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Monomial Nonnegativity and the Bruhat Order

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Abstract

We show that five nonnegativity properties of polynomials coincide when restricted to polynomials of the form $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$, where π and σ are permutations in S_n. In particular, we show that each of these properties may be used to characterize the Bruhat order on S_n.

1 Introduction

Let $x = (x_{ij})$ be a generic square matrix and define $\Delta_{I,I'}(x)$ to be the (I, I') minor of x, i.e., the determinant of the submatrix of x corresponding to rows I and columns I'. A real matrix is called *totally nonnegative* (TNN) if each of its minors is nonnegative. (See e.g. [9].) A polynomial $p(x_{11}, \ldots, x_{nn})$ in n^2 variables is called totally nonnegative if it satisfies

$$p(A) \underset{\text{def}}{=} p(a_{1,1}, \ldots, a_{n,n}) \geq 0 \tag{1}$$

for each $n \times n$ totally nonnegative matrix $A = (a_{i,j})$. Some recent interest in total nonnegativity concerns a set of polynomials known in quantum Lie theory as the *dual canonical basis* of $\mathcal{O}(SL(n, \mathbb{C}))$. (See e.g. [25].) In particular, Lusztig [17] has proved that these polynomials are TNN.

A polynomial $p(x)$ which is equal to a subtraction-free rational expression in matrix minors must be TNN. (By a result of Whitney [24], we need not be concerned that the denominator vanishes for some TNN matrices.) We shall say that such a polynomial $p(x)$ has the *subtraction-free rational function* (SFR) property. If this subtraction-free rational expression may be chosen so that the denominator is a monomial in matrix minors, we shall say that $p(x)$ has the *subtraction-free Laurent* (SFL) property. One example of a polynomial having the SFL property is

$$x_{1,2}x_{2,1}x_{3,3} - x_{1,2}x_{2,3}x_{3,1} - x_{1,3}x_{2,1}x_{3,2} + x_{1,3}x_{2,2}x_{3,1} \\ = \frac{\Delta_{13,23}(x)\Delta_{23,13}(x) + \Delta_{1,3}(x)\Delta_{3,1}(x)\Delta_{23,23}(x)}{\Delta_{3,3}(x)}.$$

Analogous classes of polynomials may be defined in terms of symmetric functions. (See [21, Ch. 7] for basic definitions concerning symmetric functions.) In particular, any finite submatrix of the infinite matrix $H = (h_{j-i})_{i,j\geq 0}$, where h_k is the kth complete homogeneous symmetric function and $h_k = 0$ for $k < 0$, is called a *Jacobi-Trudi matrix*. We define a polynomial $p(x_{1,1}, \ldots, x_{n,n})$ to be *monomial nonnegative* (MNN) if for each Jacobi-Trudi matrix $A = (a_{i,j})$ the symmetric function $p(A)$ is equal to a nonnegative linear combination of monomial symmetric functions. Defining *Schur nonnegative* (SNN) polynomials analogously, we have that every SNN polynomial is MNN. Some recent interest in SNN polynomials is motivated by problems in algebraic geometry [8, Conj. 2.8, Conj. 5.1], [1].

2 Main result

The five nonnegativity properties defined in Section 1 have been applied most often to *immanants*, polynomials which belong to $\mathrm{span}_{\mathbb{C}}\{x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} \,|\, \sigma \in S_n\}$. (See [11], [12], [13], [20], [19], [22], [23]. The results of [7] may also be stated in these terms.) Curiously, the TNN, MNN, and SNN properties coincide when applied to immanants in the main theorems of the above papers. It is also curious that none of these immanants is known *not* to have the SFL property. It would be interesting to identify immanants which have some of these nonnegativity properties and fail to have others. Nevertheless, our main result shows that the five properties coincide when applied to immanants of the form

$$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}.$$

We shall use the following well-known characterizations of the Bruhat order on S_n. The Bruhat order on S_n is often defined by comparing two permutations $\pi = \pi(1)\cdots\pi(n)$ and $\sigma = \sigma(1)\cdots\sigma(n)$ according to the following criterion: $\pi \leq \sigma$ if σ is obtainable from π by a sequence of transpositions (i, j) where $i < j$ and i appears to the left of j in π. (See e.g. [14, p. 119].) A second well-known criterion compares permutations in terms of their defining matrices. Let $M(\pi)$ be the matrix whose (i, j) entry is 1 if $j = \pi(i)$ and zero otherwise. Defining $[i] = \{1, \ldots, i\}$, and denoting the submatrix of $M(\pi)$ corresponding to rows I and columns J by $M(\pi)_{I,J}$, we have the following.

