-1} many pairs, continuing to extend in a descending sequence, so that in the end a new condition $r''_1 \leq r'_1$ has
"chosen" one node from each pair in this way.

Form $r_{0,1}$ by amalgamating $r_{0,0}$ and r''_1 as follows:

$$r_{0,1}(i) = \begin{cases} r''_1(i) \cup \bigcup\{r_{0,0}(i)\restriction\sigma : \sigma \in split_{n-i+1}(p_n(i)), \sigma \neq \vec{\sigma}_1(i)\}, & i \leq n+1; \\ r''_1(i) & \text{otherwise.} \end{cases}$$

(Note that σ may not be a splitting point of $r_{0,0}$ — these are used to preserve the *previous* splitting level.) Group the remaining 2^{k-1} many remaining elements of a_0 into pairs, and extend $r_{0,1}$ to "choose" between elements of each pair, arriving at

$r''_2 \le r_{0,1}$, and amalgamate again to get $r_{0,2}$. After constructing $r_{0,0} \ge r_{0,1} \ge \cdots \ge r_{0,k}$,

there will be only one element s_0 of a_0 remaining, but the condition $r_{0,k}$

will force that infinitely many elements of $I_n \smallsetminus B_{n+1}$

are incomparable to s_0. Now s_0 can be included in S_{n+1} while preserving the induction hypothesis.

Extend $r_{0,k} \restriction \vec{\sigma}_1$ to decide another finite

set a_1 so that $a_1 \cup \{s_0\}$ is an antichain, a_1 has size 2^k, $a_1 \cap B_{n+1} = \emptyset$, and this extension forces $a_1 \cup \{s_0\} \subseteq I_n$; amalgamate the result back into $r_{0,k}$ to form $r_{1,1}$. Group a_1 into pairs and extend $r_{1,1} \restriction \vec{\sigma}_0$ to decide between the elements of the pairs as above, amalgamating the result back into $r_{1,1}$ to obtain $r_{1,0}$. Repeat as above to form $r_{1,2}$, $r_{1,3}$, etc. Notice that the last condition $r_{1,k}$ leaves a single element s_1 from the set a_1, and

the condition $r_{1,k}$ force that infinitely many nodes in $I_n \smallsetminus B_{n+1}$ are incomparable with *both* s_0 and s_1, and both can be included in S_{n+1}.

In general, for each $i \le k$ form $r_{i,i}$ first to decide a finite set a_i of size 2^k so that $a_i \cup \{s_0, \ldots, s_{i-1}\}$ forms an antichain, then form $r_{i,j}$ for $j \ne i$ in a descending sequence below $r_{i,i}$ by grouping the set a_i into pairs and extending to decide between

the elements of the pairs until arriving at a single element s_i, with the the $r_{i,j}$ conditions forcing now that infinitely many nodes in $I_n \smallsetminus B_{n+1}$ are incomparable with *all* of the nodes $s_0, s_1, \ldots, s_i$.

At the very bottom of this finite (length $(k+1)^2$) descending sequence of conditions $r_{i,j}$,

the last condition will be $r_{k,k-1}$ — let $p_{n+1} = r_{k,k-1}$ and $S_{n+1} = \{s_i : i \le k\}$.

It is now easy to check that p_{n+1} and S_{n+1} are as desired — in particular, $p_{n+1} \Vdash I_{n+1}$ infinite. Letting $A_\alpha = \bigcup_n S_n$, it is also easy to check that A_α is almost disjoint from all previous A_β (for $\beta < \alpha$) — this is what the enumeration $\{B_i : i < \omega\}$ accomplishes. Finally, letting q be the coordinatewise fusion of the sequence

$\langle p_n : n < \omega\rangle$ gives $q \Vdash |\tau_\alpha \cap A_\alpha| = \omega$ since for each n $q \le p_n$ and by construction

$p_n \Vdash \tau_\alpha \cap S_n \ne \emptyset$. This is exactly the q needed for the density argument.

Now $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ is an antichain family in V. So suppose G is $\mathbb{P}_\omega$-generic over V, $\rho \in V[G]$ is an infinite antichain, and $\dot\rho$ is a name for ρ. Then $D_{\dot\rho}$ is dense, so there is a $q \in G$

and an $\alpha < \omega_1$ such that $q \Vdash \dot\rho \cap A_\alpha$ infinite. Since $q \in G$, $\rho \cap A_\alpha$ is infinite in $V[G]$. But ρ was arbitrary, so $\mathcal{A}$ is maximal in $V[G]$. ■

Recall that this Theorem deals with antichain families and $\bar{\mathfrak{o}}$, giving a stronger result than if it had dealt with off-branch families and $\mathfrak{o}$.

7.3 **Corollary.**
Con$(\omega_1 = \mathfrak{b} = \mathfrak{a} = \mathfrak{d} = \bar{\mathfrak{o}} < 2^\omega = \kappa)$.

Proof. Start with a model $V \vDash ZFC + CH$. Force to add κ-many Sacks reals.

By the above Theorem, together with previous results concerning the effects of Sacks forcing on the other cardinal invariants [vD, BS],

this will give the desired model. ■

8 Open Questions.

The most important of the questions seems to be the following:

8.1 *Question.* $Con(\mathfrak{o} < \mathfrak{d})$?

However, the following also remains open:

8.2 *Question.* $Con(\mathfrak{o} < \bar{\mathfrak{o}})$?

I would like to suggest a line of inquiry for answering both of these questions:
Shelah's model of $Con(\mathfrak{b} < \mathfrak{s})$ (see [Sh])
gives a context in which $\mathfrak{b} = \mathfrak{a} = \omega_1$ and
$\mathfrak{d} = \mathfrak{a}_s = \bar{\mathfrak{o}} = 2^\omega = \omega_2$, obtained by countable support
iterated proper forcing. (The invariant $\mathfrak{s}$ is the "splitting number" — see [vD] for a definition and basic results, [BS] for the proof that $\mathfrak{s} \leq \mathfrak{a}_s$.) If $\mathfrak{o} = \omega_1$ in this construction, or there is some modification which will guarantee this, then both questions are answered in the affirmative. I conjecture that this technique will work.

Another avenue of inquiry open at this writing is the effects of various other standard forcing notions on off-branch families (the referee suggested Miller's superperfect forcing as an example).

While such inquiries would prove interesting in their own right,
they are not likely to directly address either of the above questions.

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Thomas E. Leathrum
Department of Mathematics and Computer Science
Dartmouth College
Hanover, NH 03755
Current address:
Division of Mathematical and Natural Sciences
Berry College
Mount Berry, GA 30165
E-mail address: moth@bluejay.atl.ga.us