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# Two New Criteria for Comparison in the Bruhat Order

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**Abstract**

We give two new criteria by which pairs of permutations may be compared in defining the Bruhat order (of type $A$). One criterion utilizes totally nonnegative polynomials and the other utilizes Schur functions.

The Bruhat order on $S_n$ is often defined by comparing permutations $\pi = \pi(1)\cdots\pi(n)$ and $\sigma = \sigma(1)\cdots\sigma(n)$ according to the following criterion: $\pi \leq \sigma$ if $\sigma$ is obtainable from $\pi$ by a sequence of transpositions $(i, j)$ where $i < j$ and $i$ appears to the left of $j$ in $\pi$. (See e.g. [7, p. 119].) A second well-known criterion compares permutations in terms of their defining matrices. Let $M(\pi)$ be the matrix whose $(i, j)$ entry is 1 if $j = \pi(i)$ and zero otherwise. Defining $[i] = \{1, \ldots, i\}$, and denoting the submatrix of $M(\pi)$ corresponding to rows $I$ and columns $J$ by $M(\pi)_{I,J}$, we have the following.

**Theorem 1** *Let $\pi$ and $\sigma$ be permutations in $S_n$. Then $\pi$ is less than or equal to $\sigma$ in the Bruhat order if and only if for all $1 \leq i, j \leq n-1$, the number of ones in $M(\pi)_{[i],[j]}$ is greater than or equal to the number of ones in $M(\sigma)_{[i],[j]}$.*

(See [1], [2], [3], [6, pp. 173-177], [8] for more criteria.) Using Theorem 1 and our defining criterion we will state and prove the validity of two more criteria.

Our first new criterion defines the Bruhat order in terms of totally nonnegative polynomials. A matrix $A$ is called *totally nonnegative* (TNN) if the determinant of each square submatrix of $A$ is nonnegative. (See e.g. [5].) A polynomial in $n^2$ variables $f(x_{1,1}, \ldots, x_{n,n})$ is called *totally nonnegative* (TNN) if for each TNN matrix $A = (a_{i,j})$

the number $f(a_{1,1}, \dots, a_{n,n})$ is nonnegative. Some recent interest in TNN polynomials is motivated by problems in the study of canonical bases. (See [10].)

**Theorem 2** *Let $\pi$ and $\sigma$ be two permutations in $S_n$. Then $\pi$ is less than or equal to $\sigma$ in the Bruhat order if and only if the polynomial*

$$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} \tag{1}$$

*is totally nonnegative.*

*Proof:* ($\Rightarrow$) If $\pi = \sigma$ then (1) is obviously TNN. Suppose that $\pi$ is less than $\sigma$ in the Bruhat order. If $\pi$ differs from $\sigma$ by a single transposition $(i, j)$ with $i < j$, then we have $\pi(i) = \sigma(j) < \pi(j) = \sigma(i)$, and the polynomial (1) is equal to

$$\frac{x_{1,\pi(1)} \cdots x_{n,\pi(n)}}{x_{i,\pi(i)} x_{j,\pi(j)}} (x_{i,\pi(i)} x_{j,\pi(j)} - x_{i,\pi(j)} x_{j,\pi(i)}) \tag{2}$$

which is clearly TNN. If $\pi$ differs from $\sigma$ by a sequence of transpositions, then the polynomial (1) is equal to a sum of polynomials of the form (2) and again is TNN.

($\Leftarrow$) Suppose that $\pi$ is not less than or equal to $\sigma$ in the Bruhat order. By Theorem 1 we may choose indices $1 \leq k, \ell \leq n-1$ such that $M(\sigma)_{[k],[\ell]}$ contains $q+1$ ones and $M(\pi)_{[k],[\ell]}$ contains $q$ ones. Now define the matrix $A = (a_{i,j})$ by

$$a_{i,j} = \begin{cases} 2 & \text{if } i \leq k \text{ and } j \leq \ell, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that $A$ is TNN, since all square submatrices of $A$ have determinant equal to $0, 1$, or $2$. Applying the polynomial (1) to $A$ we have

$$a_{1,\pi(1)} \cdots a_{n,\pi(n)} - a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} = -2^q,$$

and the polynomial (1) is not TNN. □

Our second new criterion defines the Bruhat order in terms of Schur functions. (See [9, Ch. 7] for definitions.) Any finite submatrix of the infinite matrix $H = (h_{j-i})_{i,j \geq 0}$, where $h_k$ is the $k$th complete homogeneous symmetric function and $h_k = 0$ for $k < 0$, is called a *Jacobi-Trudi matrix*. Let us define a polynomial in $n^2$ variables $f(x_{1,1}, \dots, x_{n,n})$ to be *Schur nonnegative* (SNN) if for each Jacobi-Trudi matrix $A = (a_{i,j})$ the symmetric function $f(a_{1,1}, \dots, a_{n,n})$ is equal to a nonnegative linear combination of Schur functions. Some recent interest in SNN polynomials is motivated by problems in algebraic geometry [4, Conj. 2.8, Conj. 5.1].