Theorem 1 *Let π and σ be two permutations in S_n. Then π is less than or equal to σ in the Bruhat order if and only if for all $1 \leq i, j \leq n-1$, the number of ones in $M(\pi)_{[i],[j]}$ is greater than or equal to the number of ones in $M(\sigma)_{[i],[j]}$.*

(See [2], [3], [4], [6], [10, pp. 173-177], [16], [15], [18]. for more characterizations.)

Our result, combined with those of our previous paper [5], is the following list of nonnegativity criteria with which one may define the Bruhat order.

Theorem 2 *Let π and σ be permutations in S_n. The following conditions on π and σ are equivalent.*

1. *$\pi \leq \sigma$ in the Bruhat order.*
2. *$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ is totally nonnegative.*
3. *$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ is Schur nonnegative.*
4. *$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ is monomial nonnegative.*
5. *$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ has the subtraction-free rational function property.*
6. *$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ has the subtraction-free Laurent property.*

Proof: The implications $(3 \Rightarrow 4)$ and $(6 \Rightarrow 5 \Rightarrow 2)$ are immediate. The implication $(2 \Rightarrow 1)$ was estblished in [5, Thm. 2], and the implication $(1 \Rightarrow 6)$ follows trivially from that proof. The implication $(1 \Rightarrow 3)$ was established in [5, Thm. 3]. It will suffice therefore to prove the implication $(4 \Rightarrow 1)$.

Suppose that π is not less than or equal to σ in the Bruhat order. By Theorem 1 we may choose indices $1 \leq k, \ell \leq n-1$ such that $M(\sigma)_{[k],[\ell]}$ contains $q+1$ ones and $M(\pi)_{[k],[\ell]}$ contains q ones. Keeping n fixed, let b be a large nonnegative integer which satisfies

$$\binom{2b}{b} > (2b+2n)^{2n^2},$$

(which is possible because $\binom{2b}{b}$ grows exponentially) and consider the $n \times n$ Jacobi-Trudi matrix

$$B = \begin{bmatrix} h_{b+k-1} & \cdots & h_{b+k+\ell-2} & h_{2b+k-1} & \cdots & h_{2b+n+k-\ell-2} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_b & \cdots & h_{b+\ell-1} & h_{2b} & \cdots & h_{2b+n-1-\ell} \\ h_{n-k-1} & \cdots & h_{n-k+\ell-2} & h_{b+n-k-1} & \cdots & h_{b+2n-k-\ell-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_0 & \cdots & h_{\ell-1} & h_b & \cdots & h_{b+n-\ell-1} \end{bmatrix},$$

defined by the skew shape $(2b+k-\ell-1)^k(b+n-\ell-1)^{n-k}/(b-\ell)^\ell$. Let

$$s = k(2b+k-\ell-1) + (n-k)(b+n-\ell-1) - \ell(b-\ell)$$

be the number of boxes in this skew shape.

The polynomial $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ applied to B may be expressed as $h_\lambda - h_\mu$ for some appropriate partitions λ, μ of s, which depend on π, σ, respectively. We claim that the coefficient of m_{1^s} in the monomial expansion of $h_\lambda - h_\mu$ is negative.

Note that the ratio of the coefficients of m_{1^s} in the monomial expansions of h_λ and h_μ is

$$\frac{\binom{s}{\lambda_1,\ldots,\lambda_n}}{\binom{s}{\mu_1,\ldots,\mu_n}} = \frac{\mu_1!\cdots\mu_n!}{\lambda_1!\cdots\lambda_n!}.$$

By the locations of ones in the matrices $M(\pi)$ and $M(\sigma)$, this ratio is less than or equal to

$$\frac{(2b+2n)!^{k-q-1}}{(2b)!^{k-q}}\frac{(b+2n)!^{n-k-\ell+2q+2}}{b!^{n-k-\ell+2q}}\frac{(2n)!^{\ell-q-1}}{0!^{\ell-q}},$$

which in turn is less than or equal to

$$\begin{aligned}\frac{(2b+2n)^{2n(k-q-1)}}{(2b)!}(b+2n)!^2(2b+2n)^{2n(n-k+q-1)} &= \frac{(b+2n)!^2}{(2b)!}(2b+2n)^{2n(n-2)} \\ &\leq \frac{(2b+2n)^{2n^2}}{\binom{2b}{b}},\end{aligned}$$

which is less than 1 by our choice of b. It follows that the coefficient of m_{1^s} in the monomial expansion of $h_\lambda - h_\mu$ is negative and the polynomial $x_{1,\pi(1)}\cdots x_{n,\pi(n)} - x_{1,\sigma(1)}\cdots x_{n,\sigma(n)}$ is not MNN. □

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