**Theorem 3** *Let $\pi$ and $\sigma$ be permutations in $S_n$. Then $\pi$ is less than or equal to $\sigma$ in the Bruhat order if and only if the polynomial*

$$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} \tag{3}$$

*is Schur nonnegative.*

*Proof:* ($\Rightarrow$) If $\pi = \sigma$ then (3) is obviously SNN. Let $A$ be an $n \times n$ Jacobi-Trudi matrix and suppose that $\pi$ is less than $\sigma$ in the Bruhat order. If $\pi$ differs from $\sigma$ by a single transposition $(i, j)$, then for some partition $\nu$ and some $k, \ell, m$ ($\ell, m > 0$), the evaluation of the polynomial (3) at $A$ is equal to

$$h_\nu(h_{k+\ell}h_{k+m} - h_{k+\ell+m}h_k), \tag{4}$$

and (3) is clearly SNN. If $\pi$ differs from $\sigma$ by a sequence of transpositions, then the evaluation of (3) at $A$ is equal to a sum of polynomials of the form (4) and again (3) is SNN.

($\Leftarrow$) Suppose that $\pi$ is not less than or equal to $\sigma$ in the Bruhat order. By Theorem 1 we may choose indices $1 \leq k, \ell \leq n-1$ such that $M(\sigma)_{[k],[\ell]}$ contains $q+1$ ones and $M(\pi)_{[k],[\ell]}$ contains $q$ ones. Now define the nonnegative number $r = (k-q)(n+k-\ell-2)$ and consider the Jacobi-Trudi matrix $B$ defined by the skew shape $(n-1+2r)^k(n-1+r)^{n-k}/r^\ell$,

$$B = \begin{bmatrix} h_{n-1+r} & \cdots & h_{n+\ell-2+r} & h_{n+\ell-1+2r} & \cdots & h_{2n-2+2r} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_{n-k+r} & \cdots & h_{n-k+\ell-1+r} & h_{n-k+\ell+2r} & \cdots & h_{2n-k-1+2r} \\ h_{n-k-1} & \cdots & h_{n-k+\ell-2} & h_{n-k+\ell-1+r} & \cdots & h_{2n-k-2+r} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_0 & \cdots & h_{\ell-1} & h_{\ell+r} & \cdots & h_{n-1+r} \end{bmatrix}.$$

The polynomial (3) applied to $B$ may be expressed as $h_\lambda - h_\mu$ for some appropriate partitions $\lambda, \mu$ depending on $\pi, \sigma$, respectively. We claim that $\lambda$ is incomparable to or greater than $\mu$ in the dominance order. Since $M(\pi)_{[k],[\ell+1,n]}$ contains $k-q$ ones we have that

$$\lambda_1 + \cdots + \lambda_{k-q} \geq (k-q)(n-k+\ell+2r). \tag{5}$$

Similarly, we have

$$\mu_1 + \cdots + \mu_{k-q} \leq (k-q-1)(2n-2+2r) + \max\{n+\ell-2+r, 2n-k-2+r\}. \tag{6}$$

Subtracting (6) from (5), we obtain

$$(\lambda_1 + \cdots + \lambda_{k-q}) - (\mu_1 + \cdots + \mu_{k-q}) \geq n - \max\{\ell, n-k\} > 0,$$

as desired.

Recall that the Schur expansion of $h_\mu$ is

$$h_\mu = s_\mu + \sum_{\nu > \mu} K_{\nu,\mu} s_\nu,$$

where the comparison of partitions $\nu > \mu$ is in the dominance order and the nonnegative *Kostka numbers* $K_{\nu,\mu}$ count semistandard Young tableaux of shape $\nu$ and content $\mu$. (See e.g. [9, Prop. 7.10.5, Cor. 7.12.4].) It follows that the coefficient of $s_\mu$ in the Schur expansion of $h_\lambda - h_\mu$ is $-1$ and the polynomial (3) is not SNN. $\square$

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